Traveling Salesman Problem with Time Specific Profit on Resource

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Abstract

In general, the classical Traveling Salesman Problem (TSP) with assumptions of uniform travel times does not fully apply due to factors like fluctuating traffic conditions, diverse transportation modes, and varying resource availability. This research seeks to tackle the optimization of time-specific profit within the TSP framework while considering travel times dependent on available resources. The study also introduces an approach to extend the classical TSP model, accommodating resource-dependent travel times to maximize profit from visiting a specific set of locations within a defined timeframe. A mathematical formulation is presented, integrating the variables of travel time variability, resource availability, and profit generation into the TSP framework. This method acknowledges the dynamic nature of travel times and efficiently utilizes resources to achieve optimal profit. Furthermore, it is ensured that identifying the optimal solution using this approach will not pose a greater computational challenge than solving the classical TSP.

Keywords: TSP, resource, optimal profit, travel time

Introduction

The Traveling Salesman Problem (TSP) is a classic and fundamental challenge within the realm of operations research, representing a key combinatorial optimization problem. At its core, the TSP involves finding the most efficient route that connects a specified set of locations, with the requirement of making a single stop at each location before returning to the starting point. A central challenge of the TSP revolves around determining the most cost-effective sequence for the salesman to visit the various locations. The primary objective is to construct a tour that, at the lowest possible cost, includes every designated site. Real-world scenarios introduce variability in travel times due to factors such as traffic congestion, diverse transportation modes, and fluctuating resource availability. In contrast, the classical TSP assumes constant travel times between sites, adding a new layer of complexity to the optimization process: the need to minimize travel distance while also optimizing for the time-specific profit generated from these journeys.

The Traveling Salesman Problem (TSP) is a well-known NP-hard problem, widely recognized for its significant computational complexity. Various problem-solving approaches exist, including cutting planes, branch and bound algorithms, dynamic programming algorithms, and other recommended practices. Additionally, employing approximation techniques is a viable strategy for tackling the TSP. Solution procedures for the problem in some dimensions can be found in works by Hartley (1985), Williams (1999), Cook et al. (2008), and Cook (2011) etc.

The Traveling Salesman Problem (TSP) is traditionally based on the assumption that the distance (or travel time or cost) between any pair of cities is fixed. However, in practical settings such as manufacturing applications or transportation systems, processing or travel times can be influenced and managed using finite, expendable resources such as financial budgets, overtime, energy, fuel, subcontracting, or manpower. It's common for increasing the available resources to yield diminishing reductions in travel time.

This study introduces a unique extension of the Traveling Salesman Problem (TSP) where travel times are influenced by available resources, with a constraint on the total resource capacity. The objective goes beyond determining the sequence of city visits; it also aims to optimize resource allocation to maximize profit per unit of time, termed the profit rate. In contrast to typical TSP variants with predefined edge weights (representing travel times), this research assumes that edge weights are contingent on available resources, with a limit on total resource availability. Our problem has significance in various engineering and scientific domains requiring a delicate balance between time and cost, aligning with the primary objective of organizations focused on maximizing profit.

The influence of resource allocation on travel (or processing) time is commonly depicted through a resource consumption function. In studies involving sequencing problems, a linear resource consumption function is frequently assumed, as demonstrated in prior research by Vickson (1980), Van Wassenhove and Baker (1982), Daniels and Sarin (1989), Janiak and Kovalyov (1996), and Cheng et al (1998). Nevertheless, this linear assumption overlooks the marginal value product rule, which dictates that productivity rises at a diminishing rate as resources are augmented. Hence, in this research, we opt to use the resource consumption function formulated by Monma et al (1990) to characterize traveling time. This choice is made because it more accurately mirrors the realistic correlation between resources and productivity.

$$t_{ij} = \left(\frac{w_{ij}}{r_{ij}}\right)^k \tag{1}$$

In equation (1), w_{ij}, r_{ij} and t_{ij} respectively represent the workload, resource allocated and travelling time of the edge {i, j} joining vertex i and vertex j. The equation reveals that each traveling time t_{ij} solely depends upon the allocated resource r_{ij} and it decreases as the amount of resource allocated increases. The marginal traveling time diminishes with higher resource consumption, and when no resources are allocated, the traveling time becomes infinite.

Problem description and notation

We consider the notations t_{ij} and r_{ij} as described in equation (1), that represent travelling time and resource allocated to the edge {i, j}. Let C_{ij} represents cost (or distance) for traveling from vertex i to vertex j and R signifies the total resource consumption per tour, while V represents the contribution to profit per tour, measured in resource units. We may consider a binary decision variable x_{ij} defined as,

$$x_{ij} = \begin{cases} 1, & \text{if tour passes through edge \{i, j\}} \\ 0, & \text{otherwise} \end{cases}$$

The classic travelling salesman problem is formulated as follows (**P1**):

$$\text{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \ \ \forall \ i$$

$$\sum_{i=1}^{n} x_{ij} = 1 \ \ \forall \ j$$

$$x_{ij} = \{0, 1\} \ \ \forall \ i, j$$

No any subtours are allowed

The constraints in above problem guarantee that every chosen tour covers all vertices exactly once. To eliminate subtours, various methods can be employed. The method is described in the work of Miller et al (1960). As the main goal of the problem outlined in this paper is to optimize the profit per unit of time. This profit is obtained by subtracting the resource consumption from V and then dividing it by the total time taken to complete the tour. It's important to note that the travel time t_{ij} changes based on the resource allocation. The problem is now formulated based on the work of Zofi et al (2017) as follws (**P2**):

$$\text{Min} \ \frac{V - \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} x_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_{ij}} \tag{2}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \ \ \forall \ i; \ i = 1, 2, ..., n \tag{3}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \ \ \forall \ j; \ j = 1, 2, ..., n \tag{4}$$

$$x_{ij} = \{0, 1\} \ \ \forall \ i, j \tag{5}$$

No any subtours are allowed

Based on resource consumption function defined by Monma et al (1990), the objective function for profit rate (P) can be expressed as,

$$\text{Profit P} = \frac{V - R}{T} \tag{6}$$

where,

$$R = \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} x_{ij} \tag{7}$$

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{w_{ij}}{r_{ij}} \right)^{k} x_{ij} \tag{8}$$

Solution Approach

Now, let's take into account any possible feasible tour. This tour can be depicted as a sequence of connected edges forming a one-path graph. Monma et al. (1990) introduced the optimal resource allocation method to minimize the duration of a path in a series-parallel graph.

In the scenario, any feasible tour can be expressed as a series of interconnected graphs, where G_1 represents the edge linking the first and second vertices in the tour, G_2 represents the edge linking the second and third vertices in the tour, and so forth. And therefore $G = G_1 \rightarrow G_2 \rightarrow \cdots \rightarrow G_n$ represents the entire tour. Initially, our focus will be on solving the following problem to find the optimal resource allocation when we have a fixed, known quantity of nonrenewable resource R

$$\text{Min T} = \sum_{j=1}^{n} \left(\frac{w_j}{r_j} \right)^k \tag{9}$$

subject to

$$\sum_{j=1}^{n} r_j \leq R \tag{10}$$

The suitable Lagrange function is

$$L(r_1, r_2, \ldots, r_n, \lambda) = \sum_{j=1}^{n} \left(\frac{w_j}{r_j} \right)^k + \lambda \left(\sum_{j=1}^{n} r_j - R \right) \tag{11}$$

The sufficient conditions for optimal conditions are:

$$\frac{\partial L(r_1, r_2, \ldots, r_n, \lambda)}{\partial r_j} = 0 \qquad \forall\, j = 1, \ldots n \tag{12}$$

and

$$\frac{\partial L(r_1, r_2, \ldots, r_n, \lambda)}{\partial \lambda} = 0 \tag{13}$$

But from (11), we have

$$\frac{\partial L(r_1, r_2, \ldots, r_n, \lambda)}{\partial r_j} = k \left(\frac{w_j}{r_j} \right)^{k-1} \cdot \left(-\frac{w_j}{r_j^2} \right) + \lambda \tag{14}$$

and

$$\frac{\partial L(r_1, r_2, \ldots, r_n, \lambda)}{\partial \lambda} = \sum_{j=1}^{n} r_j - R \tag{15}$$

Hence from (12) and (14), we get

$$\lambda = \frac{k \left(\frac{w_j}{r_j} \right)^k}{r_j} \tag{16}$$

Since this condition is identical to all graphs $G_1,\ G_2,\ \ldots,\ G_n$, we may write

$$\frac{k\left(\frac{w_i}{r_i^*}\right)^k}{r_i^*} = \frac{k\left(\frac{w_j}{r_j^*}\right)^k}{r_j^*} = \cdots = \frac{k\left(\frac{w_n}{r_n^*}\right)^k}{r_n^*}$$

So, we conclude that

$$r_i^* = \frac{r_j^* w_i^{\frac{k}{k+1}}}{w_j^{\frac{k}{k+1}}} \tag{17}$$

Also using optimal condition (13) and equation (15), we can have

$$R = \sum_{i=1}^{n} r_i^* = \sum_{i=1}^{n} \frac{r_j^* w_i^{\frac{k}{k+1}}}{w_j^{\frac{k}{k+1}}} = \frac{r_j^*}{w_j^{\frac{k}{k+1}}} \sum_{i=1}^{n} w_i^{\frac{k}{k+1}} \tag{18}$$

The optimal resource allocation r_j^* for G_j in terms of R is

$$r_j^* = \frac{R\, w_j^{\frac{k}{k+1}}}{\sum_{i=1}^{n} w_i^{\frac{k}{k+1}}} \tag{19}$$

Also the optimal time to complete the entire tour is:

$$T^* = \sum_{j=1}^{n} \left(\frac{w_j}{r_j^*}\right)^k = \sum_{j=1}^{n} \frac{w_j^k}{\left(\frac{R\, w_j^{\frac{k}{k+1}}}{\sum_{i=1}^{n} w_i^{\frac{k}{k+1}}}\right)^k} = \frac{\left(\sum_{i=1}^{n} w_i^{\frac{k}{k+1}}\right)^k}{R^k} \sum_{j=1}^{n} w_j^{\frac{k}{k+1}}$$

Hence we can generalize the optimal time T as

$$T^* = \left(\frac{\left(\sum_{j=1}^{n} w_j^{\frac{k}{k+1}}\right)^{\frac{k+1}{k}}}{R}\right)^k \tag{20}$$

If we assume the workload w_G as

$$w_G = \left(\sum_{j=1}^{n} w_j^{\frac{k}{k+1}}\right)^{\frac{k+1}{k}} \tag{21}$$

Then, for a given total resource consumption R, the optimal time T of graph G is:

$$T^* = \left(\frac{w_G}{R}\right)^k \tag{22}$$

So, the profit function (2) can be written as

$$\text{Profit P} = \frac{V-R}{\left(\frac{w_G}{R}\right)^k} = \frac{(V-R)R^k}{w_G^k} \tag{23}$$

Using the marginal approach to profit with respect to total consumption resources R, we can have

$$\frac{\partial P}{\partial R} = \frac{\partial}{\partial R}\left(\frac{(V-R)R^k}{w_G^k}\right) = 0$$

As it is known that the workload w_G is independent of total resource consumption R, so we can have

$$R^* = V\frac{k}{k+1} \tag{24}$$

The condition $R^* < V$ ensures the positivity of the optimal profit P. Also the profit function is a unimodal function with a single maximum point. As observed in Equation (24), R^* is not affected by the chosen tour. Additionally, when R^* is utilized, the profit rate becomes positive. Furthermore, if there is an extra constant cost per unit of time, the optimal resource amount R^* remains unchanged. Also to optimize the profit rate, we need to minimize w_G. The value of w_G is directly determined by the chosen tour, as shown in Equation (21). The tour that achieves the minimum w_G value is the one that minimizes $\sum_{j=1}^{n}(w_j)^{\frac{k}{k+1}}$. Consequently, we introduce new edge weights $S_{ij} = (w_{ij})^{\frac{k}{k+1}}$. We can now minimize W by addressing problem **P1**, either through an optimal procedure or an approximation method. The formal statement of our optimization procedure for this problem is as follows

i. Find R^* by using equation (24)

ii. Set $S_{ij} = (w_{ij})^{\frac{k}{k+1}}$

iii. Solve TSP (**P1**) by any method

iv. Calculate w_G using equation (21)

v. Calculate the optimal resource allocation R by using equation (19)

The travelling time and the profit are calculated by equations (22) and (23) respectively. If step 3 is solved by a $(1+\epsilon)-$approximation algorithm, the resulting workload of the tour can be

$$w_{g_{approx}} \leq \left((1+\epsilon)\sum_{j=1}^{n}(w_j)^{\frac{k}{k+1}}\right)^{\frac{k+1}{k}} = (1+\epsilon)^{\frac{k+1}{k}}\, w_g^*$$

and so the time of tour will be

$$T_{approx} \leq (1+\epsilon)^{k+1}\left(\frac{w_g^*}{R}\right)^k$$

Also the approximated profit per unit time is

$$P_{approx} = \left(\frac{1}{(1+\epsilon)^{k+1}}\right)P^*$$

where, w_g represents the workload of the optimal tour. Hence, depending on the work of Zofi et al. (2017), the whole procedure of the model yields an $\frac{1}{(1+\epsilon)^{k+1}}$ approximation for sub-problem **P2**.

Conclusion

In today's applications, the traditional Traveling Salesman Problem (TSP) model's assumptions of uniform travel times may fall short when confronted with the complexities of contemporary transportation and logistical situations. For example, consider the hurdles encountered by delivery services, where the task is to create efficient routes ensuring on-time deliveries within designated time frames. Likewise, industries involving mobile services or maintenance activities must consider travel times dependent on resources to efficiently allocate resources and fulfill service obligations. In these contexts, the objective is not solely to minimize travel distance but also to optimize profit within the confines of time-sensitive windows and fluctuating travel conditions.

In the extension introduced in this paper based on the work of Zofi et al. (2017), deals the condition where the travel times are influenced by resources and the objective is to maximize profit per unit of time. The optimal resource allocation for any tour was determined using the equivalent load method, revealing that the required resource, R^*, is independent of the chosen tour and depends on V, which represents the profit contribution per tour. To maximize profit, the tour with the minimum equivalent workload had to be identified. This was achieved by transforming edge workloads and solving the classic TSP. Additionally, the optimal resource allocation for the selected tour and R^* were derived analytically. The model defined here may contribute to advancing the field of combinatorial optimization by addressing the complex interplay between time-sensitive profit optimization and resource-dependent travel times in the context of the Traveling Salesman Problem. By extending the classical TSP, we seek to bridge the gap between traditional optimization models and the practical demands of contemporary industries.

References

[1] Cheng, T. E., Janiak, A. and Kovalyov, M. Y. (1998), Bicriterion single machine scheduling with resource dependent processing times, *SIAM Journal on Optimization*, 8(2), 617-630.

[2] Cook, W. J. (2012), In Pursuit of the Travelling Salesman: Mathematics at the Limits of Computation, Princeton University Press: Princeton, NJ.

[3] Cook, W. J., Cunningham, W. H., Pulleyblank, W. R. and Schrijver, A. (1998), Combinatorial optimisation *Wiley-Interscience Series in Discrete Mathematics and Optimization*, USA, 1, 998.

[4] Daniels, R. L. and Sarin, R. K. (1989), Single machine scheduling with controllable processing times and number of jobs tardy, *Operations Research*, 37(6), 981-984.

[5] Hartley, R. (1985), Linear and nonlinear programming: an introduction to linear methods in mathematical programming, Ellis Horwood Ltd.: Chichester.

[6] Janiak, A. and Kovalyov, M. Y. (1996), Single machine scheduling subject to deadlines and resource dependent processing times, *European Journal of Operational Research*, 94(2), 284-291.

[7] López-Ibáñez, M., Blum, C., Ohlmann, J. W. and Thomas, B. W. (2013), The travelling salesman problem with time windows: Adapting algorithms from travel-time to makespan optimization, *Applied Soft Computing*, 13(9), 3806-3815.

[8] Monma, C. L., Schrijver, A., Todd, M. J. and Wei, V. K. (1990), Convex resource allocation problems on directed acyclic graphs: duality, complexity, special cases, and extensions, *Mathematics of Operations Research*, 15(4), 736-748.

[9] Van Wassenhove, L. N. and Baker, K. R. (1982), A bicriterion approach to time/cost trade-offs in sequencing, *European Journal of Operational Research*, 11(1), 48-54.

[10] Vickson, R. G. (1980), Two single machine sequencing problems involving controllable job processing times, *AIIE transactions*, 12(3), 258-262.

[11] Williams, H. P. (1999), Model building in mathematical programming, Wiley: Chichester.

[12] Zofi, M., Teller, R. and Kaspi, M. (2017), Maximizing the profit per unit of time for the TSP with convex resource-dependent travelling times, *Journal of the Operational Research Society*, 68, 1177-1182.