# The product of two hypergeometric function (I) by using Bailey's formula

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**Abstract**

*Hypergeometric functions are being used recently in many numerical analysis. Our aim of this note is to provide some products of two generalized hypergeometric functions* ${}_1F_1$. *A corollary is also established as a special case.*

**Keywords**: Generalized hypergeometric series, Kummer's type II transformation, Bailey's identity, Contiguous results

## 1.0 Introduction and Preliminaries

The hypergeometric functions are the extension of the geometric series (Poudel et al. 2023d,). The generalized hypergeometric function ${}_pF_q$ having p number of numerator and q number of denominator parameters (Poudel et al., 2023c; Poudel et al., 2024, Prudnikov et al.,1990) is given by

$$ {}_pF_q\left[\begin{matrix} h_1,...,h_p \\ k_1,...,k_q \end{matrix}; \quad z\right] = \sum_{n=0}^{\infty} \frac{(h_1)_n...(h_p)_n}{(k_1)_n...(k_q)_n} \frac{z^n}{n!} \qquad ... (1.1) $$

The Pochhammer symbol $(\lambda)_n$, commonly represented in terms of Gamma function, is defined by

$$ (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} $$

$$ = \begin{Bmatrix} 1 & (n=0; \lambda \in C \setminus \{0\}) \\ \lambda(\lambda+1)(\lambda+2)......(\lambda+n-1) & (n \in N, \lambda \in C) \end{Bmatrix} \qquad ... (1.2) $$

Through the theory of differential equations, Kummer (1836) has obtained the solutions of hypergeometric differential equations (Poudel et al., 2023a,b). Further he obtained the following results (Kummer, 1836; Rainville, 1971):

$$ e^{-\frac{x}{2}} {}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha \end{matrix}; \quad x\right] = {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix}; \quad \frac{x^2}{16}\right] \qquad ... (1.3) $$

Several studies are done in the product of generalized hypergeometric functions (Bailey, 1928; Kim & Rathie, 2017; Poudel et al., 2023; Prudnikov et al., 1990). Bailey (1928) has derived the product of two ${}_0F_1$ functions as the identity given below;

$$ {}_0F_1\left[\begin{matrix} - ; \\ \alpha \end{matrix}\; x\right] {}_0F_1\left[\begin{matrix} - ; \\ \beta \end{matrix}\; x\right] = {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta), \; \frac{1}{2}(\alpha+\beta-1); \\ \alpha, \; \beta, \; \alpha+\beta-1 \end{matrix}\; 4x\right] \quad \text{... (1.4)} $$

Rathie and Nagar (1995) established the following two results contiguous to that of Kummer's second Theorem:

$$ e^{-\frac{x}{2}} {}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x\right] = {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] - \frac{x}{2(2\alpha+1)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right]. \quad \text{.. (1.5)} $$

$$ e^{-\frac{x}{2}} {}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha-1 \end{matrix} ; x\right] = {}_0F_1\left[\begin{matrix} - \\ \alpha-\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] + \frac{x}{2(2\alpha-1)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] \quad \text{... (1.6)} $$

In 2023, Kim et al. (2023) developed the theorems by using the contiguous functions relations

$$ e^{-\frac{x}{2}} {}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha+2 \end{matrix} ; x\right] = {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] - \frac{x}{2(\alpha+1)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right] + \frac{\alpha x^2}{4(\alpha+1)(2\alpha+1)(2\alpha+3)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{x^2}{16}\right] \quad \text{... (1.7)} $$

Kim et al. (2019) and Kim and Kim (2023) generalized Bailey's result (1.4) which was previously obtained by Rathie and Choi (1998) in the following form:

$$ \left\{ {}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x\right] \right\}^2 = e^x \left\{ {}_1F_2\left[\begin{matrix} \alpha \\ \alpha+\frac{1}{2}, \; 2\alpha \end{matrix} ; \frac{x^2}{4}\right] + \frac{x}{(2\alpha+1)} {}_1F_2\left[\begin{matrix} \alpha \\ \alpha+\frac{3}{2}, \; 2\alpha+1 \end{matrix} ; \frac{x^2}{4}\right] + \frac{x}{(2\alpha+1)^2} {}_1F_2\left[\begin{matrix} \alpha+1 \\ \alpha+\frac{3}{2}, \; 2\alpha+2 \end{matrix} ; \frac{x^2}{4}\right] \right\} \quad \text{... (1.8)} $$

In this paper, we shall establish the results. We shall also mention some known results as the special cases.

**2.0 Main Results**

**2.1 Thorems**

**Theorem 2.1.1**: The following relation holds true.

$$ {}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha+2 \end{matrix} ; x\right] {}_1F_1\left[\begin{matrix} \beta \\ 2\beta+2 \end{matrix} ; x\right] = e^x \left\{ {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), \; \frac{1}{2}(\alpha+\beta) \\ \alpha+\frac{1}{2}, \; \beta+\frac{1}{2}, \; \alpha+\beta \end{matrix} ; \frac{x^2}{4}\right] \right. $$

$$-\frac{x}{2}\left\{\frac{1}{(\beta+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) & \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1\end{matrix}\;;\;\frac{x^2}{4}\right]\right.$$

$$\left.+\frac{1}{2(\alpha+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) & \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1\end{matrix}\;;\;\frac{x^2}{4}\right]\right\}$$

$$+\frac{x^2}{4}\left\{\frac{\beta}{(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) & \\ \alpha+\frac{1}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+2\end{matrix}\;;\;\frac{x^2}{4}\right]\right.$$

$$+\frac{\alpha}{(\alpha+1)(2\alpha+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) & \\ \alpha+\frac{5}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+2\end{matrix}\;;\;\frac{x^2}{4}\right]$$

$$\left.+\frac{1}{(\alpha+1)(\beta+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) & \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2\end{matrix}\;;\;\frac{x^2}{4}\right]\right\}$$

$$-\frac{x^3}{8(\alpha+1)(\beta+1)}\left\{\frac{\beta}{(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3) & \\ \alpha+\frac{3}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+3\end{matrix}\;;\;\frac{x^2}{4}\right]\right.$$

$$\left.+\frac{\alpha}{(2\alpha+1)(2\alpha+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3) & \\ \alpha+\frac{5}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+3\end{matrix}\;;\;\frac{x^2}{4}\right]\right\}$$

$$\left.+\frac{\alpha\beta x^4}{16(\alpha+1)(2\alpha+1)(2\alpha+3)(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+5), & \frac{1}{2}(\alpha+\beta+4) & \\ \alpha+\frac{5}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+4\end{matrix}\;;\;\frac{x^2}{4}\right]\right\} \quad ...(2.1)$$

**Proof:**

To prove the theorem let us consider the sum

$$S = \quad e^{-x}\left\{{}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha+2\end{matrix}\;;\;x\right]{}_1F_1\left[\begin{matrix}\beta\\ 2\beta+2\end{matrix}\;;\;x\right]\right\}$$

$$=\left\{e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha+2\end{matrix}\;;\;x\right]e^{-\frac{x}{2}}\,{}_1F_1\left[\begin{matrix}\beta\\ 2\beta+2\end{matrix}\;;\;x\right]\right\}$$

By using the property (1.7), we have

S

$$=\left\{{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{1}{2}\end{matrix}\;;\;\frac{x^2}{16}\right]-\frac{x}{2(\alpha+1)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{3}{2}\end{matrix}\;;\;\frac{x^2}{16}\right]+\frac{\alpha x^2}{4(\alpha+1)(2\alpha+1)(2\alpha+3)}\,{}_0F_1\left[\begin{matrix}-\\ \alpha+\frac{5}{2}\end{matrix}\;;\;\frac{x^2}{16}\right]\right\}$$

$$\left\{ {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] - \frac{x}{2(\beta+1)} {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right] + \frac{\beta x^2}{4(\beta+1)(2\beta+1)(2\beta+3)} {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{5}{2} \end{matrix} ; \frac{x^2}{16}\right]\right\}$$

$$= {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right]$$

$$- \frac{x}{2(\beta+1)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right]$$

$$+ \frac{\beta x^2}{4(\beta+1)(2\beta+1)(2\beta+3)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right] {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{5}{2} \end{matrix} ; \frac{x^2}{16}\right]\right\}$$

$$- \frac{x}{2(\alpha+1)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right] {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right]$$

$$+ \frac{x^2}{4(\alpha+1)(\beta+1)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right] {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right]$$

$$- \frac{\beta x^3}{8(\alpha+1)(\beta+1)(2\beta+1)(2\beta+3)} {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{5}{2} \end{matrix} ; \frac{x^2}{16}\right] {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right]$$

$$+ \frac{\alpha x^2}{4(\alpha+1)(2\alpha+1)(2\alpha+3)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{x^2}{16}\right]\right\} {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{1}{2} \end{matrix} ; \frac{x^2}{16}\right]$$

$$- \frac{\alpha x^3}{8(\alpha+1)(\beta+1)(2\alpha+1)(2\alpha+3)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{x^2}{16}\right] {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{3}{2} \end{matrix} ; \frac{x^2}{16}\right]$$

$$+ \frac{\alpha x^2}{16(\alpha+1)(2\alpha+1)(2\alpha+3)(\beta+1)(2\beta+1)(2\beta+3)} {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{5}{2} \end{matrix} ; \frac{x^2}{16}\right] {}_0F_1\left[\begin{matrix} - \\ \beta+\frac{5}{2} \end{matrix} ; \frac{x^2}{16}\right]\right\}$$

Now, using the Bailey's identity (1.4)

$$= {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), \ \frac{1}{2}(\alpha+\beta) \\ \alpha+\frac{1}{2}, \ \beta+\frac{1}{2}, \ \alpha+\beta \end{matrix} ; \frac{x^2}{4}\right] - \frac{x}{2(\beta+1)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+2), \ \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{1}{2}, \ \beta+\frac{3}{2}, \ \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]$$

$$+ \frac{\beta x^2}{4(\beta+1)(2\beta+1)(2\beta+3)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+3), \ \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{1}{2}, \ \beta+\frac{5}{2}, \ \alpha+\beta+2 \end{matrix} ; \frac{x^2}{4}\right]$$

$$- \frac{x}{2(\alpha+1)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+2), \ \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{3}{2}, \ \beta+\frac{1}{2}, \ \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]$$

$$+\frac{x^2}{4(\alpha+1)(\beta+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2\end{matrix};\ \frac{x^2}{4}\right]$$

$$-\frac{\alpha x^3}{8(\alpha+1)(\beta+1)(2\alpha+1)(2\alpha+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3) \\ \alpha+\frac{5}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+3\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{\alpha x^2}{4(\alpha+1)(2\alpha+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{5}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+2\end{matrix};\ \frac{x^2}{4}\right]$$

$$-\frac{\beta x^3}{8(\alpha+1)(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3) \\ \alpha+\frac{3}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+3\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{\alpha\beta x^4}{16(\alpha+1)(2\alpha+1)(2\alpha+3)(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+5), & \frac{1}{2}(\alpha+\beta+4) \\ \alpha+\frac{5}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+4\end{matrix};\ \frac{x^2}{4}\right]\Bigg\}$$

$$={}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta) \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta\end{matrix};\ \frac{x^2}{4}\right]-\frac{x}{2(\beta+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1\end{matrix};\ \frac{x^2}{4}\right]$$

$$-\frac{x}{2(\alpha+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{\beta x^2}{4(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{1}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+2\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{\alpha x^2}{4(\alpha+1)(2\alpha+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{5}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+2\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{x^2}{4(\alpha+1)(\beta+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2\end{matrix};\ \frac{x^2}{4}\right]$$

$$-\frac{\beta x^3}{8(\alpha+1)(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3) & & ; & \frac{x^2}{4}\\ \alpha+\frac{3}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+3 & & \end{matrix}\right]$$

$$-\frac{\alpha x^3}{8(\alpha+1)(\beta+1)(2\alpha+1)(2\alpha+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3) & & ; & \frac{x^2}{4}\\ \alpha+\frac{5}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+3 & & \end{matrix}\right]$$

$$\left.+\frac{\alpha\beta x^4}{16(\alpha+1)(2\alpha+1)(2\alpha+3)(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+5), & \frac{1}{2}(\alpha+\beta+4) & & ; & \frac{x^2}{4}\\ \alpha+\frac{5}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+4 & & \end{matrix}\right]\right\}$$

$$={}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta) & & ; & \frac{x^2}{4}\\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta & & \end{matrix}\right]-\frac{x}{2}\left\{\frac{1}{(\beta+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) & & ; & \frac{x^2}{4}\\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1 & & \end{matrix}\right]\right.$$

$$\left.+\frac{1}{2(\alpha+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) & & ; & \frac{x^2}{4}\\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1 & & \end{matrix}\right]\right\}$$

$$+\frac{x^2}{4}\left\{\frac{\beta}{(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) & & ; & \frac{x^2}{4}\\ \alpha+\frac{1}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+2 & & \end{matrix}\right]\right.$$

$$+\frac{\alpha}{(\alpha+1)(2\alpha+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) & & ; & \frac{x^2}{4}\\ \alpha+\frac{5}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+2 & & \end{matrix}\right]$$

$$\left.+\frac{1}{(\alpha+1)(\beta+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) & & ; & \frac{x^2}{4}\\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2 & & \end{matrix}\right]\right\}$$

$$-\frac{x^3}{8(\alpha+1)(\beta+1)}\left\{\frac{\beta}{(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3) & & ; & \frac{x^2}{4}\\ \alpha+\frac{3}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+3 & & \end{matrix}\right]\right.$$

$$+\frac{\alpha}{(2\alpha+1)(2\alpha+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3) & ; & \frac{x^2}{4}\\ \alpha+\frac{5}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+3\end{matrix}\right]\Bigg\}$$

$$+\frac{\alpha\beta x^4}{16(\alpha+1)(2\alpha+1)(2\alpha+3)(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+5), & \frac{1}{2}(\alpha+\beta+4) & ; & \frac{x^2}{4}\\ \alpha+\frac{5}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+4\end{matrix}\right]$$

Now, shifting $e^{-x}$ to the right hand side we prove the theorem 2.1. The following theorems can also be proved in the similar manner.

**Theorem 2.1.2**: The following relation holds true.

$$ {}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha-1\end{matrix};\ x\right]{}_1F_1\left[\begin{matrix}\beta\\ 2\beta+2\end{matrix};\ x\right]$$

$$=e^{x}\Bigg\{{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta-1) & ; & \frac{x^2}{4}\\ \alpha-\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta-1\end{matrix}\right]$$

$$-\frac{x}{2}\Bigg\{\frac{1}{(\beta+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta) & ; & \frac{x^2}{4}\\ \alpha-\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta\end{matrix}\right]-\frac{1}{(2\alpha-1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta) & ; & \frac{x^2}{4}\\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta\end{matrix}\right]\Bigg\}$$

$$+\frac{x^2}{4(\beta+1)}\Bigg\{\frac{\beta}{(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) & ; & \frac{x^2}{4}\\ \alpha-\frac{1}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+1\end{matrix}\right]$$

$$-\frac{1}{(2\alpha-1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) & ; & \frac{x^2}{4}\\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1\end{matrix}\right]\Bigg\}$$

$$+\frac{\beta x^3}{8(2\alpha-1)(\beta+1)(2\beta+1)(2\beta+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) & ; & \frac{x^2}{4}\\ \alpha+\frac{1}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+2\end{matrix}\right]\Bigg\}$$

**Theorem 2.1.3**: The following relation holds true.

$$ {}_1F_1\left[\begin{matrix}\alpha\\ 2\alpha+1\end{matrix};\ x\right]{}_1F_1\left[\begin{matrix}\beta\\ 2\beta-1\end{matrix};\ x\right]$$

$$= e^{x}\left\{ {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta-1) \\ \alpha+\frac{1}{2}, & \beta-\frac{1}{2}, & \alpha+\beta-1 \end{matrix} ; \frac{x^2}{4}\right]\right.$$

$$+\frac{x}{2(2\beta-1)}\, {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta) \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta \end{matrix} ; \frac{x^2}{4}\right]$$

$$-\frac{x}{2(2\alpha+1)}\, {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta) \\ \alpha+\frac{3}{2}, & \beta-\frac{1}{2}, & \alpha+\beta \end{matrix} ; \frac{x^2}{4}\right]$$

$$\left.-\frac{x^2}{4(2\alpha+1)(2\beta-1)}\, {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]\right\}$$

**Theorem 2.1.4**: The following relation holds true.

$${}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha+1 \end{matrix} ; x\right] {}_1F_1\left[\begin{matrix} \beta \\ 2\beta+2 \end{matrix} ; x\right]$$

$$= e^{x}\left\{ {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta) \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta \end{matrix} ; \frac{x^2}{4}\right]\right.$$

$$-\frac{x}{2(\beta+1)}\, {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{1}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]$$

$$+\frac{\beta x^2}{4(\beta+1)(2\beta+1)(2\beta+3)}\, {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{1}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+2 \end{matrix} ; \frac{x^2}{4}\right]$$

$$-\frac{x}{2(2\alpha+1)}\, {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]$$

$$+\frac{x^2}{4(2\alpha+1)(\beta+1)}\, {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{3}{2}, & \beta+\frac{3}{2}, & \alpha+\beta+2 \end{matrix} ; \frac{x^2}{4}\right]$$

$$-\frac{\beta x^3}{8(2\alpha+1)(\beta+1)(2\beta+1)(2\beta+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+4), & \frac{1}{2}(\alpha+\beta+3) \\ \alpha+\frac{3}{2}, & \beta+\frac{5}{2}, & \alpha+\beta+3\end{matrix};\ \frac{x^2}{4}\right]\Bigg\}$$

**Theorem 2.1.5**: The following relation holds true.

$${}_1F_1\left[\begin{matrix}\alpha \\ 2\alpha+2\end{matrix};\ x\right]{}_1F_1\left[\begin{matrix}\beta \\ 2\beta-1\end{matrix};\ x\right]$$

$$=e^x\Bigg\{{}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta), & \frac{1}{2}(\alpha+\beta-1) \\ \alpha+\frac{1}{2}, & \beta-\frac{1}{2}, & \alpha+\beta-1\end{matrix};\ \frac{x^2}{4}\right]+\frac{x}{2(2\beta-1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta) \\ \alpha+\frac{1}{2}, & \beta+\frac{1}{2}, & \alpha+\beta\end{matrix};\ \frac{x^2}{4}\right]$$

$$-\frac{x}{2(\alpha+1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), & \frac{1}{2}(\alpha+\beta) \\ \alpha+\frac{3}{2}, & \beta-\frac{1}{2}, & \alpha+\beta\end{matrix};\ \frac{x^2}{4}\right]$$

$$-\frac{x^2}{4(\alpha+1)(2\beta-1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{3}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+1\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{\alpha x^2}{4(\alpha+1)(2\alpha+1)(2\alpha+3)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+2), & \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{5}{2}, & \beta-\frac{1}{2}, & \alpha+\beta+1\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{\alpha x^3}{8(\alpha+1)(2\alpha+1)(2\alpha+3)(2\beta-1)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+3), & \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{5}{2}, & \beta+\frac{1}{2}, & \alpha+\beta+2\end{matrix};\ \frac{x^2}{4}\right]\Bigg\}$$

### 2.2 Corollaries

**Corollary 2.2.1**: If $\alpha = \beta$ in (2.1), then we get

$$\left\{{}_1F_1\left[\begin{matrix}\alpha \\ 2\alpha+2\end{matrix};\ x\right]\right\}^2$$

$$=e^x\Bigg\{{}_2F_3\left[\begin{matrix}\frac{1}{2}(2\alpha+1), & \alpha \\ \alpha+\frac{1}{2}, & \alpha+\frac{1}{2}, & 2\alpha\end{matrix};\ \frac{x^2}{4}\right]-\frac{x}{(\alpha+1)}\ {}_2F_3\left[\begin{matrix}(\alpha+1), & \frac{1}{2}(2\alpha+1) \\ \alpha+\frac{1}{2}, & \alpha+\frac{3}{2}, & 2\alpha+1\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{x^2}{4(\alpha+1)}\left\{\frac{2\alpha}{(2\alpha+1)(2\alpha+3)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(2\alpha+3), & (\alpha+1) & \\ \alpha+\frac{1}{2}, & \alpha+\frac{5}{2}, & 2(\alpha+1)\end{matrix};\ \frac{x^2}{4}\right]\right.$$

$$\left.+\frac{1}{(\alpha+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(2\alpha+3), & (\alpha+1) & \\ \alpha+\frac{3}{2}, & \alpha+\frac{3}{2}, & 2(\alpha+1)\end{matrix};\ \frac{x^2}{4}\right]\right\}$$

$$-\frac{x^3}{4(\alpha+1)^2}\frac{\alpha}{(2\alpha+1)(2\alpha+3)}\,{}_2F_3\left[\begin{matrix}(\alpha+2), & \frac{1}{2}(2\alpha+3) & \\ \alpha+\frac{3}{2}, & \alpha+\frac{5}{2}, & 2\alpha+3\end{matrix};\ \frac{x^2}{4}\right]$$

$$+\frac{\alpha^2x^4}{16(\alpha+1)^2(2\alpha+1)^2(2\alpha+3)^2}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}(2\alpha+5), & (\alpha+2) & \\ \alpha+\frac{5}{2}, & \alpha+\frac{5}{2}, & 2(\alpha+2)\end{matrix};\ \frac{x^2}{4}\right]\Bigg\}\Bigg\} \qquad ...(3.1)$$

**3.0 Conclusion**

In this paper, we have established the results on product of two generalized hypergeometric functions ${}_1F_1\left[\begin{matrix}\alpha\\2\alpha+n\end{matrix};\ x\right].{}_1F_1\left[\begin{matrix}\beta\\2\beta+m\end{matrix};\ x\right]$ for $n=1,2$ and with $m=1,\ 2.$ These results may be useful in mathematics, engineering and some branches of sciences.

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