Publisher: TUTA, Dhawalagiri Multiple Campus, Baglung ISSN: 2392-4438

Article information

Received: 15 February 2025

Review: 18 February - 25 March 2025

Revision 26 March - 30March 2025

Accept: 1 April 2025

Published: 10 April 2025

DOI: https://doi.org/10.3126/ps.v23i1.77528

Available: https://www.nepjol.info/index.php/ps

Symmetry in the context of the Almost Contact 3-Structure and Almost Quaternion Structure

Nanda Kishor Kumar
Department of Mathematics
Tri-Chandra Campus, Tribhuvan University
ORCID iD: https://orcid.org/0000-0003-3844-809X
Email: nandkishorkumar2025@gmail.com

Abstract

The aim of this paper is to explores the characteristics, relationships, and applications of nearly contact 3-structure and nearly quaternion structure in current mathematical study, as well as their foundations. The Interplay between almost contact 3-structure and almost quaternion structure and significance and future direction have also been described. The study of symmetry in Almost Contact 3-Structures and Almost Quaternion Structures provides deep insights into differential geometry and mathematical physics. These structures serve as a bridge between classical Riemannian geometry and modern physical theories, with ongoing research continuing to explore their rich algebraic and geometric properties.

Keywords: Almost Contact 3-Structure, Almost Quaternion Structure

Introduction

Symmetry plays a fundamental role in differential geometry, particularly in the study of geometric structures such as Almost Contact 3-Structures and Almost Quaternion Structures. These structures arise naturally in different mathematical and physical contexts, including Riemannian and complex geometry, string theory, and theoretical physics. Structures suggests a framework for understanding and classifying spaces in the field of differential geometry. Some of these fascinating structures is the Almost Quaternion Structure and the Almost Contact 3-Structure. These structures have applications in many fields of mathematics and theoretical physics in addition to providing insights into the geometry of spaces.

The product of a manifold with almost contact 3- structure and a straight line admits an almost quaternion structure (Kuo,1970). Recently, Ako and one of the present authors Yano, & Ako (1972) have showed that, if for a almost quaternion structure (F, G, H) the Nijenhuis tensors [F, F] and [F, G] vanish, then also other Nijenhuis vanish. The almost contact 3-structure has been described by Kuo (Kashiwada (1972), Tachibana (1970, 1965), Yu (1970), Eum[ ],Kashiwada [3 (1972),,Ki (1973),,Sasaki (1972), Yano (1973),], Ishihara, Konishi (1972,1973).

Kruglikov and Winther determined the gap problem for almost quaternion-Hermitian structures. They determined maximal and submaximal symmetry dimensions, both with Lie algebras and Lie groups. They categorize all structures with such symmetry dimensions. They also studied Geometric properties of the sub-maximally symmetric spaces in specific. They classify locally conformally quaternion-Kähler structures with quaternion-Kähler with torsion (Kruglikov and Winther, 2020).

Ivanov and Petkov have proveed quaternionic contact (qc) versions of the so-called Almost Schur Lemma, which, under specific positive conditions, provide estimates of the qc scalar curvature on a compact qc manifold as a constant in terms of the norm of the [−1]-component, the norm of the trace-free part of the (3)-component of the horizontal qc Ricci tensor, and the torsion endomorphism (Ivanov & Petkov, 2022)

We inspect the characteristics, relationships, and applications of Almost Contact 3-Structure and Almost Quaternion Structure in modern mathematics study as we inspect their foundations in this article. The main objective of the present article is to study the symmetric relation between Almost Contact 3-Structure and Almost Quaternion Structure. The goal of this study is to create a symmetric connection between the Almost Contact 3-Structure and the Almost Quaternion Structure. In terms of structure, this study differs from others.

Understanding Almost Contact 3-Structure

The field of odd-dimensional manifolds gives birth to the research of Almost Contact 3-Structure. It is composed of a triple (υ, ξ, η), where η is a vector field, η is a one-form, and υ is a tensor field of type (1, 1) that satisfies specific compatibility requirements. These geometric structures are higher-dimensional generalizations of contact structures. In contact geometry, the tensor ϕ serves as a natural generalization of the Reeb vector field, and ξ and η have comparable functions.

Almost contact 3-structure

Let M be an n-dimensional differential manifold, and let f, U, and u be, respectively, a vector field, a 1-form, and a tensor field of type (1,1). If this tensor field satisfy

f^2 = -I + u ⊗ U, f U = 0, u∘ f= 0, u(u) = 1, (1) where $(u\circ f)$ (x) = u(fx) and I is the identity of this tensor field and (f, U, u) is termed an almost contact structure.

Suppose f_1 , f_2 are tensor field of type (1,1) for vector fields U_1 , U_2 and u_1 , u_2 of 1-form in M. Now, let (f_1, U_1, u_1) and (f_2, U_2, u_2) are almost contact structure satisfying following axioms:

$$f_1 \ f_2 \ + \ f_2 \ , \ f_1 \ = \ u_1 \ \otimes U_2 + \ u_2 \ \otimes U_1 \ , \ \ f_1 \ U_2 \ + \ f_2 \ \ U_1 \ = 0 \tag{2}$$

$$u_1 \ \circ f_2 + u_2 \ \circ f_1 \ = 0, \quad u_1(U_2) \ = 0, \ \ u_2(U_1) = 0 \tag{3}$$

If both Equations (2) and (3) satisfied then the sets (f_1, U_1, u_1) and (f_2, U_2, u_2) define an almost contact 3-structure in M.

If both sets define an almost contact 3-structure, then

$$f_3 = \ f_1 f_2 \text{ - } u_2 \otimes U_1 = \text{ - } f_2 f_1 \text{ - } u_1 \otimes U_2$$

$$U_3 = \ f_1 U \text{ - } f_2 \otimes U_1, \quad u_3 = \ u_1 \ \circ f_2 = \text{-} \, u_2 \ \circ f_1 \tag{4}$$

We can verify that (f_3, U_3, u_3) also defines an almost contact structure as (f_1, U_1, u_1). Now verifying that

$$f_1 = \ f_2 f_3 \text{ - } u_3 \otimes U_2 \ = \ f_3 f_2 \text{ - } u_2 \otimes U_3 \ , \quad f_2 = \ f_3 f_1 \text{ - } u_1 \otimes U_3 \ = \text{-} \, f_1 f_3 \text{ - } u_3 \otimes U_1$$

$$U_1 = f_2 U_3 = \text{-} \, f_3 U_2, \qquad U_2 = f_3 U_1 = \text{-} \, f_1 U_3$$

$$u_1 = \ \ u_2 \circ f_3 = \text{-} \, u_3 \circ f_1 \qquad u_2 = \ \ u_3 \circ f_1 = \text{-} \, u_1 \circ f_3$$

$$u_2(U_3) \ = 0, \quad u_3(U_2) \ = 0, \ \ u_3(U_1) \ = 0, \quad u_1(U_3) \ = 0$$

This is why, any two of (f_1, U_1, u_1) , (f_2, U_2, u_2) , and (f_3, U_3, u_3) may define the same almost contact 3-structure.In this way, the almost contact 3-structures (f_λ, U_λ, u_λ) , for $\lambda =$ 1,2,3 describe in M an almost contact 3-structure.

Exploring Almost Quaternion Structure

Equally, Almost Quaternion Structure arises as a higher-dimensional extension of quaternionic geometry. To put it simply, it is a tensor field J of type (1, 1) that satisfies several constraints similar to those of the quaternionic architecture. From a geometric perspective, Almost Quaternion Structure extends the rich algebraic and geometric characteristics of quaternionic geometry to higher-dimensional environments, capturing its core.

Almost quaternion structure

Let three tensor fields F_λ ($\lambda = 1{,}2\ 3$) of type (1,1) in a manifold satisfying

$$F'^2_\lambda = \text{-} \ I \ , \ F_\lambda F_\mu = \ -F_\mu F_\lambda = F_\nu \ , \tag{5}$$

where (λ, μ, ν) is an even permutation of type (1, 2, 3). Then the set { F_λ;$\lambda = 1{,}2{,}3$} is called an almost quaternion structuring in $\overline{M}$, is necessarily in 4m- dimensional.

For two tensor fields P and Q of type (1,1) in $\overline{M}$, the Niehues tensor [P, Q] of P and Q is, by definition, tensor field of type (1, 2) such that

2[P, Q] (X, Y) = [PX, QY] – P [QX, Y] – Q [X, PY] + [QX, PY] – Q [PX, Y] -P [X, QY]+

(PQ+QP) [X, Y] (6)

and so, the Nijenhuis tensor [P, P] of P is defined by

$[P, P](X,Y) = [PX,PY]-P[PX,Y]-P[X,PY] + P^2 [X,Y]$ (7)

X and Y are vector fields in $\bar{M}$. Ako (1972) has proved.

Theorem 1. If an almost quaternion structure { F_λ : 1,2,3} then the Nijenhuis tensors [F_1, F_1] and F_2, F_2] vanish, and make it vanish the other Nijenhuis tensors $[F_3, F_3]$, $[F_2, F_3]$, $[F_3, F_1]$, and $[F_1, F_2]$. (Yano & Ako, 1972).

Key Properties and Applications

The relationship between Almost Contact 3-Structure and Sasakian geometry, in which the underlying manifold permits a certain kind of metric, is one of its main characteristics. There are fundamental consequences in differential geometry and topology as a result of this link. Moreover, Almost Contact 3-Structures serve as a link between several branches of mathematics by being useful in the study of complex geometry and CR (Cauchy-Riemann).

Application

- Almost contact 3-structures seem in the study of certain types of almost contact manifolds, which are fundamental objects in the theory of contact geometry.
- They are used in the study of certain types of special holonomies in Riemannian geometry, such as the study of nearly Kähler manifolds.
- Almost quaternion structures are vital in the study of hypercomplex manifolds, which are simplifications of complex manifolds to higher dimensions.
- They appear in the context of special geometries, such as special holonomy metrics or in the study of supersymmetric field theories.

Nearly contact 3-structures and virtually quaternion structures are geometric frameworks that are useful in differential geometry, mathematical physics, and theoretical physics. They enable the study of certain sorts of manifolds. They offer profound insights into the underlying spaces by providing complex structures that capture intriguing geometric and algebraic characteristics (Kumar, & Yadav, 2023 & Yadav, Kumar, & Sahani,2023).

Interplay between almost contact 3-structure and almost quaternion structure

Remarkably, there exists a profound interplay between Almost Contact 3-Structure and Almost Quaternion Structure. While seemingly distinct, these structures exhibit intriguing connections, hinting at deeper underlying principles in geometry. For instance, certain classes of manifolds can admit both Almost Contact 3-Structure and Almost Quaternion Structure simultaneously, leading to a fusion of their geometric properties.

Let M is a manifold of almost contact 3-structure {(F_λ , $U_\lambda, u_\lambda), \lambda = 1,2\ 3$). The product space M× R, where R is straight line. M ∈ M× R is denoted by

$$X = \begin{pmatrix} x \\ \alpha \end{pmatrix} \quad (8)$$

Now, defining this tensor fields of type (1,1) in M× R as

$$F_\lambda X = F_\lambda \begin{pmatrix} x \\ \alpha \end{pmatrix} = \begin{pmatrix} f_\lambda x & \alpha U_\lambda \\ & u_\lambda(x) \end{pmatrix} \quad (9)$$

From Equation (9), we have

$$F_\lambda^2 = -I, \quad F_\lambda F_\mu = - F_\mu\ F_\lambda = F_\upsilon \quad (10)$$

(λ, μ, υ) = Even permutation of (1,2,3)

(F_λ , $\lambda = 1{,}2\ 3$) = An almost quaternion structure in M$\times$ R.

New Finding

Almost Contact 3-Structure on a (4n+3)-dimensional manifold can be seen as symmetric of an Almost Quaternion Structure on a (4n)-dimensional manifold. Both mathematicians and scientists are captivated by Almost Contact 3-Structure and Almost Quaternion Structure as they search for better understanding of the structure of space and geometry.

Significance and Future Directions

Almost Contact 3-Structure and Almost Quaternion Structure are fields of study that lie at the interface of topology, algebra, and geometry. These structures are a valuable resource for studying a variety of mathematical phenomena and also pay to our understanding of manifold geometry. Also, they have an impact on subjects outside of mathematics, such theoretical physics and mathematical biology.

New borders in geometry and its applications are sure to arise as scholars solve their mysteries and consider their ramifications, influencing the direction of mathematical study in the years to come.

Conclusion

We study the foundation of virtually contact 3-structure and almost quaternion structure, as well as their characteristics, relationships, and applications in current mathematics research. Analyzing Almost Quaternion Structure's connection to Almost Contact 3-Structure.

Ethical approval for the research: Not applicable

Consent for publication: Not applicable

Conflict of interest: The author does not have any conflict of interest with any institutions concerning this research

Ethical conduct of research: This paper is written ethically

References

Ako, M., & Yano, K. (1972). Integrability conditions for almost quaternion structures. *Hokkaido Mathematical Journal, 1*, 63–86.

Ishihara, S., & Konishu, M. (1972). On f-3-structure. *Hokkaido Mathematical Journal, 1,* 127–135.

Ishihara, S., & Konishu, M. (1972). Fibred Riemannian spaces with Sasakian 3-structure. In *Differential Geometry, in Honour of K. Yano* (pp. 179–194). Kinokuniya, Tokyo.

Ishihara, S., & Konishu, M. (1973). Fibred Riemannian space with triple of Killing vectors. *Kodai Mathematical Seminar Reports, 25*.

Ivanov, S., & Petkov, A. (2022). The almost Schur lemma in quaternionic contact geometry. *arXiv*. https://doi.org/10.48550/arXiv.2204.04482

Kashiwada, T. (1972). On three framed f-structures with some relations. *Natural Science Reports of Ochanomizu University, 22,* 91–99.

Kruglikov, B., & Winther, H. (2020). Submaximality symmetric quaternion Hermitian structures. *arXiv*. https://arxiv.org/abs/1812.11229v2

Kumar, N. K., & Yadav, D. P. (2023). Concept of bilinear transformation, Jacobian, and conformal mapping with applications. *Mikailalsys Journal of Advanced Engineering International, 1*(1), 1–13. https://doi.org/10.58578/mjaei.v1i1.1970

Kuo, Y. Y. (1970). On almost contact 3-structure. *Tohoku Mathematical Journal, 22,* 235–332.

Kuo, Y. Y., & Tachibana, S. (1970). On the distribution appeared in contact 3-structure. *Taita Journal of Mathematics, 2,* 17–24.

Obata, M. (1965). Affine connection of manifolds with almost complex, quaternion, or Hermitian structure. *Japan Journal of Mathematics, 26*, 91–99.

Sasaki, S. (1972). Spherical space forms with normal contact metric 3-structure. *Journal of Differential Geometry, 6*, 307–315.

Tachibana, S., & Yu, W. N. (1970). On a Riemannian space admitting more than one Sasakian structure. *Tohoku Mathematical Journal, 22*, 536–540.

Yadav, D. P., Kumar, N. K., & Sahani, S. (2023). Distance metrics for machine learning and its relation with other distances. *Mikailalsys Journal of Mathematics and Statistics, 1*(1), 15–23. https://doi.org/10.58578/mjms.v1i1.1990

Author Bio-note

Nand Kishor Kumar, M.Sc in Mathematics and PhD in Economics from Tribhuvan University
Lecturer, Tri-Chandra Campus, Tribhuvan University