Weak compatibility and E.A. property for a common fixed point theorem in Menger space

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Abstract

One of the generalizations of Metric space is a Menger space which was introduced in 1942 by Menger who used distribution functions also called probabilistic distance instead of non-negative real numbers as values of the metric. In this paper, we obtain a unique common fixed point for four pairwise weakly compatible self-mappings which use the conditions of E.A. property and also satisfy certain sufficient conditions in setting of Menger space.

Keywords: Menger space, Probabilistic distance, Probabilistic metric space, Distribution function, t-norm

1.0 Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space, introduced in 1942 by Menger (1942), who used distribution functions instead of non-negative real numbers as values of the metric. Schweizer and Sklar (1983) studied this concept, and the important development of Menger space theory was due to Sehgal and Bharucha-Reid (1972).

Sessa (1982) introduced weakly commuting maps in metric spaces. Jungck (1976) enlarged this concept to compatible maps. The notion of compatible maps in Menger spaces was introduced by Mishra (1991). Singh and Jain (2005) generalized the results of Mishra (1991) using the concept of weak compatibility and compatibility of pairs of self-maps. In 2002, Aamri and Moutawakil (2002) introduced E.A. property for two self-mappings in metric space. In 2010, Manro and Bhatia (2010) used E.A. property in intuitionistic fuzzy metric space to obtain a fixed point for four self-mappings. Inspired by the result of Manro and Bhatia (2010), we apply E.A. property under contractive conditions to obtain a unique common fixed point for four weakly compatible self-mappings in Menger space.

2.0 Preliminaries

Definition 1: A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;
(c) $a*1 = a$ for all $a \in [0,1]$;
(d) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Examples of t −norms are $a*b = max\{a + b - 1, 0\}$ and $a*b = min\{a, b\}$.

Definition 2: A distribution function is a function $F: [-\infty, \infty] \to [0,1]$ which is left continuous on R, non-decreasing and $F(-\infty) = 0, F(\infty) = 1$.

We will denote Δ by the family of all distribution functions on $[-\infty, \infty]$. H is a special element of Δ defined by

$$H(t) = \begin{cases} 0, & if\ t \leq 0, \\ 1, & if\ t > 0 \end{cases}.$$

If X is a nonempty set, $F: X \times X \to \Delta$ is called a probabilistic distance on X and F(x, y) is usually denoted by F_{xy}.

Definition 3: The ordered pair (X, F) is called a probabilistic metric space (shortly PM-space) if X is a non-empty set and F is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$ (Sehgal & Bharucha-Reid, 1972):

(i) $F_{xy}(t) = 1 \Leftrightarrow x = y$;
(ii) $F_{xy}(0) = 0$;
(iii) $F_{xy} = F_{yx}$;
(iv) $F_{xz}(t) = 1;\ F_{zy}(s) = 1 \Rightarrow F_{xy}(t + s) = 1$.

The ordered triple $(X, F, *)$ is called Menger space if (X, F) is a PM-space, $*$ is a t −norm and the following condition is also satisfied: for all $x, y, z \in X$ and $t, s > 0$;

(v) $F_{xy}(t + s) \geq F_{xz}(t) * F_{zy}(s)$.

Proposition 1: Let (X, d) be a metric space, then the metric d induces a distribution function F defined by $Fxy(t) = H(t - d(x, y))$ for all $x, y \in X$ and $t > 0$. If t −norm $*$ is $a*b = min\{a, b\}$ for all $a, b \in [0,1]$, then (X, F, $*$) is a Menger space. Further, $(X, F, *)$ is a complete Menger space if (X, d) is complete (Sehgal & Bharucha-Reid, 1972).

Definition 4: For a Menger space $(X, F, *)$ with $*$ being a continuous t-norm (Schweizer & Sklar, 1983),

(a) A sequence $\{x_n\}$ in X is said to converge to a point x in X (written $x_n \to x$) iff for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.
(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n x_{n+p}}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$.
(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition 5: Self maps A and B of a Menger space $(X, F, *)$ are said to be weakly compatible (or coincidently commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Definition 6: Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible if $F_{ABx_nBAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to x$ for some x in X as $n \to \infty$ (Mishra, 1991).

Definition 7: Self maps A and B of a Menger space $(X, F, *)$ are called semi compatible if $F_{ABx_nBx}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to x$ for some x in X.

Lemma 1: Let $\{x_n\}$ be a sequence in a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t *$ t $\geq t$. If there exists a constant $k \in (0,1)$ such that $F_{x_nx_{n+1}}(kt) \geq F_{x_{n-1}x_n}(t)$ for all t>0 and $n = 1, 2 \dots$, then $\{x_n\}$ is a Cauchy sequence in X (Singh & Jain, 2005).

Lemma 2: Let $(X, F, *)$ be a Menger space. If there exists $k \in (0,1)$ such that $F_{xy}(kt) \geq F_{xy}(t)$ for all $x, y \in X$ and $t > 0$, then $x = y$ (Singh & Jain, 2005).

Definition 8: Let A and B be two self maps of a metric space(X, d), we say that A and B satisfy E.A. property, if there exists a sequence $\{x_n\}$ in X such that Ax_n, $Bx_n \to x_0$ for some $x_0 \in X$ as n→ ∞ (Aamri & Moutawakil, 2002).

Example 1: Let $X = [0, \infty)$. Let $F_{xy}(t) = \frac{t}{t+|x-y|}$ for all $t > 0$.

Define $A, B: X \to [0, \infty)$ by $Ax = \frac{x}{5}$ and $Bx = \frac{2x}{5}$ for all $x \in X$. Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 0$, where $x_n = \frac{1}{n}$.

3.0 Main Results

In this result, we are proving a theorem to obtain a unique common fixed point for four weakly compatible self-mappings using E.A. property.

Theorem 1: Let A, B, T and S be self-mappings of a Menger space $(X, F, *)$ with $t *$ t $\geq t$ such that

(i) For each $x \neq y$ in X and $t > 0$, $F_{AxBy}(t) \geq \max\{F_{SxTy}(t), F_{AxSx}(t), F_{ByTy}(t), F_{BySx}(t), F_{AxTy}(t)\}$,
(ii) (A, S) and (B, T) are weakly compatible,
(iii) (A, S) or (B, T) satisfies E.A. property,
(iv) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

If any of the ranges of A, B, T and S is complete subspace of X, then A, B, T and S have a unique common fixed point in X.

Proof: Suppose that (B, T) satisfies the E.A. property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = x_0$ for some $x_0 \in X$.

Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$.

Hence $\lim_{n\to\infty} Sy_n = x_0$.

Now, we will show that $\lim_{n\to\infty} Ay_n = x_0$.

By Condition (i) of the theorem

$F_{Ay_nBx_n}(t) \geq \max\{F_{Sy_nTx_n}(t), F_{Ay_nSy_n}(t), F_{Bx_nTx_n}(t), F_{Bx_nSy_n}(t), F_{Ay_nTx_n}(t)\}$,

Letting limit $n \to \infty$; we get

$\lim_{n\to\infty} F_{Ay_nBx_n}(t) \geq \max\{F_{x_0x_0}(t), F_{Ay_nx_0}(t), F_{x_0x_0}(t), F_{x_0x_0}(t), F_{Ay_nx_0}(t)\}$,

$= \max\{1, F_{Ay_nx_0}(t), 1, 1, F_{Ay_nx_0}(t)\}=1$

$\therefore \lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = x_0$.

Suppose that $S(X)$ is a complete subspace of X, then $x_0 = Su$ for some $u \in X$. Subsequently we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = x_0 = Su$.

Now, we shall show that $Au = Su$.

By (i), we have

$F_{AuBx_n}(t) \geq \max\{F_{SuTx_n}(t), F_{AuSu}(t), F_{Bx_nTx_n}(t), F_{Bx_nSu}(t), F_{AuTx_n}(t)\}$,

Taking limit $n \to \infty$; we get

$F_{AuSu}(t) \geq \max\{F_{SuSu}(t), F_{AuSu}(t), F_{SuSu}(t), F_{SuSu}(t), F_{AuSu}(t)\}$,

$= \max\{1, F_{AuSu}(t), 1, 1, F_{AuSu}(t)\}=1$

$\Rightarrow Au = Su$.

Case I: As the pair of mappings (A, S) is weakly compatible, so $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

Since $A(X) \subset T(X)$, there exists $v \in X$ such that $Au = Tv$.

Now we shall show that $Tv = Bv$.

$F_{AuBv}(t) \geq \max\{F_{SuTv}(t), F_{AuSu}(t), F_{BvTv}(t), F_{BvSu}(t), F_{AuTv}(t)\}$,

$F_{AuBv}(t) \geq \max\{F_{TvTv}(t), F_{TvTv}(t), F_{BvTv}(t), F_{BvTv}(t), F_{TvTv}(t)\}$,

$= \max\{1, , 1, F_{BvTv}(t), F_{BvTv}(t), 1\}=1$

$\Rightarrow Au = Bv$.

$\therefore\ Tv = Bv$.

Hence, we have $Au = Su = Tv = Bv$.

Case II: As the pair of mappings (B, T) is weakly compatible, so $BTv = TBv$ and hence $BBv = BTv = TBv = TTv$.

Finally, we show that Au is common fixed point of A, B, T and S.

Putting $x = u$ and $y = Au$ in (1), we get

$F_{AuAAu}(t) = F_{AAuBv}(t)$

$$\geq \max\{F_{SAuTv}(t), F_{AAuSAu}(t), F_{BvTv}(t), F_{BvSAu}(t), F_{AAuTv}(t)\},$$

$$= \max\{F_{AAuAu}(t), F_{AAuAAu}(t), F_{BvTv}(t), F_{AuAAu}(t), F_{AAuAu}(t)\},$$

$$= \max\{ F_{AAuAu}(t), 1, 1, F_{AuAAu}(t), F_{AuAAu}(t)\}=1$$

$\Rightarrow AAu = Au.$

$\therefore Au = AAu = SAu.$

Hence Au is the common fixed point of A and S.

Similarly, we can prove that Bv is the common fixed point of B and T i.e. $BBv = TBv = Bv$. Since $Au = Bv$, therefore $BAu = TAu = Au$.

Hence, Au is the common fixed point of A, B, T and S.

The cases in which $A(X)$ or $B(X)$ is a complete subspace of X are similar to the cases in which $T(X)$ or $S(X)$, respectively is complete subspace of X, since $A(X) \subset T(X)$ and

$B(X) \subset S(X)$.

Uniqueness: Let u and v be two common fixed point of A, B, T and S, then

$F_{uv}(t) = F_{AuBv}(t)$

$$\geq \max\{F_{SuTv}(t), F_{AuSu}(t), F_{BvTv}(t), F_{BvSu}(t), F_{AuTv}(t)\},$$

$$= max\{1, 1, 1, 1, 1\} = 1$$

which implies $u = v$.

Therefore A, B, T and S have a unique common fixed point.

Hence completes the proof.

Corollary 1: Let A, B and S be self-mappings of a Menger space $(X, F, *)$ with $t * t \geq t$ such that

(i) For each $x \neq y$ in X and t>0 $F_{AxBy}(t) \geq \max\{F_{SxSy}(t), F_{AxSx}(t), F_{BySy}(t), F_{BySx}(t), F_{AxSy}(t)\}$,
(ii) (A, S) and (B, S) are weakly compatible,
(iii) (A, S) or (B, S) satisfies E.A. property,
(iv) $A(X) \subset S(X)$ and $B(X) \subset S(X)$,

If any of the ranges of A, B and S is complete subspace of X, then A, B and S have a unique common fixed point in X.

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