Modified Generalized Rayleigh Distribution: Model and Properties

Lal Babu Sah Telee
Department of Statistics
Nepal Commerce Campus, Kathmandu, Nepal
Email:lalbabu3131@gmail.com (Corresponding)
Vijay Kumar
Department of Mathematics and Statistics
DDU Gorakhpur University, Gorakhpur, India
Email: vkgkp@rediffmail.com

Abstract
This article is based on formulation of a new three parameters continuous distribution. Some important properties are discussed here. Different methods of estimation are used for parameter estimation. Validity of the model is tested using different information criteria as well as some graphical plots. For applicability testing, a real data set is taken. For model comparison we have considered some previously introduced probability models and is found that proposed model fits the real life data better compared to considered models.

Keywords: *Parameter estimations, Model validation, Model comparison, Information criteria, Maximum likelihood estimations.*

Cite this paper
Sah Telee, L. B., & Kumar, V. (2022). Modified Generalized Rayleigh Distribution: Model and Properties. *Pravaha, 28*(1), 11-22.

Introduction
Different forms of cumulative distribution functions was introduced by Burr (1942) for modeling lifetime data among which Burr Type X was one of most important. Generalized Rayleigh distribution also called Burr Type X having two parameters was also introduced by Surles and Padgett (2001). This two parameter Rayleigh distribution is a particular case of Generalized Weibull distribution given by Mudholkar and Srivastav (1993). In this article we have modified two parameters Burr Type X distribution called Generalized Rayleigh (GR) distribution having cumulative distribution function as

$$F(x;\alpha,\lambda)=\left(1-\exp\left(-\left(\lambda x\right)^{2}\right)\right)^{\alpha}\ ;x>0,\alpha>0,\lambda>0 \tag{1}$$

There are many techniques of getting new distribution using existing distribution. One of the methods is modification of the existing distribution. Treyer (1964) gave inverse Rayleigh distribution. The cumulative distribution function of inverse Rayleigh distribution is

$$G(x,\theta)=\exp(-\theta(1/x)^{2}) \tag{2}$$

Above model is modified [Khan, 2014] by adding one extra parameter α as

$$F(x,\alpha,\theta)=\exp((-\alpha/x)-\theta(1/x)^{2}) \tag{3}$$

In literature we can also find many modifications of Weibull distribution. Two parameter Weibull distributions is given as

$$\overline{F}(x,\alpha,\beta)=\exp[-(\alpha,x)]^{\beta} \tag{4}$$

This distribution is modified to generate several distributions that possess bathtub hrf. Modifications of Weibull distribution is exponentiated Weibull distribution [Mudholkar & Srivastava, 1993]. As given by [Lai. et al., 2003] to get new lifetime distributions as

$$\overline{F}(x)=\exp[ax^{b}.\exp(\lambda x)] \tag{5}$$

Here we have modified the Generalize Rayleigh Distribution defined in (1) by adding a new parameter β as

$$F(x;\alpha,\beta,\lambda)=\left(1-e^{-(\lambda x)^{2}\exp(\beta x)}\right)^{\alpha} \quad ;x>0,\alpha>0,\beta>0,\lambda>0$$

Corresponding probability density function of the proposed model is given as

$$f(x;\alpha,\beta,\lambda)=\alpha\lambda^{2}xe^{-(\lambda x)^{2}e^{\beta x}}e^{\beta x}(2+\beta x)\left(1-e^{-(\lambda x)^{2}\exp(\beta x)}\right)^{\alpha-1}$$

Objective of The Study

Main objective of this study is to introduce a new three parameter probability distribution having bathtub hazard rate function. Other objective of the model is to formulate a model having more flexibility, more applicability for real set of data compared with some already defined models.

Methodology

MGRL distribution is derived using theoretical concept of Generalized Rayleigh distribution. Estimation of the parameters of the model is done using three methods, maximum likelihood estimation, least square methods and Cramer-von Mises method. A real set data is taken to check the applicability of the model. Different information criteria such as Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are use for testing the model validation. For testing the goodness of fit, three methods, Kolmogrov-Smirnov (KS), Anderson-darling (AD), and Cramer-Von Mises (CVME) are used. All the graphs and the computations are done by using the R-programming language.

Model and Statistical Properties

Let X is continuous random variable following the Modified Generalized Rayleigh (MGRL) Distribution then Cumulative distribution function (CDF) and probability density function (PDF) are given by

$$F(x;\alpha,\beta,\lambda)=\left(1-e^{-(\lambda x)^{2}\exp(\beta x)}\right)^{\alpha} \quad ;x>0,\alpha>0,\beta>0,\lambda>0 \tag{5}$$

Corresponding probability density function of the proposed model is given as

$$f(x;\alpha,\beta,\lambda)=\alpha\lambda^{2}xe^{-(\lambda x)^{2}e^{\beta x}}e^{\beta x}(2+\beta x)\left(1-e^{-(\lambda x)^{2}\exp(\beta x)}\right)^{\alpha-1} \tag{6}$$

Reliability Function: Reliability function of MGRL is given as

$$R(x;\alpha,\beta,\lambda)=1-\left(1-e^{-(\lambda x)^{2}\exp(\beta x)}\right)^{\alpha} \quad ;x>0,\alpha>0,\beta>0,\lambda>0 \tag{7}$$

Hazard Rate Function (HRF): The Hazard Rate function of MGRL is given by

$$h(x)=\left(\alpha\lambda^{2}xe^{-(\lambda x)^{2}\,\mathrm{e}^{\beta x}}\;e^{\beta x}\,(2+\beta x)\left(1-e^{-(\lambda x)^{2}\exp(\beta x)}\right)^{\alpha-1}\right)\left(1-\left(1-e^{-(\lambda x)^{2}\exp(\beta x)}\right)^{\alpha}\right)^{-1} \tag{8}$$

Quantile Function: The quantile function of the model is

$$-\beta(x-2)\log x+\log(\lambda^{-2} * (-\log(1-(p[i])^{(1/\alpha)}))=0 \tag{9}$$

Skewness and Kurtosis: Skewness and kurtosis are the measure that describes the nature of distribution. Bowely's skewness of the MGRL distribution based on quartiles has form

$$Sk(B)=\frac{Q(3/4)-2Q(1/2)+Q(1/4)}{Q(3/4)-Q(1/4)}, \tag{10}$$

Kurtosis of the MGRL distribution based on octiles has form [Moors, 1988].

$$K(moors)=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}, \tag{11}$$

We have taken different values of the parameters taking θ = 0.7 of MGRL distribution and plotted probability density function as well as hazard rate function given below in figure 1.

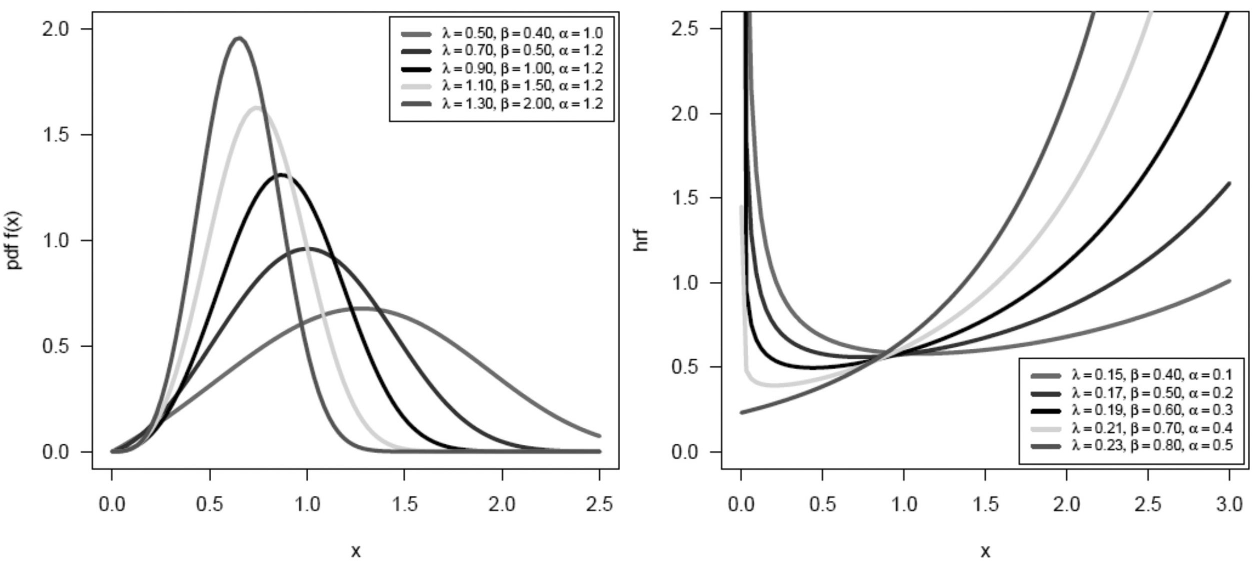

Figure1. The PDF (Left panel) and HRF (Right panel) of the MGRL

Estimation of Parameter

Here we have used three important methods of parameter estimations. The methods are

Maximum Likelihood Estimation (MLE): Suppose that $\underline{x} = (x_1,\ldots,x_n)$ is a random sample from MGRL with sample size n having log likelihood function as,

$$\begin{aligned}\ell = 2n\log\lambda + n\log\alpha - \lambda\sum_{i=1}^{n} x_i e^{\beta x_i} + (\alpha-1)\sum_{i=1}^{n}\log\left\{1-e^{-(\lambda x_i)^2\, e^{\beta x_i}}\right\} \\ + \sum_{i=1}^{n}\log x_i + \beta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}(2+\beta x_i)\end{aligned} \tag{12}$$

Differentiating above log likelihood function (12) with respect to unknown parameters λ, β and α respectively, we get

$$\frac{\partial \ell}{\partial \lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i e^{\beta x_i} + 2\lambda(\alpha-1)\sum_{i=1}^{n} x_i^2 e^{\beta x_i} e^{-(\lambda x_i)^2 e^{\beta x_i}} \left\{1 - e^{-(\lambda x_i)^2 e^{\beta x_i}}\right\}^{-1}$$

$$\frac{\partial \ell}{\partial \beta} = -\lambda \sum_{i=1}^{n} x_i^2 e^{\beta x_i} + (\alpha-1)\sum_{i=1}^{n} \lambda^2 x_i^3 e^{\beta x_i} e^{-(\lambda x_i)^2 e^{\beta x_i}} \left\{1 - e^{-(\lambda x_i)^2 e^{\beta x_i}}\right\}^{-1}$$

$$+ \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i (2 + \beta x_i)^{-1}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left\{1 - e^{-(\lambda x_i)^2 e^{\beta x_i}}\right\}$$

By setting $\frac{\partial \ell}{\partial \lambda} = \frac{\partial \ell}{\partial \beta} = \frac{\partial \ell}{\partial \alpha} = 0$ and solving those for λ, β and α we get the ML estimators of the $MGRL$ $(\lambda,\ \beta,\ \alpha)$ distribution. But normally, it is not possible to solve above non-linear equations so using R programming we can compute them easily. Suppose $\underline{\Theta}(\lambda,\ \beta,\ \alpha)$ is the parameter vector of $MGRL$ $(\lambda,\ \beta,\ \alpha)$ with respective MLE of $\underline{\Theta}$ as $(\hat{\lambda}, \hat{\beta}, \hat{\alpha})$. The asymptotic normality is given as,

$\left(\hat{\underline{\Theta}} - \underline{\Theta}\right) \rightarrow N_3\left[0, \left(I(\underline{\Theta})\right)^{-1}\right]$ where,

$$I(\underline{\Theta}) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \beta}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \alpha}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \beta^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta \partial \alpha}\right) \\ E\left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \beta}\right) & E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) \end{pmatrix}$$

Practically $\underline{\Theta}$ is unknown so MLE has an asymptotic variance $\left(I(\underline{\Theta})\right)^{-1}$ so, it is useless. By plugging in the estimated value of the parameters, we approximate the asymptotic variance. Here, observed fisher information matrix $O(\hat{\underline{\Theta}})$ is used as an estimate of the information matrix

$$I(\underline{\Theta}) \text{ given by } O(\hat{\underline{\Theta}}) = -\begin{pmatrix} \left(\frac{\partial^2 l}{\partial \hat{\lambda}^2}\right) & \left(\frac{\partial^2 l}{\partial \hat{\lambda} \partial \hat{\beta}}\right) & \left(\frac{\partial^2 l}{\partial \hat{\lambda} \partial \hat{\alpha}}\right) \\ \left(\frac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\lambda}}\right) & \left(\frac{\partial^2 l}{\partial \hat{\beta}^2}\right) & \left(\frac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\alpha}}\right) \\ \left(\frac{\partial^2 l}{\partial \hat{\alpha} \partial \hat{\lambda}}\right) & \left(\frac{\partial^2 l}{\partial \hat{\alpha} \partial \hat{\beta}}\right) & \left(\frac{\partial^2 l}{\partial \hat{\alpha}^2}\right) \end{pmatrix}_{\left(\hat{\lambda}, \hat{\beta}, \hat{\alpha}\right)} = -H(\underline{\Theta})_{\left(\underline{\Theta} = \hat{\underline{\Theta}}\right)}$$

Here H is called the Hessian matrix.

We can use Newton-Raphson technique for optimizing the likelihood can produce the observed information matrix. The variance covariance matrix is,

$$\left[-H\left(\underline{\Theta}\right)_{|(\Theta=\hat{\Theta})}\right]^{-1}=\begin{pmatrix}\text{var}(\hat{\lambda}) & \text{cov}(\hat{\lambda},\hat{\beta}) & \text{cov}(\hat{\lambda},\hat{\alpha})\\ \text{cov}(\hat{\beta},\hat{\lambda}) & \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta},\hat{\alpha})\\ \text{cov}(\hat{\alpha},\hat{\lambda}) & \text{cov}(\hat{\alpha},\hat{\beta}) & \text{var}(\hat{\alpha})\end{pmatrix}$$

Hence the *100(1-δ) %* CI for the parameters λ, β, and α are given by relations,

$$\hat{\lambda}\pm Z_{\delta/2}\sqrt{\text{var}(\hat{\lambda})},\ \hat{\beta}\pm Z_{\delta/2}\sqrt{\text{var}(\hat{\beta})}\text{ and }\hat{\alpha}\pm Z_{\delta/2}\sqrt{\text{var}(\hat{\alpha})}$$

Where $Z_{\delta/2}$. is upper percentile of standard normal variate.

Least-Square Estimation (LSE): Consider $F(X_{(i)})$ is the CDF of the variables $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$. Here $\{X_1, X_2, \ldots, X_n\}$ is a random sample having size n with a distribution function F (.). LSE of the unknown parameters λ, β and α of $MGRL\ (\lambda,\ \beta,\ \alpha)$ distribution can be obtained by minimizing (13) with respect to unknown parameters α, λ and θ.

$$A\left(x;\lambda,\beta,\alpha\right)=\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^2=\sum_{i=1}^{n}\left[\left(1-e^{-\left(\lambda x_{(i)}\right)^2 e^{\beta x_{(i)}}}\right)^{\alpha}-\left(\frac{i}{n+1}\right)\right]^2 \tag{13}$$

Differentiating equation (13) with respect to parameters λ, β and α respectively, we get

$$\frac{\partial A}{\partial\lambda}=4\lambda\alpha\sum_{i=1}^{n}x^2_{(i)}e^{\beta x_{(i)}}e^{-\left(\lambda x_{(i)}\right)^2 e^{\beta x_{(i)}}}\left(1-e^{-\left(\lambda x_{(i)}\right)^2 e^{\beta x_{(i)}}}\right)^{\alpha-1}\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

$$\frac{\partial A}{\partial\beta}=2\alpha\lambda^2\sum_{i=1}^{n}x^3_{(i)}\ e^{\beta x_{(i)}}\ e^{-\left(\lambda x_{(i)}\right)^2 e^{\beta x_{(i)}}}\left(1-e^{-\left(\lambda x_{(i)}\right)^2 e^{\beta x_{(i)}}}\right)^{\alpha-1}\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

$$\frac{\partial A}{\partial\alpha}=2\sum_{i=1}^{n}\left(1-e^{-\left(\lambda x_{(i)}\right)^2 e^{\beta x_{(i)}}}\right)^{\alpha}\log\left(1-e^{-\left(\lambda x_{(i)}\right)^2 e^{\beta x_{(i)}}}\right)\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

We can also obtain the weighted least square estimators by minimizing the expression

$$D\left(X;\lambda,\beta,\alpha\right)=\sum_{i=1}^{n}W_i\left[F(X_{(i)})-\frac{i}{n+1}\right] \tag{14}$$

The weights W_i are $W_i=\frac{1}{Var\ (X_{(i)})}=\frac{(n+1)^2\ (n+2)}{i(n-i+1)}$

We can minimize expression (14) to find the weighted least square estimators of the parameters λ, β, and α.

$$D(X;\lambda,\beta,\alpha)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left(1-e^{-(\lambda x_{(i)})^2 e^{\beta x_{(i)}}}\right)^{\alpha}-\frac{i}{n+1}\right]^2 \tag{15}$$

Cramer - von Mises Estimation (CVME): We can minimize the function (16) to obtain the Cramer-von Mises estimators of the parameters λ, β and α.

$$Z(x;\lambda,\beta,\alpha)=\sum_{i=1}^{n}\left[F(X_{(i)})-\left(\frac{2i-1}{2n}\right)\right]^2=\sum_{i=1}^{n}\left[\left(1-e^{-(\lambda x_{(i)})^2 e^{\beta x_{(i)}}}\right)^{\alpha}-\left(\frac{2i-1}{2n}\right)\right]^2 \tag{16}$$

$$\frac{\partial Z}{\partial \lambda}=4\lambda\alpha\sum_{i=1}^{n}x^2_{(i)}e^{\beta x_{(i)}}e^{-(\lambda x_{(i)})^2 e^{\beta x_{(i)}}}\left(1-e^{-(\lambda x_{(i)})^2 e^{\beta x_{(i)}}}\right)^{\alpha-1}\left[F(X_{(i)})-\left(\frac{2i-1}{2n}\right)\right]$$

$$\frac{\partial Z}{\partial \beta}=2\alpha\lambda^2\sum_{i=1}^{n}x^3_{(i)}\, e^{\beta x_{(i)}}\, e^{-(\lambda x_{(i)})^2 e^{\beta x_{(i)}}}\left(1-e^{-(\lambda x_{(i)})^2 e^{\beta x_{(i)}}}\right)^{\alpha-1}\left[F(X_{(i)})-\left(\frac{2i-1}{2n}\right)\right]$$

$$\frac{\partial Z}{\partial \alpha}=2\sum_{i=1}^{n}\left(1-e^{-(\lambda x_{(i)})^2 e^{\beta x_{(i)}}}\right)^{\alpha}\log\left(1-e^{-(\lambda x_{(i)})^2 e^{\beta x_{(i)}}}\right)\left[F(X_{(i)})-\left(\frac{2i-1}{2n}\right)\right]$$

We can obtain CVM estimators by solving $\frac{\partial Z}{\partial \lambda}=0, \frac{\partial Z}{\partial \beta}=0$, and $\frac{\partial Z}{\partial \alpha}=0$ simultaneously.

Application To Real Data Set

In this section, we demonstrate the applicability of the $MGRL\ (\lambda,\ \beta,\ \alpha)$ distribution using other real dataset. The data represents the number of deaths per day during the end of December 2020 due to COVID-19 in first wave from 23 January to 24 December [Government of Nepal Ministry of Health and Population (2020)]

2, 2, 2, 2, 2, 2, 2, 3, 2, 3, 3, 4, 2, 5, 5, 3, 2, 4, 4, 8, 4, 4, 3, 2, 3, 7, 6, 6, 11, 9, 3, 8, 7, 11, 8, 12, 12, 14, 7, 11, 12, 6, 14, 9, 9, 11,6, 6, 5, 5, 14, 9, 15, 11, 8, 4, 7 , 11, 10, 16, 2, 7, 17, 6, 8, 10, 4, 10, 7, 11, 11, 8, 7, 19, 9, 15, 12, 10, 14, 22, 9, 18, 12, 19, 21, 12, 12, 18, 8, 26, 21, 17, 13, 5, 15, 14, 11, 17, 16, 17, 23, 24, 20, 30, 18, 18, 17, 21, 18, 22, 26, 15, 13, 13, 6, 9, 17, 12, 17, 22, 7, 16, 16,2 4, 28, 23, 23, 19, 25, 29, 21, 9, 13, 16, 10, 17, 20, 23, 14, 12, 11, 15, 9, 18, 14, 13, 6, 16, 12, 11, 7, 3, 5, 5.

Exploratory data analysis:

Exploratory data analysis refers to the critical procedure of initial calculation of data to determine the pattern with help of summary statistics and graphical representation. It is an approach of statistical analysis that attempts to maximize insight into data [Tukey (1977)]. Exploratory data analysis uncovers underlying structure and extracts important variables of the data. Figure 2 represent the box plot and the TTT plot of the given data.

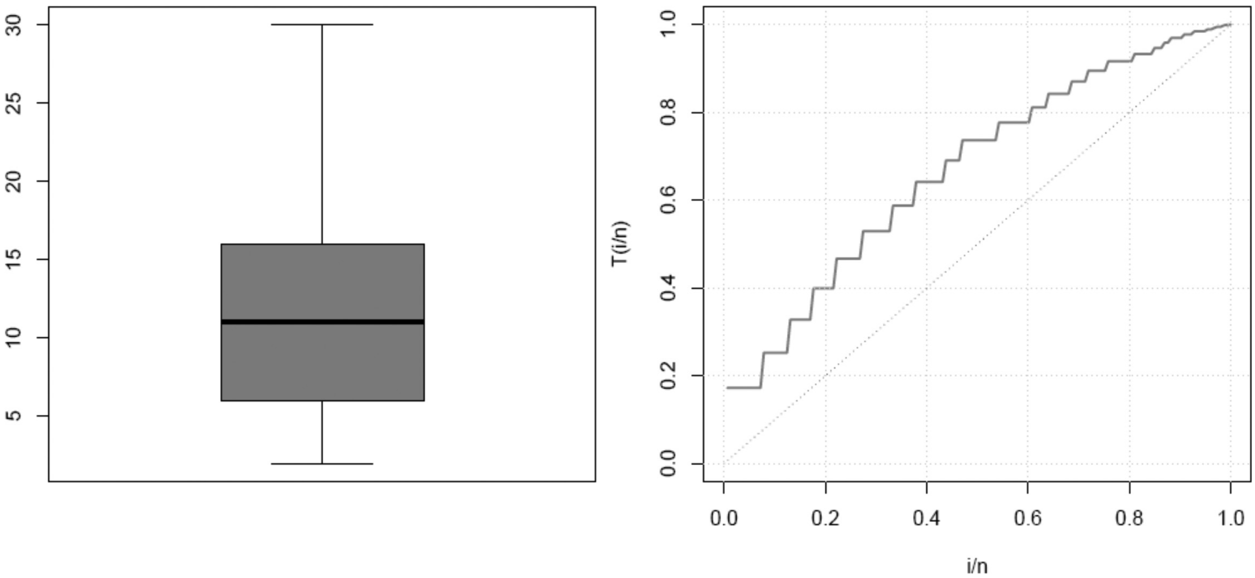

Figure 2. Boxplot (Left panel) and TTT plot (Right panel) of MGRL

Table 1

Summary Statistics:

Minimum	Q1	Median	Mean	Q3	SD	Skewness	Kurtosis
2.00	6.00	11.00	11.61	16	6.7591	0.508327	2.547717

The data set is positively skewed and non normal in shape.

By employing the optim () function in R software (R Core Team, 2020), we have estimated the parameters using MLE, LSE and CVM of MGRL distribution and are tabulated below in table 2.

Table 2

Parameters using MLE, LSE and CVM of MGRL

Parameter	MLE	LSE	CVM
Lambda	0.063364	0.05742150	0.05781137
Beta	0.007936	0.01156316	0.01177607
Alpha	0.767525	0.66836639	0.67687053

Figure 3 includes the graph of P-P plot and Q-Q plot of the proposed model MGRL

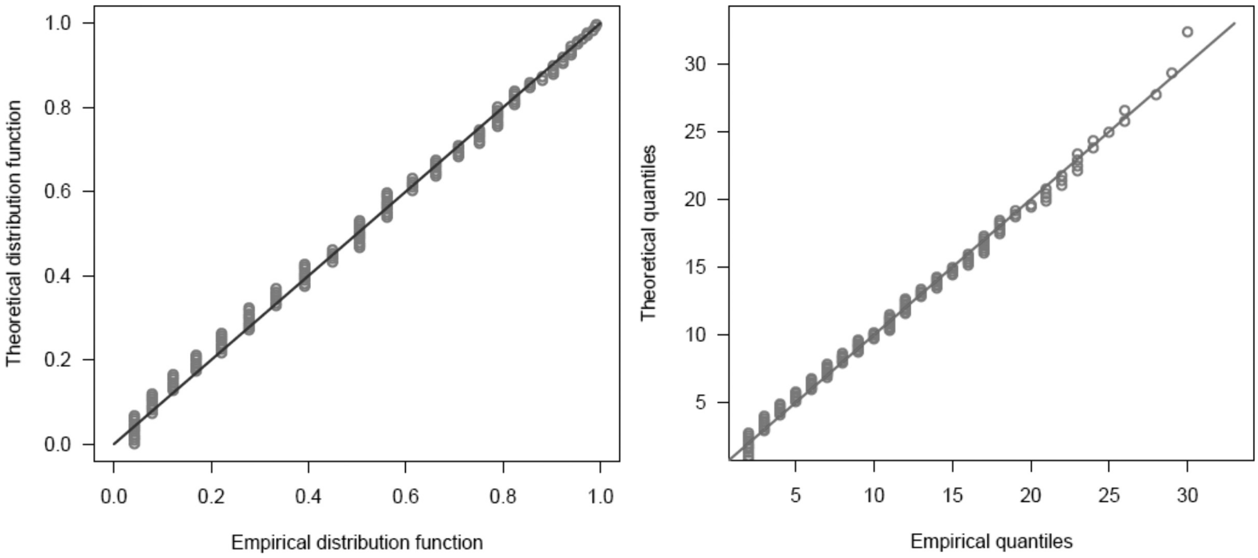

Figure 3 P-P plot (left panel) and Q-Q plot (Right panel) of MGRL

Values of log-likelihood, AIC, BIC, CAIC, and HQIC criteria are tabulated in Table 3.

Table 3

Estimated parameters, log-likelihood, AIC, BIC, CAIC, and HQIC

Method	LL	AIC	BIC	CAIC	HQIC
MLE	-496.7941	999.5882	1008.679	999.7492	1003.281
LSE	-497.4529	1000.906	1009.997	1001.067	1004.599
CVE	-497.2900	1000.580	1009.671	1000.741	1004.273

The KS, W and A^2 statistic with their corresponding p-value of MLE, LSE and CVE estimates we have presented in Table 4.

Table 4

The KS, W and A^2 statistic with a p-value

Method	KS(p-value)	W(p-value)	A^2(p-value)
MLE	0.049793(0.8425)	0.060717(0.8102)	0.52646(0.7197)
LSE	0.056023(0.7229)	0.041567(0.9255)	0.44764(0.8003)
CVE	0.054515(0.7535)	0.041105(0.928)	0.43312(0.8151)

We have plotted the histogram of the data and the fitted density curve of the model for different methods of estimations. Similarly we have plotted the ecdf of the model along with the fitted distribution function using different methods of estimation.

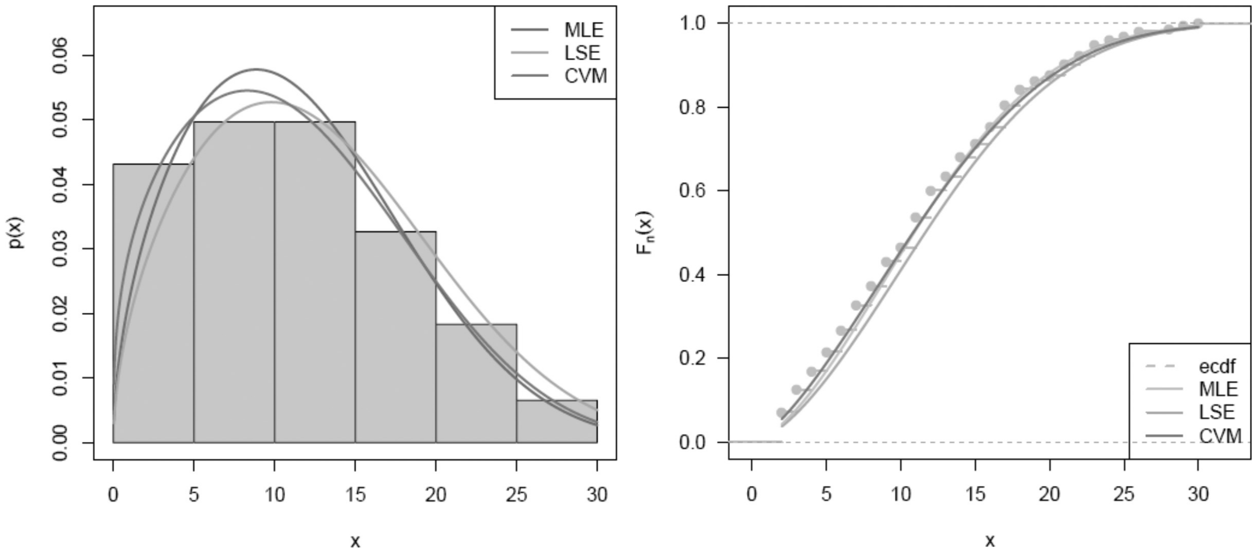

Figure 4. Histogram and density curve (left panel) & ecdf versus the fitted distribution function curve using different methods of estimations.

In this section, we have presented the applicability of MGRL distribution using a real dataset used by earlier researchers. To compare the potentiality of the proposed model, we have considered the following four distributions.

I. Exponentiated half logistic exponential (EHLE) distribution :
Exponentiated half logistic exponential distribution with statistical properties and applications was given by Almarashi et al.(2018).

$$f_{EHLE}(x) = \frac{2a\lambda\alpha e^{-\alpha\lambda x}[1-e^{-\alpha\lambda x}]^{a-1}}{[1+e^{-\alpha\lambda x}]^{a+1}}; x>0, \mathrm{a}>0, \alpha>0, \lambda>0.$$

II. Marshall-Olkin logistic Exponential (MOLE):
The density function of Marshall-Olkin logistic Exponential (MOLE) is given below which was introduced by Monsoor et al. (2019).

$$f_{MOLE}(x) = \frac{\alpha\theta\lambda e^{\lambda x}(e^{\lambda x}-1)^{-\alpha-1}}{[1+\theta(e^{\lambda x}-1)^{-\alpha}]^2}; x>0, \alpha>0, \theta>0, \lambda>0.$$

III. Marshall-Olkin power generalized Weibull:
The probability density function of Marshall-Olkin power generalized Weibull [Afify et al (2020)] is given by

$$f_{MOPGW}(x) = \alpha\beta\lambda x^{\beta-1}(1+\lambda x^{\beta})^{\alpha-1}\,\mathrm{e}^{1-(1+\lambda x^{\beta})^{\alpha}}; x>0, \alpha>0, \beta>0, \lambda>0.$$

IV. Odd Lomax Exponential(OLE) distribution:
Odd Lomax Exponential (OLE) distribution is given by Ogunsanya et al.(2019)

$$f_{OLE}(x) = \alpha\beta^{\alpha}\lambda e^{\lambda x}\left[\beta+\left(e^{\lambda x}-1\right)\right]^{-(\alpha+1)}; x>0, \alpha>0, \beta>0, \lambda>0.$$

Table 5 represents the estimated parameter values of the proposed model along with the models taken in consideration using maximum likelihood estimation.

Table 5

Estimated parameter values using MLE of different models

Model	Lambda	Beta	Alpha	Theta
MGRL	0.0634(0.0125)	0.0079(0.0156)	0.76753(0.1243)	

Model	Lambda	Beta	Alpha	Theta
EHLE	1.7502(7.4483)	1.9055(0.2216)	0.0954(0.4055)	
MOLE	0.1468(0.0449)		1.2874(0.2403)	5.5874(3.0400)
MOPGW	0.0104(0.0037)	1.9197(0.2191)	0.7836(0.2434)	
OLE	7.9304(5.5727)	0.0678(0.0645)	3.6849(3.3443)	

We have computed the log likelihood values along with different information criteria values for different models and are tabulated in table 6 below. It is shown that the proposed model has least information criteria values showing that model fits better to real data sets respective to other models taken in considerations.

Table 6

Different information criteria values of MGRL and other models

Model	-LL	AIC	BIC	CAIC	HQIC
MGRL	496.794	999.588	1008.68	999.749	1003.281
EHLE	499.708	1005.418	1014.51	1005.580	1009.110
MOLE	499.713	1005.430	1014.52	1005.590	1009.118
MOPGW	497.937	1001.870	1010.97	1002.035	1005.565
OLE	499.5843	1005.169	1014.26	1005.33	1008.862

We have also plotted the histogram and fitted density curve of the MGRL and the other models taken in consideration. We have also plotted the ecdf of the model and the fitted distribution function of other models taken in consideration and are shown in figure 5.

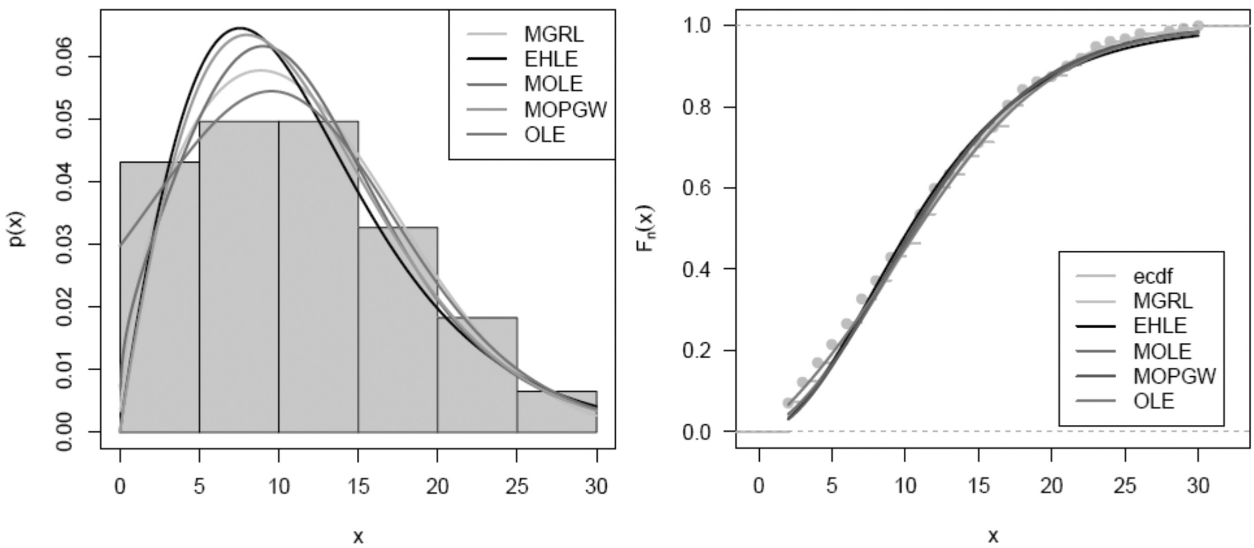

Figure 5 Histogram and fitted PDF of models (left panel) and ecdf with fitted distribution functions of the model.

To compare the goodness-of-fit of the MGRL distribution with other competing distributions, we have also displayed the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistic in Table 7. It is observed that the MGRL distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the MGRL distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 7

KS, W and AD values along with respective p values of MGRL and other models

Model	KS(p-value)	W(p-value)	AD(p-value)
MGRL	0.0498(0.8425)	0.0607 (0.8102)	0.5264(0.7197)
EHLE	0.0777(0.3145)	0.1228 (0.4850)	0.9346 (0.3933)
MOLE	0.0540(0.7634)	0.0826 (0.6777)	0.6938 (0.5638)
MOPGW	0.0648(0.5411)	0.1055(0.5592)	0.8565(0.4418)
OLE	0.0685(0.4699)	0.0581(0.8266)	0.6130(0.6355)

Conclusion

This article explains a new distribution called Modified Generalized Rayleigh distribution having three parameters. A detailed study of different statistical characteristics of the proposed are explained. That is; the derivation of precise expressions for its hazard rate function, survival function, the quantile function and skewness and kurtosis are presented. Three well-known estimation methods namely maximum likelihood estimation (MLE), Cramer-Von-Mises estimation (CVME), and least-square estimation (LSE) methods are used to estimate the parameter and we found that the CVM are relatively better than LSE and MLE methods. The curves of the PDF of the proposed distribution have shown that it can have various shapes like increasing-decreasing and right skewed and flexible for modeling real-life data. Also, the graph of the hazard function is inverted bathtub. The applicability and suitability of the proposed distribution has been evaluated by considering a real-life dataset and the results exposed that the proposed distribution is much flexible as compared to some other fitted distributions.

References

Afify, A. Z., Kumar, D., & Elbatal, I. (2020). Marshall–Olkin power generalized Weibull distribution with applications in engineering and medicine. *Journal of Statistical Theory and Applications*, *19*(2), 223-237.

Almarashi, A. M., Khalil, M. G., Elgarhy, M., & ElSehetry, M. M. (2018). Exponentiated half logistic exponential distribution with statistical properties and applications. *Advances and applications in statistics*, *53*(4), 423-440.

Burr, I. W. (1942). Cumulative frequency functions. *The Annals of mathematical statistics*, *13*(2), 215-232.

Government of Nepal Ministry of Health and Population. (2020). Health sector responsehttps://heoc.mohp.gov.np | https://portal.edcd.gov.np/

Khan, M. S. (2014). Modified inverse Rayleigh distribution. *International Journal of Computer Applications*, *87*(13), 28-33.

Lai, C.D., Xie, M., & Murthy, D. (2003). A modified Weibull distribution. *IEEE Transactions on Reliability* 52(1), 33-37.

Mansoor, M., Tahir, M. H., Cordeiro, G. M., Provost, S. B., & Alzaatreh, A. (2019). The Marshall-Olkin logistic-exponential distribution. *Communications in Statistics-Theory and Methods*, *48*(2), 220-234.

Moors, J. J. A. (1988). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, *37*(1), 25-32.

Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE transactions on reliability*, *42*(2), 299-302.

Mudholkar, G.S. & Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability,* 42(2), 299–302.

Ogunsanya, A. S., Sanni, O. O., & Yahya, W. B. (2019). Exploring some properties of odd Lomax-exponential distribution. *Annals of Statistical Theory and Applications (ASTA)*, *1*, 21-30.

R Core Team (2020). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URLhttps://www.R-project.org/.

Surles, J. G., & Padgett, W. J. (2001). Inference for reliability and stress-strength for a scaled Burr Type X distribution. *Lifetime data analysis*, *7*(2), 187-200.

Surles, J. Q., & Padgett, W. J. (2004). Some properties of a scaled Burr type X model. *Journal Statistical Planning and Inference*, *7*(2), 187-200.

Treyer, V. N. (1964). Doklady Acad, Nauk, Belours. U.S.S.R.

Tukey, J. W. (1977). *Exploratory data analysis* (Vol. 2, pp. 131-160).