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Some new examples of nuclear spaces

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Abstract

In this paper, we study about nuclear spaces of continuous functions and find a new example of nuclear spaces of continuous functions which help us for better understanding of the nuclear spaces of continuous functions. We show that the space of infinitely differential functions on an open set in R^2 is nuclear.

Keywords: continuous functions, infinitely differential functions, nuclear space

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Introduction

The concept of a nuclear space arose in an investigation of the question : for what spaces are the analogous Schwarz's kernel theorem valid? The fundamental results in the theory of nuclear spaces are due to A. Grothendieck [4,5]. The function spaces used in analysis are, as a rule, Banach or nuclear spaces. Nuclear spaces play an important role in the spectral analysis of operators on Hilbert spaces. Nuclear spaces are closely connected with measure theory on locally convex spaces [3,6].

The definition of nuclear locally convex spaces which was introduced by A. Grothendieck in 1951 in setting of his theory of topological tensor products [1,4].

G.Kothe has shown that every complete quasi-barreled co-nuclear space has the approximation property. He also asked whether the same is true for any co-nuclear space [2,8].

In the present paper, we study about nuclear space of continuous functions and find a new example of nuclear spaces of continuous functions such as the space of infinitely differential functions on an open set in R^2 is nuclear.

Let g be an open domain in R^n and let C(g) denotes the space of all continuous functions on g. The topology of C(g) is given by seminorms

$$\|f\|_k = \sup_{t \in k} \left| f(t) \right| .$$

Since we can find $k_1 \subseteq k_2 \subseteq \cdots \subseteq g$ with $\bar{k}_n \subseteq$ interior (K_{n+1}) and $\bigcup_{n=1}^{\infty} k_n$ = g, C(g) is an (F)-space.

We now determine which subspace E(g) of C(g) are nuclear. Let E(g) be a subspace of C(g) and k a compact subset of g. consider $u(k) = \{ f \in E(g) : \|f\|_k \leq 1\}$.

Now $E_{u(k)}$= E/ N(u(k)) and $\|\hat{f}\|_k = \|f\|_k$.

Let $E_{u(k)}$ be the space of all restrictions f_k of f ϵ E(g) to k. Two functions f and g in E(g) are in the same residue class in $E_{u(k)}$ iff $\|f-g\|_k = 0$ i.e. iff $f_k = g_k$.So$E_{u(k)} \cong E/N(u(k)) \cong$ E(k).

The canonical mapping $E_{u(H)} \to E_{u(k)}$ where u(H) < u(k) is just the restriction mapping E(H) → E(k), where H ⊇ k. Thus we have:

Definitions

Definition: Let E and F are two arbitrary real or complex locally convex spaces. A function T from E into F is called *continuous* if the pre- image

$$G = T^{-1}(G') = \{ x \in E : T x \in G' \}$$

Of each open subset G' of F is an open subset of E [12].

Definition: Let f is a function defined around a point c ϵ R. We say that f is *infinitely differentiable* at c if all higher derivatives $f^{(k)}(c)$ exist as finite numbers for all nonnegative integers k[14].

Definition: The function f in E(g) are said to satisfy on a *priori estimate* of order $p \geq 1$ if

there exists a positive Radon measure μ on g such that for every compact $k \subseteq g$, there is a compact H, $k \subseteq H \subseteq g$, and a positive constant ρ(k) > 0 such that

$$\|f\|_k = \sup_{t \in k} |f(t)| \leq \rho(k)\left(\int_H |f(s)|^p d\mu(s)\right)^{1/p} \text{ for all } f \in E(g) \text{ [7].}$$

Definition: The functions of E(g) are said to have a *kernel representation* if there exists a positive Radon measure on g such that for every compact k, there exists a compact H, $k \subseteq H \subseteq$ g such that $f(t) = \int_H f(s)M(s,t)d\mu(s)$ for all t ϵ k and f ϵ E(g). M(s, t) is defined on $H \times k$, is μ measurable and essentially bounded on $H \times k$, and depend only on k and H[6,9].

Definition: A locally convex space E is called *nuclear* if there is a fundamental system $\mathcal{V}$ of neighborhoods of 0 in E such that:

(N) for every neighborhood of 0, $U \in \mathcal{V}$ there exists a neighborhood of 0, V with V < U (i.e. V is absorbed by U) such that the canonical mapping from E_V onto E_U is nuclear or quasi-nuclear or absolutely summing or equivalently.

(N_1) for every neighborhood of 0, U in $\mathcal{V}$ there exists a neighborhood of 0,V with V < U such that the canonical mapping from $E'(U^0)$ into $E'(V^0)$ is nuclear or quasi-nuclear or absolutely summing[5,10].

Propositions

Proposition: E(g) is nuclear if and only if for every compact k ⊆ g there exists a compact H, k ⊆ H ⊆ g, such that the restriction of f ϵ E(H) to k is a nuclear mapping, i.e.$f_H \rightarrow f_k$ is a nuclear map for all f ϵ E(g).

Proposition: A space E(g) is nuclear if and only if its functions satisfy on a priori estimate . of order p for some p, 1≤ p ≤ 2 [7].

Proposition: The space E(g) of all local $w_i^{'}$-solutions of $L^{\varphi} = 0$ is nuclear.

Proof: Let k be a compact subset of g. Cover k with a finite number of balls $|t - t_i| < \nu$ with t_i ϵ k, whose radius ν is sufficiently small so that

$$H = \bigcup_{i=1}^{n}\{t: |t - t_i| \leq 2\nu\} \subseteq g.$$

From the local a priori estimates given above we have

$$\sup_{|t-t_i| \leq \nu} |\varphi(t)| \leq \sigma_i \left(\int_{|t-t_0| \leq 2\nu} |\varphi(t)|^2 \mu(t)dt \right)^{1/2}$$

It follows

$$\sup_{t\epsilon k} |\varphi(t)| \leq \sigma \left(\int_H |\varphi(t)|^2 \mu(t)dt \right)^{1/2}$$

σ = max. σ_i.

The result now follows from prop.(3.2) by letting μ be the positive Radon measure defined by $< \varphi, \mu > = \int_g \varphi(t)\mu(t)dt$.

Proposition: E(g) is nuclear if and only if its functions have a kernel representation[1].

Proof: If the functions of E(g) have a Kernel representation, then

$$\sup_{t\in k} |f(t)| \leq M \int_H |f(s)| d\mu(s)$$

and we have on a priori estimate of order p =1. So by prop.(3.2), E(g) is nuclear.

Suppose conversely E(g) is nuclear. Then as we proved in prop.(3.2), there exists on a priori estimate of order p =1,

$$\sup_{t\in k} |f(t)| \leq \rho(k) \int_H |f(s)| d\mu(s).$$

If we define $\| f \|_H^{(i)} = \int_H |f(s)| d\mu(s)$,

We see that $\| f \|_k \leq \rho(k) \| f \|_H^{(i)} \leq \rho(k) \| f \|_H \ \mu(H)$

and hence the topologies given by the semi-norms and $\| f \|_k$ and $\| f \|_H^{(i)}$ are equivalent. By nuclearity, given k there exists H so that the canonical mapping (i.e. the restriction map) $E_\mu^{(i)}(H) \rightarrow E(k)$ is nuclear. This means

$$f(t) = \sum_{i=1}^{\infty} < v_i, f > f_i(t) \text{ for all } t \in k$$

where $\mu_i \in E_\mu^{(i)}(H)' \subseteq \int_\mu(H)' = \int_\mu(H)$, $f_i \in E(k)$, and $\sum_{i=1}^{\infty} \|v_i\|_H^{\infty} \|f_i\|_k < \infty$

Each μ_i can be extended by Hahn- Banach to a continuous linear functional on $\int_\mu^1(H)$. So each μ_i is an element of $\int_\mu^\infty(H)$, and hence a bounded μ-measureable function.

$< v_i, f > = \int_H v_i(t)f(t)d\mu(t)$.

Define $M(s, t) = \sum_{i=1}^\infty v_i(r) f_i(t)$ $r \in H$, $t \in k$.

Observe $|M(s, t)| \leq \sum_{i=1}^\infty \|v_i\|_H^\infty \|f_i\|_k < \infty$….. ………….. $(*)$

We have,

$$f(t) = \sum_{i=1}^\infty < v_i, f > f_i(t) = \sum_{i=1}^\infty (\int_H v_i(s)f(s)\, d\mu(s) f_i(t)$$

$$= \int_H (\sum_{i=1}^\infty v_i(s)f_i(t)f(s)d\mu(s)$$

$$= \int_H M(s,t)\, f(s)d\mu(s)$$

The $\int_H$ and $\sum_{i=1}^\infty$ can be interchanged because of $(*)$.

Examples

Example- 1 : Let ε(I) = space of infinitely differentiable functions on I.

Let I = [a, b] be a compact interval in R . ε(I) denotes the space of all functions defined on I with infinitely many derivatives. The topology on ε(I) is given by the sequence of norms $\|f\|_N = \sup_{t \in [a,b]} \sum_{k=o}^r |f^k(t)|$ where r = 1,2,… . With this topology ε(I) is an (F)-space.

We now want to show ε(I) is nuclear. The equality

$$f^{(k)}(t) - f^{(k)}(a) = \int_a^t f^{(k+1)}(s)ds$$

yields the inequality,

$$|f^k(t)| \leq \int_a^b |f^{(k+1)}(s)|ds + |f^k(a)|$$

From this we obtain,

$$\| f \|_N \leq \sum_{k=0}^N (\int_a^b |f^{(k+1)}(s)|ds + |f^k(a)|)\ldots\ldots\ldots\ldots\ldots(*)$$

Using $(*)$ we shall establish that the criterion of nuclearity is satisfied. We shall show that given $U = \{f: \| f \| \leq 1\}$, there exists $V = \{f: \| f \|_{r+1} \leq 1\}, V < U$, and a positive Radon measure μ on V^0 such that

$\| f \|_{\mathcal{V}} = p_u(f) \leq \int_{V^0} |f(v)|\, d\mu(v\ )$ for all f ϵ ε(I) .

Define $< \delta_s^{(k)}, f > = f^{(k)}(s)$. Then $\delta_s^{(k)} \in V^0$ for s ϵ[a,b] and for k = 0,1, 2,…, r + 1, since

$$| < \delta_s^{(k)}, f >= |f^{(k)}(s)| \leq \| f \|_{r+1}.$$

Now define a continuous linear functional on C(V^0) as follows:

For $\varphi(\mathcal{V})$ ϵ C(V^0), set

$$\mu(\varphi) = \int_{V^0} \varphi\,(v)\,d\mu(v) = \sum_{k=0}^r \left(\int_a^b \varphi(\delta_s^{(k+1)})ds + \varphi(\delta_a^{(k)})\right)$$

μ is continuous on C(V^0) since

$$\mu(\varphi)\sum_{k=o}^{r}(\int_{a}^{b}\sup_{\mu\varepsilon V^o}|\varphi(v)|\,ds+\sup_{\mu\varepsilon V^0}|\varphi(v)|\leq K\parallel\varphi\parallel_\infty$$

So μ is Radon measure on V^o and we have,

$$\parallel f\parallel_r\leq\sum_{k=o}^{r}(\int_{a}^{b}\left|f^{(k+1)}(s)\right|ds+\left|f^k(a)\right|)\qquad \text{by } (*)$$

$$=\sum_{k=o}^{r}\left(\int_{a}^{b}|f\left(\delta_s^{(k+1)}\right)ds+\left|f\left(\delta_a^{(k)}\right)\right|\right)$$

$$=\int_{V^o}|f(v)|\,d\mu\,(v).$$

This proves ε(I) is nuclear.

Example – 2 : Let ε(g) = space of infinitely differentiable functions on g, where g is an open set in R^2. Then the topology on ε(g) is given by the seminorms

$p_{k,r}(f)=\sup_{t\varepsilon k}\sum_{k=o}^{r}|f^{(k)}(t)|$, k is compact set in g, r an integer.

Given a compact set k ⊆ g, cover k with a grid of compact squares k_1, k_2,.. k_n such that $k\subseteq\bigcup_{i=1}^{n}k_i\subseteq$ g. Assume now the seminorms

$$\parallel f\parallel_{i,r}=\sup_{t\varepsilon k_i}\sum_{k=o}^{r}|f^{(k)}(t)|$$

The restriction of a function in ε(g) to k_i is a function in $\varepsilon(k_i)$. As the techniques outlined in the proof of example (4.1), we set

$V_i=\{f:\parallel f\parallel_{i,r+1}\leq 1\}$

and obtain a Radon measure μ_i on V_i^0 such that

$$\parallel f\parallel_{i,r}\ \leq\int_{V_i^o}|f(\mu)|d\mu_i(\mu)\quad\text{for all } f\in\varepsilon(k_i)$$

and therefore also for all f ϵ ε(g). We can regard V_i as a nhd. in ε(g). Set $V=\bigcap_{i=1}^{n}V_i$. On V, define a Radon measure μ by

$\int_{V^o}\varphi(v)\,d\mu(v)=\sum_{i=1}^{n}\int_{V_i^o}\varphi(v)\,d\mu(v)$ for all φ ϵ $C(V^o)$.

Letting $\parallel f\parallel_{k,r}=\ \sup_{t\varepsilon k}\sum_{k=o}^{r}|f^{(k)}(t)|$ denote a seminorm corresponding to the originally given compact set k, we have

$\parallel f\parallel_{k,r}\leq\parallel f\parallel_{k_i,r}=\sup_{t\in\bigcup_{i=1}^{n}k_i}\sum_{k=0}^{r}\left|f^{(k)}(t)\right|$

$\leq\sum_{i=1}^{n}\int_{V_i^0}|f(v)|\,d\mu_i(v)=\int_{V^o}|f(v)|d\mu(v).$

Thus prove ε(g) is nuclear since we can find $k_1\subseteq k_2\subseteq\ldots\subseteq$ g with $\bigcup_{n=1}^{\infty}k_n=$ g, the topology on ε(g) can be defined by a sequence of seminorms $\parallel\ \parallel_{k_n,r}$.

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