Modified Exponential Power Distribution with Properties and Applications

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Abstract

We have developed the Modified Exponential power distribution, a new and adaptable probability distribution, in this work. An additional shape parameter is added to the exponential power distribution to create this distribution. Numerous statistical properties of suggested model are derived and analyzed. Cramer-Von-Mises (CVME), maximum likelihood (MLE), and least-squares (LSE) are used to estimate the model's parameters. P-P and Q-Q charts are used to assess the validity of the model. Several information criteria are applied in model comparisons, including the Bayesian Information Criterion (BIC), the Hannan-Quinn Information Criterion (HQIC), the Corrected Akaike Information Criterion (CAIC), and the Akaike Information Criterion (AIC). These criteria help determine the best model by balancing goodness-of-fit with model complexity. Test statistics, together with their corresponding p-values, are also utilized to evaluate the recommended model's goodness of fit. These tests consist of the Kolmogorov-Smirnov (KS), Cramer-Von Mises (CVM), and Anderson-Darling (An) tests. R programming is used for the dataset analysis and visualization.

Keywords: *Bayesian Information Criterion, Exponential Power distribution, Failure rate function, Maximum likelihood estimation, Survival function.*

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Introduction

Statistics has been getting wide scope in every field. Statistics is backbone of the research either in scientific and non-scientific studies. It is being applied in theory building, theory testing, model formulation, information gathering, programming, data analysis, actuarial science, and in environmental science etc. Probability distribution is one of the statistical tools that help in decision-making and model formulation. In modern age of statistics, researchers are formulation new probability models using different techniques. Formulating of models using family of distribution, modifying the existing probability model, adding some extra parameters to existing models, exponentiation the models and combining two or more models are the important methods.

Over the past few decades, the exponential distribution has garnered significant attention from researchers for its versatility in modeling lifetime data. This model is known to perform exceptionally well across various applications, as it offers numerous closed-form solutions for survival analysis. Even though it's reasonable to assume a constant failure rate, failure rates are often not constant in practice. Therefore, using the exponential lifetime model randomly seems impractical and inaccurate. Some of the modification of the exponential distribution found in the literature are Beta Exponential distribution (Nadarajah & Kotz, 2006), Generalized Exponential (GE) distribution with an increasing and decreasing failure rate function (Gupta & Kundu, 1999), Logistic-exponential distribution with an increasing, decreasing, bathtub (BT)-shaped, and upside-down bathtub (UBT)-shaped failure rate function (Lan &Leemis, 2008), the Exponentiated Exponential distribution (Nadarajah, 2011), an Extension of the Exponential distribution having mode at zero and allowing increasing, decreasing and constant hazard rates (Nadarajah & Haghighi, 2011), the Marshall-Olkin exponential Weibull distribution having increasing and bathtub shaped hazard rate (Pogány et al., 2015), the exponentiated generalized extended exponential distribution having the classic shapes: bathtub, inverted bathtub, increasing, decreasing and constant hazard rates (de Andrade et al., 2016), the Extended Exponential Distribution having decreasing, increasing or bathtub shaped hazard rate (Afify et al., 2018), the Marshall-Olkin logistic-exponential distribution having increasing, decreasing, bathtub and upside-down bathtub hazard rate functions (Mansoor et al., 2019), the Truncated Cauchy Power Exponential distribution having a variety of shape and monotonically increasing, increasing-decreasing, and constant hazard rate (Chaudhary et al., 2020) and the Logistic-exponential Power distribution (Joshi et al., 2020).

Chaudhary et al. (2021) developed the Exponentiated Weibull inverted exponential distribution which exhibits inverted bathtub shape and increasing – decreasing hazard rate. Using the half Cauchy family of distributions as a baseline, Chaudhary and Kumar (2022) also presented the half Cauchy modified exponential distribution.

Modified Generalized Exponential Distribution, as defined by (Telee & Kumar, 2023), is a modification of the Generalized Exponential distribution first presented by (Gupta & Kundu, 1999). Depending on the model's parameter values, it displays an inverted bathtub or a reverse j-shaped hazard rate. Telee and Kumar (2023) also introduced Modified Inverse Generalized Exponential Distribution which displays increasing and decreasing or inverted bathtub shaped hazard rate based on set of parameters. Otoo et al. (2023) proposed Odd Chen Exponential Distribution which exhibits different shapes, including the well-known bathtub shape hazard rate. Chaudhary et al. (2024) suggested novel flexible distribution called Modified Arctan Exponential distribution with revealing its remarkable flexibility in accommodating both increasing and decreasing hazard functions, as well as an inverted bathtub-shaped hazard function. The Cauchy modified generalized exponential distribution suggested by (Chaudhary et al., 2024) also displays both a rising trend and an inverted bathtub shape hazard rate. The Cauchy modified generalized exponential distribution, which displays both a rising trend and an inverted bathtub shape hazard rate, was also developed by (Chaudhary et al., 2024).

Chaudhary et al. (2023) have developed innovative model called Inverse Exponential Power (IEP) distribution by inverse transformation technique. The Inverse Exponential Power distribution's cumulative distribution function (CDF) and probability density function (PDF) are defined, respectively, by

$$F_{IEP}(x;\alpha,\lambda)=\exp\left\{1-\exp\left(\frac{\lambda}{x}\right)^{\alpha}\right\} \quad ; \; \alpha>0,\, \lambda>0, x>0\,; \tag{1}$$

$$f_{IEP}(x;\alpha,\lambda)=\frac{\alpha}{\lambda}\left(\frac{\lambda}{x}\right)^{\alpha+1}\exp\left(\frac{\lambda}{x}\right)^{\alpha}\exp\left\{1-\exp\left(\frac{\lambda}{x}\right)^{\alpha}\right\} \quad ; \alpha>0,\, \lambda>0, x>0\,. \tag{2}$$

The hazard rate function of IEP distribution displays increasing, inverted bathtub and decreasing shape.

Based on the Exponential Power (EP) lifetime distribution that was first suggested by (Smith & Bain, 1975), we suggest a unique distribution called Modified Exponential Power (MEP) distribution in this study. The CDF of EP distribution is defined by the equation (3).

$$F_{EP}(x;\alpha,\beta)=1-\exp\left\{1-\exp\left(\frac{x}{\alpha}\right)^{\beta}\right\} \quad ; \; \alpha>0,\, \beta>0, x>0\,; \tag{3}$$

And the associated PDF of EP distribution is defined by the equation (4).

$$f_{EP}(x;\alpha,\beta)=x^{\beta-1}\alpha^{-\beta}\beta\exp\left(\frac{x}{\alpha}\right)^{\beta}\exp\left\{1-\exp\left(\frac{x}{\alpha}\right)^{\beta}\right\} \quad ; \alpha>0,\, \beta>0, x>0\,. \tag{4}$$

The subsequent sections are organized as follows: Section 2 introduces the Modified Exponential Power (MEP) distribution and explores its statistical properties. In Section 3, we examine estimation techniques such as Cramer-Von-Mises (CVME), maximum likelihood (MLE)and least-squares (LSE. Section 4 applies these techniques to a real dataset, presenting the estimated model parameters and demonstrating various test criteria used to evaluate the model's goodness of fit. Finally, Section 5 provides concluding remarks.

Modified Exponential Power distribution

The Exponential Power (EP) distribution, which was first presented by (Smith & Bain, 1975), with equations (3) and (4), has been modified to create the novel distribution known as the Modified Exponential Power (MEP) distribution, which we have described in this section. The CDF of the MEP distribution may be obtained as follows if X is a non-negative random variable that follows the MEP distribution:

$$F(x;\alpha,\beta,\lambda)=\left[1-\exp\{1-\exp(\beta x)^{\alpha}\}\right]^{\lambda};\alpha>0,\beta>0,\lambda>0,x>0 \tag{5}$$

And the associated PDF of equation (5) is obtained as follows:

$$f\left(x;\alpha,\beta,\lambda\right)=\alpha\beta^{\alpha}\lambda x^{\alpha-1}e^{(\beta x)^{\alpha}}e^{[1-\exp(\beta x)^{\alpha}]}\left(1-\exp[1-\exp(\beta x)^{\alpha}]\right)^{\lambda-1};\alpha,\beta,\lambda>0,x>0 \tag{6}$$

The Survival function:

The MEP distribution’s survival function is

$$S(x;\alpha,\beta,\lambda)=1-\left[1-\exp\{1-\exp(\beta x)^{\alpha}\}\right]^{\lambda};\alpha>0,\beta>0,\lambda>0,x>0 \tag{7}$$

The Hazard rate function:

The hazard rate function of MEP distribution is

$$h(x;\alpha,\beta,\lambda)=\frac{\alpha\beta^{\alpha}\lambda x^{\alpha-1}e^{(\beta x)^{\alpha}}e^{[1-\exp(\beta x)^{\alpha}]}\left(1-\exp[1-\exp(\beta x)^{\alpha}]\right)^{\lambda-1}}{1-\left[1-\exp\{1-\exp(\beta x)^{\alpha}\}\right]^{\lambda}}\ ;\ \alpha>0,\beta>0,\lambda>0,x>0\,. \tag{8}$$

And the Cumulative hazard rate function of MEP distribution is

$$H\left(x\right)=-\log\left(1-\left[1-\exp\{1-\exp(\beta x)^{\alpha}\}\right]^{\lambda}\right);\alpha>0,\beta>0,\lambda>0,x>0$$

For the MEP distribution across different parameter sets, Figure 1 displays the PDF and hazard rate plots. The shape of the density function, as illustrated in the left panel of Figure 1, can vary significantly with different parameter values. It can be right-skewed and unimodal, among other possible shapes. The hazard rate is depicted in Figure 1's right panel as an increasing, inverted bathtub and decreasing shape.

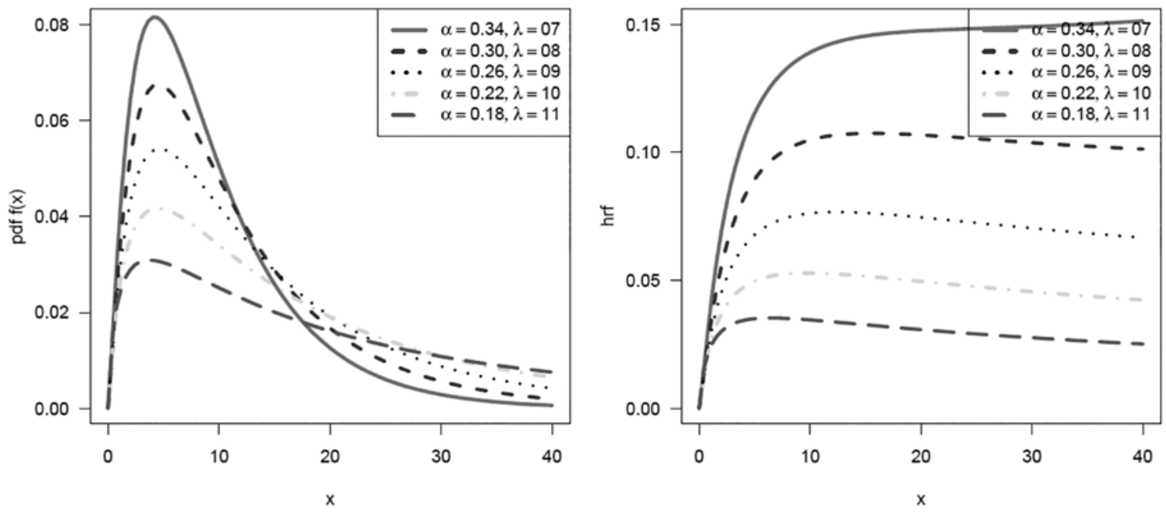

Fig. 1: PDF plots (Left) and Hazard rate plots (right) for β= 0.25

Reversed hazard rate function

The reversed hazard rate function of MEP distribution is

$$h_{rev}(x;\alpha,\beta,\lambda)=\frac{\alpha\beta^{\alpha}\lambda x^{\alpha-1}e^{(\beta x)^{\alpha}}e^{[1-\exp(\beta x)^{\alpha}]}}{1-\exp\{1-\exp(\beta x)^{\alpha}\}}\quad;\ \alpha>0,\beta>0,\lambda>0,x>0\ . \tag{9}$$

The Quantile function:

The MEP's quantile function is given by

$$x_p=\frac{1}{\beta}\left(\ln\left\{1-\ln(1-p)^{1/\lambda}\right\}\right)^{1/\alpha}\quad;0<p<1. \tag{10}$$

Random deviate generation:

Random deviate generation is given by

$$x=\frac{1}{\beta}\left(\ln\left\{1-\ln(1-u)^{1/\lambda}\right\}\right)^{1/\alpha}\quad;0<u<1. \tag{11}$$

where u has the uniform U (0, 1) distribution.

Skewness and Kurtosis:

Kenney and Keeping (1962) created the following Bowley's measure of skewness, which is based on the quartile:

$$S_k(B)=\frac{Q(1/4)+Q(3/4)-2Q(1/2)}{Q(3/4)-Q(1/4)},$$

and the coefficient of kurtosis ,as determined by (Moors ,1988) with octiles, is as follows:

$$K_u(M)=\frac{Q(0.875)-Q(0.625)-Q(0.125)+Q(0.375)}{Q(3/4)-Q(1/4)}.$$

Estimation of Parameters: Parameter estimation of the model is most essential work of the probability distribution. There are various techniques of parameter estimation. In this study, we have used three important methods namely MLE, CVE and LSE methods of estimation.

Maximum Likelihood Estimation

Here, we've demonstrated the MLE method for parameter estimation in the MEP model. Assume that $\underline{x} = (x_1, \ldots, x_n)$ is a sample containing n MEP items. The MEP's log likelihood function is given by the equation (12).

$$l(x;\alpha,\beta,\lambda) = n\log\alpha + n\alpha\log\beta + n\log\lambda + (\alpha-1)\sum_{i=1}^{n}\log x_i + \beta^{\alpha}\sum_{i=1}^{n}x_i^{\alpha} + n - \sum_{i=1}^{n}\exp(\beta x_i)^{\alpha} + n(\lambda-1) - \sum_{i=1}^{n}\exp[1-\exp(\beta x_i)^{\alpha} \quad (12)$$

By calculating the derivative of (12) in the first order w.r. to α, β, and λ and solving for $\frac{\partial l}{\partial \alpha} = \frac{\partial l}{\partial \lambda} = \frac{\partial l}{\partial \beta} = 0$,for the α, λ, and β simultaneously, we may get the MLE of the MEP.

Least-Square Estimation

Consider the random sample $\{X_1, X_2, \ldots, X_n\}$ which has n units and is drawn from an ordered random variable $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$, where the cumulative distribution function (CDF) of X_i $F(X_i)$. It is then possible to get the LSE of α, β, and λ a by minimizing the relation (13) w.r.to α, β, and λ.

$$B(X;\alpha,\beta,\lambda) = \sum_{i=1}^{n}\left[\left[1-\exp\{1-\exp(\beta x)^{\alpha}\}\right]^{\lambda} - \frac{i}{n+1}\right]^2 \qquad (13)$$

Once the relation (13) is differentiated w.r. to α, β, and λ, the simultaneous equations for α, β, and λ may be solved to get the estimated parameters.

Cramer-Von Mises estimation

Through use of the Cramer-Von Mises estimation approach, we may get estimated values of the unknown parameters α, β, and λ by minimizing relation (14).

$$C(X;\alpha,\beta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\left[1-\exp\{1-\exp(\beta x)^{\alpha}\}\right]^{\lambda} - \frac{2i-1}{2n}\right]^2 \qquad (14)$$

Equation (14) can be differentiated with respect to α, β, and λ, and the equations $\frac{\partial C}{\partial \alpha} = 0, \ \frac{\partial C}{\partial \beta} = 0, \mathit{and}\ \frac{\partial C}{\partial \lambda} = 0$ may be solved and then the CVM estimators of the parameters may be derived.

Applications to Real Dataset

We examine a real data set of waiting times (in minutes) for 100 bank clients reported by (Ghitany et al.,2008) in order to demonstrate the flexibility of the MEP distribution.

33.1, 38.5, 27.0,31.6, 0.8, 0.8, 1.5, 1.3, 3.5, 3.3, 4.0, 3.6, 6.7, 6.3, 7.1, 6.9, 7.1,7.1, 7.1, 7.6, 7.4, 5.3, 5.0, 5.7, 5.5, 6.1, 5.7, 8.0, 7.7, 8.6, 8.2, 12.5, 8.6, 13.0, 12.9, 13.3, 13.1, 8.6, 13.6, 8.8, 8.8, 1.9, 1.8, 2.1, 1.9, 2.7, 2.6, 3.1, 2.9, 8.9, 3.2, 9.5, 8.9, 9.7, 9.6, 10.9, 10.7, 11.0, 11.0, 11.2, 11.1, 11.5, 11.2, 12.4, 11.9, 4.2, 4.1, 4.3, 4.2, 4.4, 4.3, 4.4, 4.7, 4.6, 4.8, 4.7, 4.9, 4.9, 6.2, 6.2, 9.8, 6.2, 113.9, 3.7, 18.4, 21.3, 14.1, 18.9, 19.0, 21.4, 15.4, 15.4, 17.3, 19.9, 17.3, 18.2, 20.6, 18.1, 23.0,21.9.

Using the R software's optim() function as explained by the (R Core Team ,2023), maximizing (12) has allowed us to approximate the MLEs of the MEP distribution. The computed Log-Likelihood value is $l = -316.9897$. The standard errors (S.E.) and MLEs for alpha, beta, and lambda are displayed in Table 1.

Table 1

MLE and S.E., of MEP

(Parameters)	MLE	SE
α	0.3406	0.1028
β	0.2309	0.1518
λ	7.6184	5.0291

Boxplot and TTT are shown in Figure 2 to study the nature of the data.

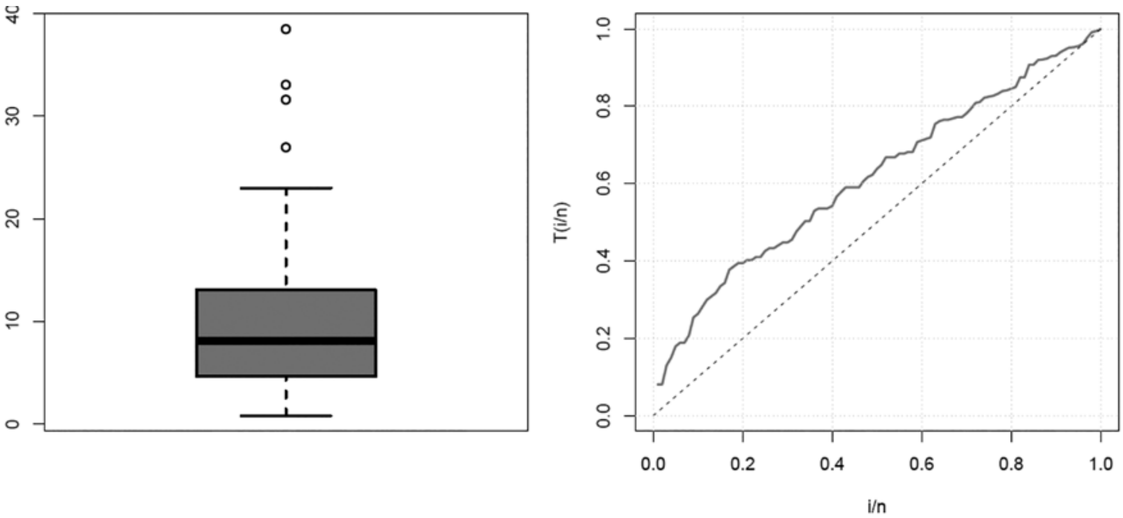

Figure 2: The box plot(left) and TTT plot(right).

The estimated parameters α, λ, and β using R are presented in Table 2.

Table 2: Estimated parameters for α, β, and λ

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	LL	AIC	KS(p-value)
MLE	0.3406	0.2309	7.6184	-316.9897	639.9793	0.0358(0.9995)
LSE	0.2969	0.3083	9.9523	-317.1191	640.5982	0.0389(0.9981)
CVE	0.2988	0.3107	10.1546	-317.0868	640.1736	0.0354 (0.9996)

A P-P plot and a Q-Q plot are shown in Figure 3 to evaluate the model's normality, respectively, in the left and right panels. The MEP distribution fits the data quite well, as demonstrated by these plots.

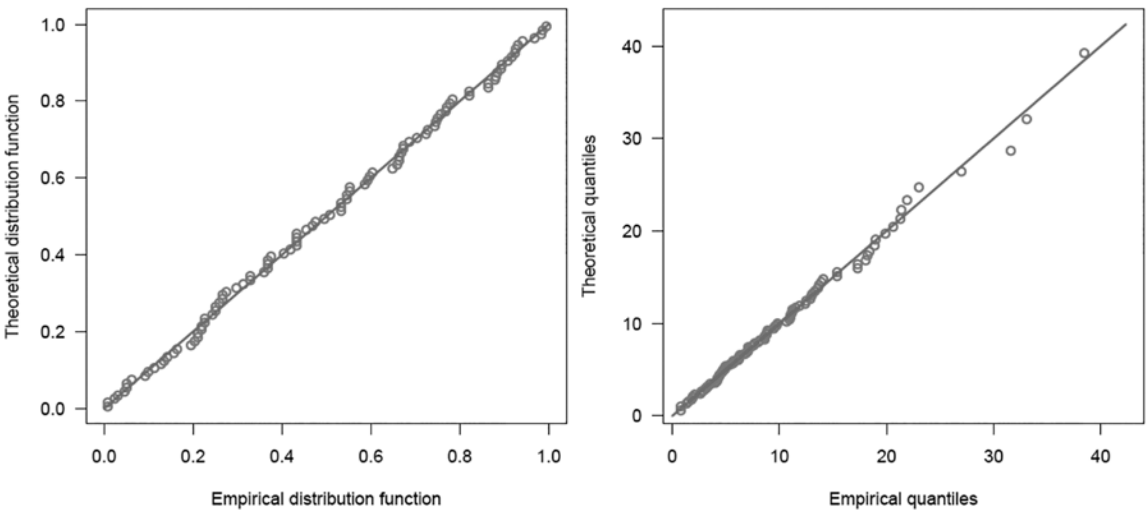

Figure 3: The MEP distribution's P-P plot in the left panel & Q-Q plot in the right panel.

Model Comparison

We assessed the MEP model's goodness of the fit in contrast to five other models utilizing the same dataset. The models compared include: Modified Inverse NHE (Chaudhary et al., 2023), Exponentiated Power Lindley (EPL) (Ashour & Eltehiwy, 2015), Marshall-Olkin Extended Exponential (MOEE) (Marshall & Olkin, 1997), Exponential Extension (EE) NHE (Nadarajah & Haghighi, 2011), Generalized Rayleigh (GR) (Kundu & Raqab, 2005) and Power Lindley (PL) (Ghitany et al., 2014). goodness of the fit

The estimated parameters of the suggested model are shown in Table 5, along with the rival models under consideration.

Table 5: Estimated parameters of MEP and other models

Estimated parameters with standard error of estimations (S.E.) for the competing models and the suggested model

Models	α	β	λ	θ	δ
MEP	0.3406(0.1028)	0.2309(0.1518)	7.6184(5.0291)		
MINH	0.4858(0.0415)	0.1099(0.0160)	37.5129(6.4768)		
EPL	2.6277(1.7794)	0.7472(0.1879)		0.5086(0.3379)	
PL	1.0834(0.0704)	0.1529(0.0282)			
MOEE	4.1180(1.3544)		0.1924(0.0260)		
GR	0.6298(0.0777)		0.0694(0.0053)		
NHE	3.4195(1.9166)		0.0205(0.0137)		

We also showed the dataset's histogram, the MEP model's goodness-of-fit graph, and the models that were taken into account in Figure 4's left panel. Fitted CDF and empirical CDF are displayed in the graph's right panel. Plots show that, when compared to the other models taken into consideration, the MEP distribution offers a superior fit to the dataset.

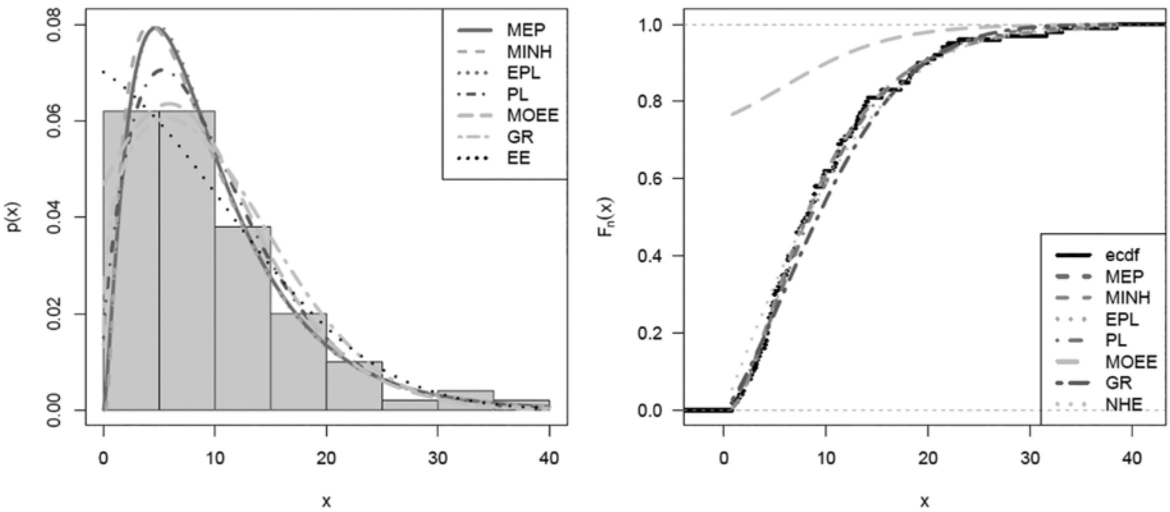

Figure 4: The MEP distribution's empirical vs. estimated CDF plots (right panel), as well as the fitted density plot & histogram (left panel).

The MEP distribution's applicability may be ascertained using examples of the HQIC, BIC, AIC, and CAIC. These figures are shown in Table 6. The validation criteria indicate that the MEP model has lower values compared to the other models considered. This suggests that, in comparison to the other models considered, the suggested model fits the data better.

Table 6: HQIC, BIC, CAIC, log-likelihood (LL) and AIC

Models	LL	AIC	BIC	CAIC	HQIC
MEP	-316.9897	639.9793	647.7948	640.2293	643.1424
MINH	-317.0699	640.1398	647.9553	640.3898	643.3029
EPL	-317.1008	640.2016	648.0171	640.4516	643.3646
PL	-318.3186	640.6372	645.8475	640.7609	642.7459
MOEE	-320.7120	645.4241	650.6344	645.5453	647.5328
GR	-321.5182	647.0364	652.2467	647.1601	649.1451
NHE	-323.4487	650.8973	656.1077	651.0185	653.0060

A crucial section of statistical modeling is determining the fitted model's goodness of fit and contrasting it with alternative models. We created the statistics for the Kolmogorov-Smirnov (KS), Cramer-Von Mises (CVM), and Anderson-Darling (AD) tests to evaluate the MEP distribution's goodness-of-fit relative to other distributions. Table 7 shows the values of these test statistics. It is discovered that in all goodness of fit methods, the MEP and MINH distribution have higher p-values and smaller test statistic values. This leads to the conclusion that, when compared to other models considered, the MEP and MINH distribution matches the real data set more consistently.

Table 7: The KS, AD, CVM statistics and their corresponding p-value

Model	KS(p-value)	AD(p-value)	CVM(p-value)
MEP	0.036 (0.9995)	0.1273 (0.9996)	0.017 (0.9989)
MINH	0.036(0.9995)	0.1274(0.9996)	0.0173(0.9990)
EPL	0.038(0.9989)	0.018(0.9987)	0.128(0.9996)
PL	0.052(0.9498)	0.046(0.9025)	0.303(0.9359)
MOEE	0.06(0.8690)	0.08(0.7164)	0.64(0.6150)
GR	0.095(0.3337)	0.204(0.2595)	1.09(0.3126)
NHE	0.107(0.2028)	0.21(0.2499)	1.554(0.1642)

Concluding Remarks

In this work, we introduce the Modified Exponential Power (MEP) distribution, a new and versatile probability distribution. An additional shape parameter is added to the Exponential Power (EP) distribution to create this distribution. The distributional and statistical characteristics of the suggested model have been thoroughly examined. There are three possible geometries for the MEP distribution's hazard function: increasing, inverted bathtub, and decreasing, while its probability density function (PDF) can be right-skewed and unimodal. As can be seen from the P-P and Q-Q plots, the MEP distribution matches the original dataset considerably better. Using a real dataset, we evaluated three popular estimation procedures: CVM, LSE, and MLE estimation. Our findings indicate that MLE outperforms both LSE and CVM. Additionally, the application demonstrates that the MEP and MINH distributions consistently surpass competing distributions in terms of fit and flexibility. This model is expected to be a valuable alternative in the fields of probability theory and applied statistics. Its applicability across various fields highlights its potential as a powerful tool for researchers and practitioners seeking to model complex data accurately. Future research could focus on extending the MEP distribution to multivariate cases or exploring its applications in new domains.

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