Identification of Algebraic and Transcendental Numbers Using Some Elementary Functions

Bal Bahadur Tamang
Lecturer, Department of Mathematics, Mahendra Ratna Multiple Campus, Ilam
Tribhuvan University, Nepal
Email: bal.tamang@mrmc.tu.edu.np

Abstract

In this paper, to identify irrational numbers while appropriately emphasizing basic functions. All rational numbers can be expressed algebraically, but not all irrational numbers. Rational, irrational, algebraic numbers and transcendental numbers are mostly concerned with irrational numbers using some elementary functions.

Keywords: Elementary function, algebraic number, rational number, irrational number, transcendental number

Introduction

Algebraic integers Hard & Wight (1993) is a natural extension of conventional integers that are fascinating in and of themselves. The investigated algebraic numbers are primarily quadratic in structure, and they satisfy basic algebraic equations of degree two. The roots of several classes of polynomials are algebraic numbers. We will be considering polynomials with a rational number of coefficients. These polynomials are over the field of rational numbers $\mathbb{Q}$. The collections of polynomials in one variable $x$are denoted by $\mathbb{Q}[x]$, all polynomials in x with integral coefficients are denoted by $\mathbb{Z}[x]$, and the set of all polynomials in x with coefficients in any set of numbers F is denoted by $F[x]$.

If a complex number ζsatisfies the equation

$$a_n x^n + a_{n-1} x^{n=1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

is said to be an algebraic number of degree n, where $a_n, \cdots a_1, a_0$ are rational numbers. The set of all algebraic number is denoted by$\overline{\mathbb{Q}}$. Otherwise, the number is called transcendental. So, every rational number$r$ is an algebraic number because $r$satisfies the equation $x - r = 0$.

If a monic polynomial equation

$$a^n + b_1 x^{n-1} + \cdots + b_n = 0$$

with integral coefficients is satisfied by an algebraic number ζ,then that number is said to be an algebraic integer.

In general, algebraic numbers are complex, but they can also be real. Use the real algebraic number $\sqrt{2}$ and the complex algebraic number ζas examples, both of which have degrees of 2.Here we identify some algebraic numbers through some of the elementary functions. As we need some ideas of rational and irrational numbers concepts we also discuss regarding this.

In mathematics, the infinite divisibility of numbers leads to irrationals. A circle's circumference-to-diameterratio π, Euler's number e, the golden ratio φ, and the square root of 2 are all instances of irrational numbers (Niven1956). The irrationals are indeterminate or immeasurable approximations of rational numbers; their value is generally unknown, similar to the concepts of uncertainty, and they contribute to the theoretical consistency of calculus.

A rational number is a real number with the form $\frac{u}{v}$ where $u, v$are integers and $v \neq 0$. A real number that cann'tbe put in that form is known as an irrational number. The Set of real numbers is countable, a measure of that set is zero. Almost all numbers in the set of real numbers are irrational. It is easy to see that every rational number is an algebraic number. Among rational numbers, the only ones that are algebraic integers are the integers $0, \pm 1, \pm 2, \pm 3, \cdots$ and suppose α and β are algebraic numbers, then $\alpha + \beta$and $\alpha.\beta$ are algebraic numbers as well as α and β are algebraic intgers, then $\alpha + \beta$and $\alpha.\beta$ are algebraic integers in(Niven , Zuckerman & Montgomery,1991). Similarly, a field is generated by the collection of all algebraic numbers. A ring is generated by the collection of all algebraic integers So, the set of all algebraic numbers and the set of algebraic integers are both closed under addition and multiplication. A complex number $\zeta$is algebraic iff its real and imaginary parts are algebraic numbers.

If a function$F(x)$with domain D can be produced from one and the same set of fundamental elementary functions using a finite number of fundamental elementary operations in one and the same method for all $x \in D$, then it is said to be an elementary function. So, all rational functions are elementary functions. An algebraic number ζsatisfies a unique irreducible monic polynomial equation $g(x) = 0$over$\mathbb{Q}$. Furthermore, every polynomial equation over $\mathbb{Q}$ satisfied by $\zeta$is divisible by $g(x)$. The equation $g(x) = 0$ is the minimum equation of an algebraic number $\zeta$and $g(x)$ is the smallest polynomial for the variable ζ. The degree of a minimal polynomial is the degree of an algebraic number.

Methodology

We explain about the difference between rational and irrational numbers. Using elementary functions, identify irrational numbers. Later, we distinguish between Algebraic and Transcendental numbers using certain basic functions of a number theoretic aspect, as contrasted to the more algebraic parts of the theory.

Results and Discussion

Algebraic number theory is an area of Mathematics that studies integers, rational numbers, and their generalizations using abstract algebra approaches in (Ribenboim, 1972). Algebraic objects such as algebraic number fields and related rings of integers, finite fields, and function fields are used to express number-theoretic concerns. These aspects, such as whether or not a ring admits unique factorization, the behavior of ideals, and the Galois groups of fields, can actually solve basic issues in number theory, such as whether or not Diophantine equations have solutions.

The Babylonians knew that $\pi$is close to $\frac{25}{8}$. Ptolemy used the approximation number $\frac{377}{120}$ in 200 AD, which is off by $0.000074\cdots$only. Archimedes proved $\frac{223}{71} < \pi < \frac{22}{7}$ approximately 300BC.Currently, π has 12.1trillion digits. One terabyte of storage is equal to one trillion digits. If compressed, the known digits might fit on one terabyte hard drive. In 1761, Johann Lambert established thatπ is irrational.

Theorem

If a polynomial equation with integral coefficients

$$c_n x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots c_1 x + c_0 = 0, c_n \neq 0 \qquad (1)$$

has a non-zero rational solution $\frac{a}{b}$,where the integers $a$and b are relatively prime, then $a|c_0$and $b|c_n$.

Proof

If we put $x = \frac{a}{b}$in (1)and we multiply by b^{n-1}, then we get

$$c_n \frac{a^n}{b} + c_{n-1}a^{n-1}b + \cdots + c_1 ab^{n-2} + c_0 b^{n-1} = 0$$

Since the expression $c_{n-1}a^{n-1} + \cdots + c_1 ab^{n-2} + c_0 b^{n-1}$is integer, so the term $c_n \frac{a^n}{b}$must be integer. Since $(a, b) = 1,$ then $b \nmid c_n$. Again, we put $x = \frac{a}{b}$ in (1) and we multiply by $\frac{b^n}{a}$, then we get

$$c_n a^{n-1} + c_{n-1}a^{n-2}b + \cdots c_1 b^{n-1} + c_0 \frac{b^n}{a} = 0$$

the expression $c_n a^{n-1} + c_{n-1}a^{n-2}b + \cdots c_1 b^{n-1}$is an integer, the term $c_0 \frac{b^n}{a}$must be an integer. Again as $(a, b) = 1,$ then $a|c_0$.

Theorem

Any integer dividing $c_0$is a solution if the polynomial (1) with the coefficients $c_n = \pm 1$has a rational solution.

Proof

If the polynomial equation (1) has a solution $x = \frac{a}{b}$, then $b|c_n$ but $c_n = \pm 1$ and so $b = \pm 1$. If the denominator is positive, $b = 1$. Therefore, the solution is $x = a$, an integer that must divide c_0.

Theorem

If there is any rational solution to the equation $x^n = c$, it is an integer solution for any integers $c$and $n > 0$. Then$x^n = c$has a rational solution if and only if c is the n^{th} power of an integer.

Proof

Assume that the equation $x^n = c$has a solution of the form $x = \frac{a}{b}$with $b > 0$. So $b|1$is equivalent to $b = 1$. For this, the integer $x = a$ is the only possible rational solution to the equation $x^n = c$. This shows that c is the n^{th}power of a.

As a result of this theorem, we can conclude that $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{8}, \cdots$are irrational because none of their integral solutions of $x^2 = 2, x^2 = 3, x^2 = 5, x^2 = 8, \cdots$exists.

Trigonometric functions with specific values can also be applied with the Theorem, we can deduce that in the case where $m$is a positive integer that is not the n^{th} power of an integer, $\sqrt[n]{m}$ is irrational.

Theorem

If $n$is a positive integer and $g(x)$ is a polynomial with integral coefficients, then $h(x) = \frac{x^n g(x)}{n!}$and all of its derivatives which are evaluated at $x = 0$ are integers. In addition, the integer $h^n(0)$ is divisible by $n + 1$ with the possible exception of the case $j = n$. If $g(x)$ is a factor of x then $j = n$does not require an exception.

Proof

Since $h(x) = \frac{x^n g(x)}{n!}$where $g(x) = c_0 x^m + \cdots + c_{m-1} x + c_m$ and c_j are integers, we have$2f^j(0) = \frac{c_j (j!)}{n!}$

If $j < n$, then $h^j(0) = 0$, which is divisible by $n = 1$. If $j = n$, then $h^n = c_m$. If $g(x)$ is a factor of x, then $g(0) = 0$. In that case $h^n(0)$is divisible by $n + 1$.

Irrational numbers can be further classified into algebraic and transcendental numbers. Suppose that θ is a non-zero rational number. Then $cos\theta, sin\theta, tan\theta$ are irrational numbers and similarly,

$\cos^{-1}\theta, \sin^{-1}\theta, \tan^{-1}\theta$ are irrational numbers for any rational number θ. Also, π is irrational number.

Theorem

Let $= \theta\pi$, where θ is a rational number. Then the functions $sin\alpha, cos\alpha, tan\alpha$ are algebraic numbers apart from the case where $tan\alpha$ is undefined.

Proof

Any positive integer n we prove the presence a polynomial $f_n(x)$ of a degree n with integral coefficients and a leading coefficient 1 such that $2cos\, n\alpha = f_n(2cos\, \alpha)$ holds for all real numbers α. According to the well-known identity $2cos2\alpha = (2cos\alpha)^2 - 2$. We observe that

$\theta f_1(x) = x$ and $f_2(x) = x^2 - 2$. By elementary trigonometry, we have

$$\theta 2\cos(n+1)\,\alpha = (2cos\alpha)(2\cos n\alpha) - 2\cos\,(n-1)\alpha$$

This gives us the formula $\theta f_{n+1}(x) = xf_n(x) - f_{n-1}(x)$. Choose a positive integer n such that $n\theta$ is also a positive integer. Given that $\alpha = \theta\pi$ it follows that

$n\theta f_n(2cos\alpha) = 2cosn\alpha = 2cosn\theta\pi = \pm 2$

This shows that $2cos\alpha$ is the solution to a polynomial equation. We can determine that $cos\alpha$ is also a polynomial equation's solution in some cases. It follows that $cos\alpha$ is an algebraic number.

If α is a rational multiple of π, then $\frac{\pi}{2} - \alpha$ is also rational. In fact that $sin\alpha$ is an algebraic number can be obtained from the identity $sin\alpha = \cos\,(\frac{\pi}{2} - \alpha)$. Since algebraic numbers form a field and $sin\alpha$ and $cos\alpha$ are algebraic numbers, then $tan\alpha$ is also an algebraic number.

Theorem

Let $\alpha = \theta\pi$, then $e^{i\theta\pi}$ is an algebraic number.

We know that a complex number ζ is algebraic if and only if its real and imaginary parts are algebraic numbers. With this result also we can conclude that $sin\theta\pi$ and $cos\theta\pi$ are algebraic as

$e^{i\theta\pi} = cos\theta\pi + isin\theta\pi$

Liouville's finding of the theory known as the Liouville approximation theorem motivated the study of transcendental numbers. Suppose that $\alpha\epsilon\mathbb{R}$ is algebraic if it is a root of a non-zero integral polynomial, which means that $a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0$ for any $a_i \in \mathbb{Z}$, not all of them zero. The degree of α is the smallest degree n of such a polynomial. The algebraic numbers of degree 1 are exactly rational numbers. Transcendental numbers are those that cannot be represented algebraically. Algebraic numbers do not have extremely close rational approximations, according to Joseph Liouville.

Theorem [Liouville Theorem]

Suppose $\alpha\epsilon\mathbb{R}$ is an irrational algebraic number of degree $n \geq 2$. There is a constant $c = c(\alpha) > 0$ that only depends α on the condition that $|\alpha - \frac{u}{v}| \geq \frac{c}{v^n}$ exists, for every $\frac{u}{v} \in \mathbb{Q}$.

Transcendental numbers can be built using Liouville's theorem. A Liouville number, defined as a non-rational number α is one for which the Liouville theorem holds for all rational $\frac{u}{v}$ for no pair $c > 0,\ n \geq 2$. Liouville's theorem gives each and every Liouville number is transcendental.

Consider the irrational number $\alpha \in \mathbb{R}$ which has the property that for every $n \in \mathbb{N}$ there is a fraction $\frac{u}{v}$ that satisfies $v \geq 2$ and

$$\left|\alpha - \frac{u}{v}\right| < \frac{1}{v^n}$$

Then α is transcendental number. It gives every Liouville number is transcendental number. The required precise rational approximations are provided by partial sums, and Liouville numbers can be generated as sums of infinite series that are rapidly convergent. It is simple to see in his way that

$$\zeta = \sum_{k=1}^{\infty} 10^{-k!} = 0.110001\,000000000000000001000\cdots$$

isLiouville number and hence it is transcendental. Liouville's theorem was further modified by Norwegian mathematician Axel Thue.

Theorem [A. Thue]

Let $\varepsilon > 0$ and suppose $\alpha \epsilon \mathbb{R}$is an algebraic number of degree $n \geq 2$. There is a constant $c = c(\alpha, \varepsilon)$ such that$|\alpha - \frac{u}{v}| \geq \frac{c}{v^{1+\varepsilon+n/2}}$holdsfor every $\frac{u}{v} \in \mathbb{Q}$.

In the modern sense, Euler was most certainly the first to define transcendental numbers (Murty&Rath1014). Liouville proved that if ζ is algebraic, then we can examine a polynomial $f(x) \in \mathbb{Z}[x]$ of least degree for which it is a root. This polynomial is unique up to a factor of an integral. The degree of $\zeta$is the same as the degree of f.As a result, algebraic numbers with degree 1 are rational numbers. If ζhas degree 2and $f(x)$ is an irreducible polynomial with integer coefficients such that $f(x) = 0$, Liouville proved that for any rational number $\frac{a}{b}$there is a positive constant Cdepending on f. We have

$$\left|\zeta - \frac{a}{b}\right| < \frac{C}{b^n}$$

Liouville claims that there is a limit to how close irrational algebraic numbers can be approximated by rational numbers. In other words, the possibility that a number is transcendental improves with the efficiency of the approximation by rational numbers. This idea can be expanded upon to establish the transcendence of e in the following situations;

Charles Hermite was the first to establish the transcendence of e.This effective, clear proof was created by David Hilbert. Hilbert's proof of transcendence is based on the fundamental characteristic of the exponential function, whereas the irrationality proof makes use of infinite series for e(Spivak2006). So, the number $e = 2.71828\cdots$ is transcendental. There is no relation of the form that $e$satisfies $a_n e^n + a_{n-1} e^{n-1} + \cdots + a_1 e + a_0$ having integral coefficients not all zero (Baker 1975).

Theorem

Let $f(x) \in \mathbb{C}[x]$ with $deg\ f = m, u \in \mathbb{C}$ and let $I(u;f) = \int_0^u e^{(u-t)} f(t)dt$ be the integral along the segment of line that runs from 0to u, then (Shidlovskii1989)

$$I(u;f) = e^u \sum_{j=0}^{m} f^j(0) - \sum_{j=0}^{m} e^j(u)$$

A transcendental number is e^a(Siegel1949), if $a$is an algebraic number.Similarly, the transcendental numbers are $e^i, e^{\sqrt[n]{m}}, e^{10}$ and π(Van 2008).

Schanuel conjecture that $tr(\text{deg}) \in \mathbb{Q}(x_1, x_2, \cdots, x_n, e^{x_1}, e^{x_2}, \cdots e^{x_n}) \geq n$

if $x_1, x_2, \cdots x_n$ are linearly independent over $\mathbb{Q}$ (Siegel1949). According to the Lindemann-Weierstrass theorem, if $x_1, x_2, \cdots x_n$ are algebraic numbers that are linearly independent, then this conjecture is correct. The Hermite-Lindemann theorem leads to Schanuel's conjecture for $n = 1$. Also, consider $x_1 = 1, x_2 = \pi$

$tr(\text{deg}) \in \mathbb{Q}(\pi, e) = tr(\text{deg}) \in \mathbb{Q}(1, \pi i, e, e^{\pi i})$

is at least 2 because these are linearly independent over $\mathbb{Q}$.It follows that $e$and $\pi$are algebraically independent. As a result, Schanuel suggests that $e + \pi$ and $e.\pi$ are both transcendental and algebraically independent (Siegel1949). The number $\gamma = [10^{1!}; 10^{2!}; \cdots]$ is transcendental.Similarly, Euler's constant e and π are irrational and $\frac{\pi}{e}$, $2^e, \pi^e, \pi^{\sqrt{2}}, log\pi$ are also irrational numbers.

Conclusion

Numerous numbers are properly studied, regardless of whether they are transcendental, algebraic, rational, or irrational. They are mostly concerned with irrational numbers using some elementary functions. As a result, it can be shown that the Liouville's numbers are a subset of the Lebesgue measure zero, not all transcendental numbers are Liouville numbers. So, π is transcendental but not a Louville number.

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