Lindley Generalized Inverted Exponential Distribution: Model and Applications

Lal Babu Sah Telee

Department of Statistics, Nepal Commerce Campus, Kathmandu, Nepal, Email:lalbabu3131@gmail.com (Corresponding)

Vijay Kumar

Department of mathematics and statistics, DDU Gorakhpur University, Gorakhpur, India, Email: vkgkp@rediffmail.com

Abstract:

In this article, we have proposed three parameters distribution using lindley family of distribution and Generalized inverted Exponential distribution called Lindley Generalized Inverted Exponential Distribution: Model and Applications(LGIE). The statistical properties as well as different characteristics like the hazard rate function (HRF), the probability density function (PDF), the cumulative distribution function (CDF), quantile function , skewness, and kurtosis of the proposed distribution are discussed. The parameters of the proposed distribution are estimated by using the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises (CVM) methods. All the computations are performed in R programming software. To assess the application of the proposed distribution, a real lifetime data set is analyzed and performed the goodness-of-fit.

Keywords: *Maximum likelihood estimation, Generalized Inverted Exponential distribution, Estimation, Hazard function, TTT plot.*

Introduction

Single parameter exponential distribution is simple exponential distribution. This one parameter exponential distribution has numerous applications in life testing data and is widley used in different literature. This distribution plays important role in development of new theory and the derivation of new probability distributions. Due to the mathematical tractability of the one parameter exponential distribution, it can be easily illustrated in real life **(Barlow and Proschan) and (Leemis)**. Application of this distribution is restricted to a constant hazard rate because time independent hazard rate system is very rare. Due to this restriction, many literature based on exponential generalization are not suitable for real life data. The gamma distribution is based on sum of independent exponential variates is one of the popular generalization of the exponential distribution. Another distribution is Weibull distribution based on power transformed distribution which is also the popular generalization of exponential distribution. One of the most important characteristics of these generalization is the distribution follows the constant, non increasing, non decreasing and bathtub hazard rates. But in real life we can face many problems where data also shows inverted bathtub hazard rate where curve initially increases and then decreases. In such a case many distribution not fit the data appropriately and generalization of the distribution becomes more essential for fitting such type of data. Numerous important extension of the exponential distribution are proposed till now in many statistical literatures. Among these extension, one parameter inverse exponential (IED) possessing inverted bathtub hazard rate function. In literatures, there are many uses of IED in survival analysis **(Lin et al.) and (Singh et al.).** There are two parameter generalizations also available in literatuere .One of the important two parameters generalization is generalized inverted exponential distribution (GIED) proposed by Abouammoh and Alshingiti showed that GIED is better than IED for real life data.

Lindley in 1958 proposed one parameter distribution named as lindley distribution in the context of baysian statistics. Recently, many studies have been done focusing on modified forms of the baseline distribution using Lindley family of distribution proposed by **Zografas and Balakrishnan (2009)**. These distributions are more flexible in context of density and hazard rate functions. A detailed study on the Lindley distribution along with properties was studied by **(Ghitany et al., 2008)**.

Objective of the Study

Main objective of this work is to find new distribution with more flexibility by adding one extra parameter to generalized inverted exponential distribution (GIE) to achieve a better fit to real data.

Methodology

LGIE is compound distribution derived by using theoretical concept of Generalized Inverted Exponential distribution and lindley family. For the estimation constants of the model, we have used three popular methods of estimation. These methods are least square estimation, Cramer von mises estimation and maximum likelihood estimation. For numerical calculation of the model, R software is used. Here we have considered a real life data to assess the application of the proposed distribution. The statistical properties of this distribution as well as different characteristics like probability density function (PDF), the hazard rate function (HRF), the cumulative distribution function (CDF), quantile function , skewness, and kurtosis of the proposed distribution are discussed. Box plots, TTT plot, density fits etc are obtained using r software. Here we have used different model validation criteria such as AIC, BIC, and CAIC are obtained and compared with some well known probability distributions.

Model Formulation

Lindley Generalized Inverted exponential (LGIE) Distribution

Let X is random variable following Lindley distribution with parameter λ having density function as

$$f(t)=\frac{\lambda^2\left(1+t\right)e^{-\lambda t}}{\lambda+1},t>1,\lambda>1 \tag{1}$$

Cumulative density function of above distribution is

$$F(t)=1-\frac{(1+\lambda+\lambda t)e^{-\lambda t}}{1+\lambda} \quad ,t>0,\lambda>0 \tag{2}$$

Adding an additional shape parameter θ on Lindley generator (Lindley-G), **Cakmakyapan and Ozel (2016)** have introduced a new class of distributions having CDF as

$$F_{L-G}\left(x;\theta,\psi\right)=1-\left[\bar{G}(x,\psi)\right]^{\theta}\left[1-\frac{\theta}{\theta+1}\ln\bar{G}(x;\psi)\right];x>0,\,\theta>0 \tag{3}$$

Density function of Lindley G family of distribution is given as

$$f_{L-G}(x;\theta,\psi)=\frac{\theta^2}{\theta+1}\;g(x;\psi)[\bar{G}(x;\psi)]^{\theta-1}\left[1-\ln\bar{G}(x;\psi)\right] \quad ;x>0,\,\theta>0 \tag{4}$$

Where, $\bar{G}(x;\psi)$ =1- $G(x;\psi)$

In literatures, the generalized inverted exponential distribution has established more attention. Singh et. al.,(2013) developed generalized inverted exponential distribution having PDF and CDF as

$$F(x)=\left(1-\left(1-e^{-\lambda/x}\right)^{\alpha}\right) \tag{5}$$

$$g(x)=\frac{\alpha\lambda}{x^2}e^{-\lambda/x}\left(1-e^{-\lambda/x}\right)^{\alpha-1} \tag{6}$$

We have,

$$\bar{F}(x)=\left(1-e^{-\lambda/x}\right)^{\alpha}$$

Using equations (3) and (4) , we can get the distribution function and density function of the LGIE as

$$F_{L-G}(\alpha,\lambda,\theta)=1-\left(1-e^{-\lambda/x}\right)^{\alpha\theta}\left[1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x}\right)^{\alpha}\right] \tag{7}$$

$$f_{L-G}(\alpha,\lambda,\theta)=\frac{\alpha\lambda\theta^2}{(\theta+1)x^2}e^{-\lambda/x}\left(1-e^{-\lambda/x}\right)^{\theta\alpha-1}\left[1-\ln\left(1-e^{-\lambda/x}\right)^{\alpha}\right] \tag{8}$$

Some statistical properties

In this section we have derived some important statistical properties of the proposed model LGIE.

Reliability Function: Reliability function of the LGIE is given as

$$R(x;\alpha,\lambda,\theta)=1-F(x;\alpha,\lambda,\theta)=\left(1-e^{-\lambda/x}\right)^{\alpha\theta}\left[1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x}\right)^{\alpha}\right] \tag{9}$$

Hazard rate function: The hazard rate function of LGIE is given as

$$h(x;\alpha,\lambda,\theta)=\frac{f(x)}{1-F(x)}=\frac{\alpha\lambda\theta^2e^{-\lambda/x}\left(1-e^{-\lambda/x}\right)^{\theta\alpha-2}\left[1-\ln\left(1-e^{-\lambda/x}\right)^{\alpha}\right]}{(\theta+1)x^2\left[1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x}\right)^{\alpha}\right]} \tag{10}$$

Quantile function: The quantile function of LGIE is given as

$$\left(1-e^{-\lambda/x}\right)^{\alpha\theta}\left[1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x}\right)^{\alpha}\right]+(p-1)=0,\ 0<p<1 \tag{11}$$

Random deviate Generation: The random deviate generate of the proposed distribution is defined by

$$\left(1-e^{-\lambda/x}\right)^{\alpha\theta}\left[1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x}\right)^{\alpha}\right]+(v-1)=0\ ,0<v<1 \tag{12}$$

Skewness and Kurtosis of LGIE distribution: Skewness and kurtosis are the measure that describes the nature of distribution .Bowely's skewness of the $LGIE(\alpha,\lambda,\theta)$ distribution based on quartiles has form

$$Sk(B)=\frac{Q(3/4)-2Q(1/2)+Q(1/4)}{Q(3/4)-Q(1/4)}, \tag{13}$$

Kurtosis of the $LGIE(\alpha,\lambda,\theta)$ distribution based on octiles has form [Moors, 1988].

$$K(moors) = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}, \tag{14}$$

We have taken different values of the parameters taking θ=0.7 of LGIE distribution and plotted probability density function as well as hazard rate function given below in figure 1.

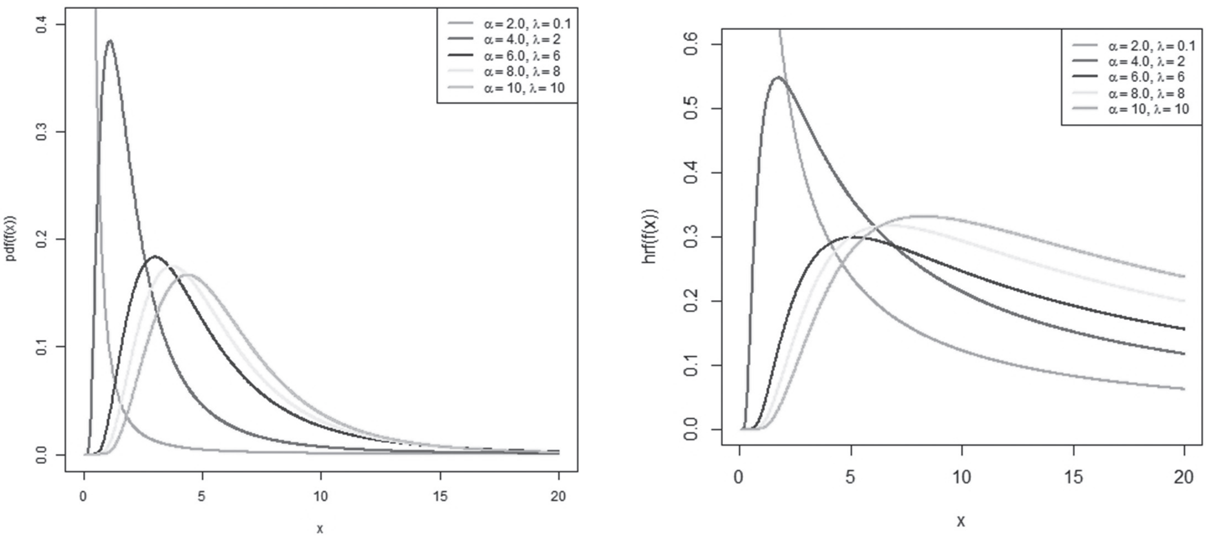

Figure 1: PDF plot (left) and HRF (right) for different values of α and λ taking θ=0.7

Estimation of the constants of model

Constants of the distribution are estimated by least square estimation, Cramer von mises estimation and maximum likelihood estimation. R programming as well as analytical methods are used for data analysis.

Maximum Likelihood estimation (MLE): In this section, we have presented the maximum likelihood estimators of the LGIE model. Suppose that $\underline{x} = (x_1, \ldots, x_n)$ is a random sample from $LGIE(\alpha, \lambda, \theta)$ with sample size n having log likelihood function as,

$$\ell = n\ln\alpha + n\ln\lambda + 2n\ln\theta - \ln(\theta+1) + \sum_{i=1}^{n}\ln(1/x_i^2) - \frac{1}{\lambda}\sum_{i=1}^{n}(1/x_i) + (\alpha\theta-1)\sum_{i=1}^{n}\ln\left(1-e^{-\lambda/x_i}\right) + \sum_{i=1}^{n}\ln\left(1-\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right) \tag{15}$$

After differentiating (15) with respect to parameters α, λ, and θ, we get

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \theta\sum_{i=1}^{n}\ln\left(1-e^{-\lambda/x_i}\right) - \sum_{i=1}^{n}\left(1-\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right)^{-1}\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\alpha} + \frac{1}{\lambda^2}\sum_{i=1}^{n}(1/x_i) + (\alpha\theta-1)\sum_{i=1}^{n}\frac{1}{x_i}e^{-\lambda/x_i}\left(1-e^{-\lambda/x_i}\right)^{-1} - \alpha\sum_{i=1}^{n}\frac{1}{x_i}e^{-\lambda/x_i}\left(1-e^{-\lambda/x_i}\right)^{-1}\left(1-\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right)^{-1}$$

$$\frac{\partial\ell}{\partial\theta} = \frac{2n}{\theta} - \frac{1}{(\theta+1)^{-1}} + (\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\lambda/x_i}\right) \tag{16}$$

By setting $\frac{\partial \ell}{\partial \alpha} = \frac{\partial \ell}{\partial \lambda} = \frac{\partial \ell}{\partial \theta} = 0$ and solving them for α, λ and θ we get the ML estimators of the $LGIE(\alpha, \lambda, \theta)$ distribution. But normally, it is not possible to solve non-linear equations (16) so with the aid of suitable computer programming we can compute them easily. Suppose $\underline{\Theta} = (\alpha, \lambda, \theta)$ is the parameter vector of with respective MLE of $\underline{\Theta}$ as $\hat{\underline{\Theta}} = (\hat{\alpha}, \hat{\lambda}, \hat{\theta})$. The asymptotic normality is given as,

$\left(\hat{\underline{\Theta}} - \underline{\Theta}\right) \rightarrow N_3\left[0, \left(I\left(\underline{\Theta}\right)\right)^{-1}\right]$.where,

$$I\left(\underline{\Theta}\right) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \alpha \partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \lambda \partial \alpha}\right) & E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \theta^2}\right) \end{pmatrix}$$

Practically $\underline{\Theta}$ is unknown so it is useless that the MLE has an asymptotic variance $\left(I\left(\underline{\Theta}\right)\right)^{-1}$. By plugging in the estimated value of the parameters, we approximate the asymptotic variance. Here, observed fisher information matrix $O(\hat{\underline{\Theta}})$ is used as an estimate of the information matrix $I\left(\underline{\Theta}\right)$ given by

$$O(\hat{\underline{\Theta}}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \hat{\alpha} \partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\alpha} \partial \hat{\theta}} \\ \frac{\partial^2 l}{\partial \hat{\lambda} \partial \hat{\alpha}} & \frac{\partial^2 l}{\partial \hat{\lambda}^2} & \frac{\partial^2 l}{\partial \hat{\lambda} \partial \hat{\theta}} \\ \frac{\partial^2 l}{\partial \hat{\theta} \partial \hat{\alpha}} & \frac{\partial^2 l}{\partial \hat{\theta} \partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\theta}^2} \end{pmatrix}_{|(\hat{\alpha}, \hat{\lambda}, \hat{\theta})} = -H\left(\underline{\Theta}\right)_{|(\underline{\Theta} = \hat{\underline{\Theta}})}$$

Here is called the Hessian matrix.

We can use Newton-Raphson technique for maximizing the likelihood gives the observed information matrix. Matrix below is the variance covariance matrix,

$$\left[-H\left(\underline{\Theta}\right)_{|(\underline{\Theta} = \hat{\underline{\Theta}})}\right]^{-1} = \begin{pmatrix} \text{variance}(\hat{\alpha}) & \text{co variance}(\hat{\alpha}, \hat{\lambda}) & \text{co variance}(\hat{\alpha}, \hat{\theta}) \\ \text{co variance}(\hat{\lambda}, \hat{\alpha}) & \text{variance}(\hat{\lambda}) & \text{co variance}(\hat{\lambda}, \hat{\theta}) \\ \text{co variance}(\hat{\theta}, \hat{\alpha}) & \text{co variance}(\hat{\theta}, \hat{\alpha}) & \text{variance}(\hat{\theta}) \end{pmatrix} \quad (17)$$

Hence using asymptotic normality of MLEs, the 100(1-a) % confidence intervals for the parameters α, λ , and θ are given by relations,

$$\hat{\alpha} \pm Z_{a/2}\sqrt{\text{Var}(\hat{\alpha})}, \hat{\lambda} \pm Z_{a/2}\sqrt{\text{Var}(\hat{\lambda})}, \text{ and } \hat{\theta} \pm Z_{a/2}\sqrt{\text{Var}(\hat{\theta})}$$

where $Z_{a/2}$ is known the upper percentile of standard normal variate.

Least-Square Estimation (LSE): Consider $F(X_i)$ is the CDF of the variables $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$. Here $\{X_1, X_2, \ldots, X_n\}$ is a random sample having size n with a distribution function F(.). LSE of the unknown parameters α, λ, and θ of $LGIE(\alpha, \lambda, \theta)$ distribution can be obtained by minimizing (18) with respect to unknown parameters α, λ and θ.

$$A(x;\alpha,\lambda,\theta,\delta)=\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]^2$$

$$=\sum_{i=1}^{n}\left[1-\left(1-e^{-\lambda/x_i}\right)^{\alpha\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right\}-\left(\frac{i}{n+1}\right)\right]^2 \quad (18)$$

Differentiating (18) with respect to α, λ , and θ we get,

$$\frac{\partial A}{\partial\alpha}=\frac{-2\theta^2}{\theta+1}\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]\left(1-e^{-\lambda/x_i}\right)^{\alpha\theta}\left(1-\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right)\ln\left(1-e^{-\lambda/x_i}\right)$$

$$\frac{\partial A}{\partial\lambda}=\frac{-2\theta^2\alpha}{\theta+1}\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]\left(1-e^{-\lambda/x_i}\right)^{\alpha\theta-1}\left(1-\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right)$$

$$\frac{\partial A}{\partial\theta}=-2\alpha\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]\left(1-e^{-\lambda/x_i}\right)^{\alpha\theta}\ln\left(1-e^{-\lambda/x_i}\right)\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}-\frac{1}{(\theta+1)^2}\right\}$$

We can also obtain the weighted least square estimators by minimizing the expression

$$D(X;\alpha,\lambda,\theta)=\sum_{i=1}^{n}W_i\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

The weights W_i are $W_i=\dfrac{1}{Var\,(X_{(i)})}=\dfrac{(n+1)^2\,(n+2)}{i(n-i+1)}$

We can minimize expression (19) to find the weighted least square estimators of the parameters α, λ, and θ.

$$D(X;\alpha,\lambda,\theta)=\sum_{i=1}^{n}\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}\left[\left[1-\left(1-e^{-\lambda/x_i}\right)^{\alpha\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right\}\right]-\frac{i}{n+1}\right]^2 \quad (19)$$

Cramer Von Mises Estimation (CVME): We can minimize the function (20) to obtain the Cramer-Von-Mises estimators of the parameters α, λ and θ

$$Z(X;\alpha,\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{in}\mid\alpha,\lambda,\theta\right)-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\left(1-e^{-\lambda/x_i}\right)^{\alpha\theta}\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right\}-\frac{2i-1}{2n}\right]^2 \quad (20)$$

Differentiating (20) with respect to α, λ , and θ as,

$$\frac{\partial Z}{\partial\alpha}=\frac{-2\theta^2}{\theta+1}\sum_{i=1}^{n}\left[F(X_i)-\frac{2i-1}{2n}\right]\left(1-e^{-\lambda/x_i}\right)^{\alpha\theta}\left(1-\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right)\ln\left(1-e^{-\lambda/x_i}\right)$$

$$\frac{\partial Z}{\partial\lambda}=\frac{-2\theta^2\alpha}{\theta+1}\sum_{i=1}^{n}\left[F(X_i)-\frac{2i-1}{2n}\right]\left(1-e^{-\lambda/x_i}\right)^{\alpha\theta-1}\left(1-\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}\right)$$

$$\frac{\partial Z}{\partial\theta}=-2\alpha\sum_{i=1}^{n}\left[F(X_i)-\frac{2i-1}{2n}\right]\left(1-e^{-\lambda/x_i}\right)^{\alpha\theta}\ln\left(1-e^{-\lambda/x_i}\right)\left\{1-\left(\frac{\theta}{\theta+1}\right)\ln\left(1-e^{-\lambda/x_i}\right)^{\alpha}-\frac{1}{(\theta+1)^2}\right\}$$

We can obtain CVM estimators by solving $\frac{\partial Z}{\partial \alpha} = 0, \frac{\partial Z}{\partial \lambda} = 0,$ and $\frac{\partial Z}{\partial \theta} = 0$ simultaneously.

Real data set application of the Model

Here we have analyzed the real data set to verify the proposed model. Due to the movement of the atoms of the conductors, failures in the conductors can occur in microcircuits .This phenomenon is also called the electro migration. We have taken the data from an accelerated life test of 59 conductors (Schaffi et al. 1987; Nelson and Doganaksoy 1995). Data represents the failure times measured in hours with no censored observations.

6.515, 6.476, 6.071, 7.543, 6.956, 5.807, 6.725, 7.974, 8.799, 6.033, 10.092, 6.538, 5.589, 6.545, 9.289, 2.997, 8.591, 11.038, 5.381, 4.137, 7.459, 7.495, 7.937, 10.491, 5.009, 7.489, 6.522, 6.573, 8.687, 6.129, 6.369, 7.024, 8.336, 7.224, 7.365, 4.706, 6.958, 6.492, 5.459, 6.869, 6.352, 6.087, 8.532, 8.120, 4.288, 9.663, 9.218, 7.496, 4.531, 7.945, 4.700, 7.398, 7.683, 6.948, 9.254, 6.923, 5.640, 5.434, 5.923.

Exploratory data analysis

Exploratory data analysis refers to the critical procedure of initial calculation of data to determine the pattern with help of summary statistics and graphical representation. Exploratory data analysis uncovers underlying structure and extracts important variables of the data. Figure 2 represent the box plot and the TTT plot of the given data.

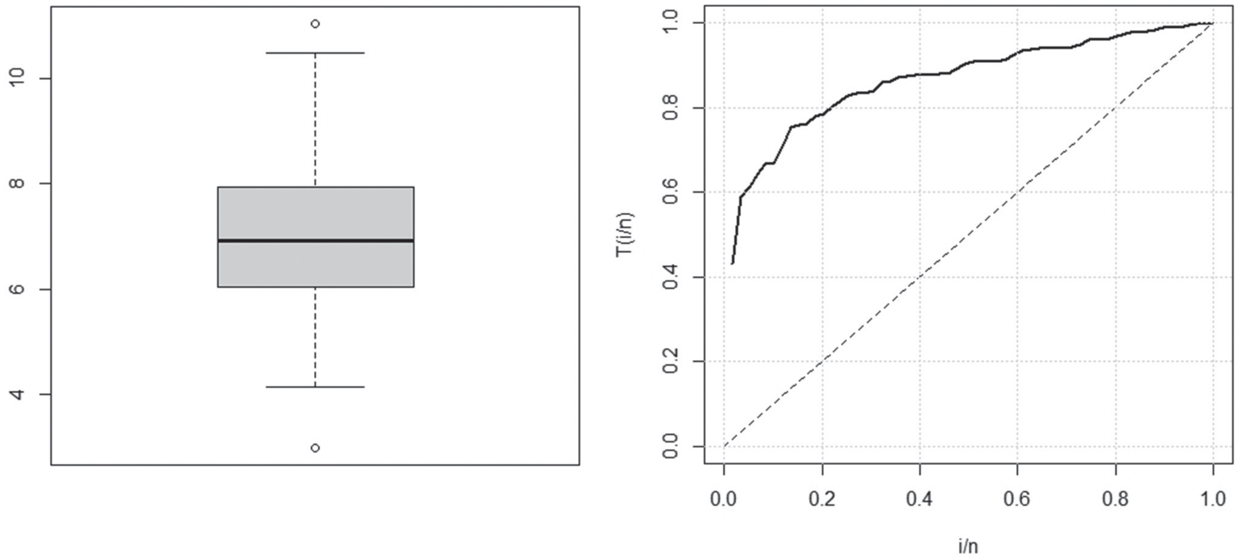

Figure 2. Boxplot (Left panel) and TTT plot (Right panel)

Table 1 represents the summary of the real life data set taken in consideration for the practical implication of the proposed distribution.

Table1

Summary Statistics

Min.	1st Qu	Median	Mean	3rd Qu	Max.	Skewness	Kurtosis
2.997	6.052	6.923	6.980	7.941	11.038	0.1931723	3.087389

Parameters: Here we have used optim () function of R language (R Core Team, 2020) and (Ming Hui, 2019), The MLEs of LGIE model are calculated by maximizing the likelihood function. We got the

Log-Likelihood value as l = -111.4192. Table 2 gives MLE's with their corresponding standard errors (SE) for α, λ, and θ.

Table 2

MLE and SE α, , and θ of LGIE

Parameter	MLE	SE
alpha	97.0105493	71.504975
lambda	29.9323509	4.8064174
theta	0.90282850	0.8502477

Figure 3 is the Q-Q plot and P-P plot .These plots show that the LGIE model fits the above real data set well.

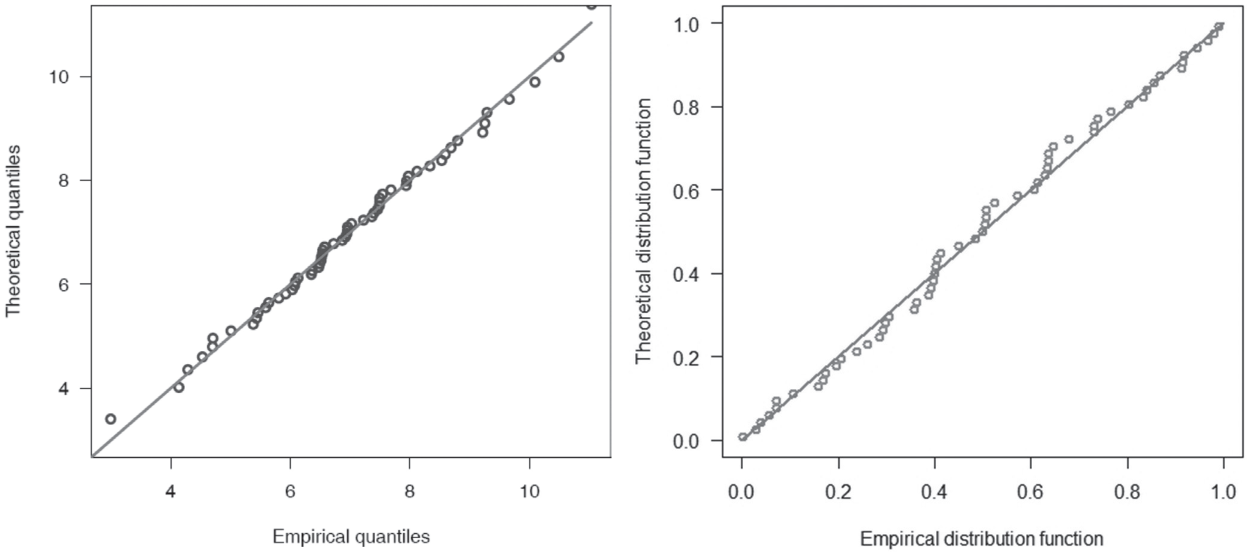

Figure 3. QQ plot (Left) and PP plot (Right) of LGIE

Table 3 gives the estimated value of the parameters of LGIE distribution using three methods of estimation with corresponding negative log-likelihood, AIC , BIC and KS statistics along with corresponding p-values.

Table 3

Estimated parameters values , log-likelihood values, AIC, BIC and KS statistic of the LGIE

Methods	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	-LL	AIC	BIC	CAIC	HQIC	KS (p-value)
MLE	97.0105	29.9324	0.9028	111.4192	228.8385	235.0711	229.2748	227.055	0.064869 (0.9513)
LSE	92.6680	31.2540	1.0940	111.4749	228.9498	235.1824	229.3861	227.1663	0.062458 (0.9644)
CVE	90.3909	33.3215	1.4362	111.7362	229.4724	235.7050	229.9088	227.6889	0.052551 (0.9940)

In figure 4: we have plotted the histogram of the data along with the fitted density function of the proposed model taking estimation using MLE, LSE and CVME.

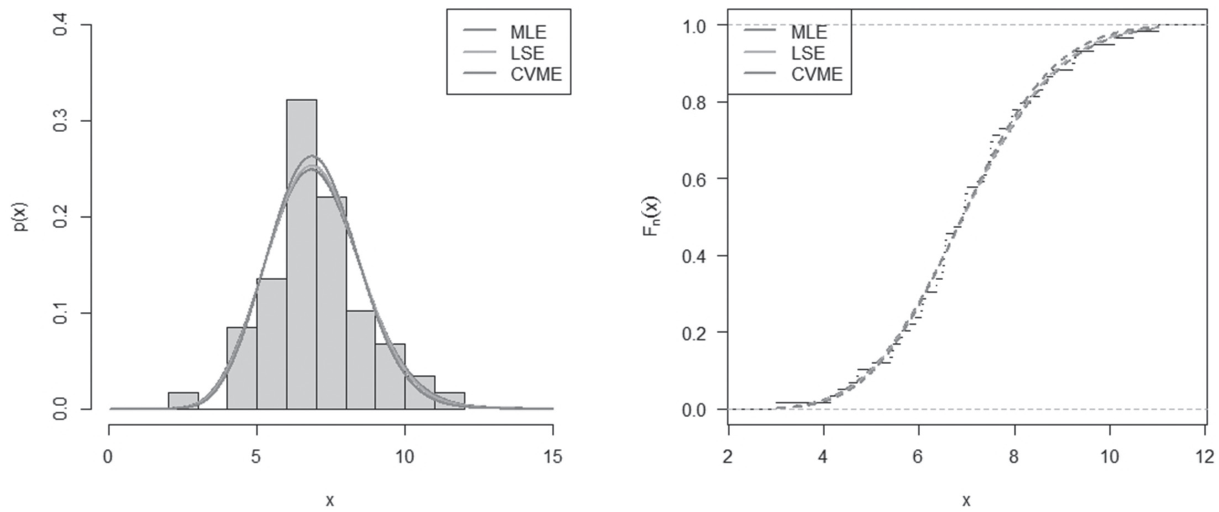

Figure 4Histogram vs. density fit (Left) and ecdf vs cdf (Right) of LGIE

To compare the potentiality of the proposed model, we have considered the following distributions.

i. Lomax Exponentiated Weibull (LEW):

This is a new extension of the exponentiated Weibull model. We used the simulation study to evaluate the performance of the maximum-likelihood method [Ansari and Nofal(2020)].

$$f_{LEW}(x) = \frac{\alpha\beta\theta x^{\beta-1}\left(1-e^{-x^{\beta}}\right)^{\alpha-1}}{\left[1-\left(1-e^{-x^{\beta}}\right)^{\alpha}\right]^{2}}\left[1+\frac{\left(1-e^{-x^{\beta}}\right)^{\alpha}}{1-\left(1-e^{-x^{\beta}}\right)^{\alpha}}\right]^{-(\theta+1)} \quad ;\alpha,\beta,\theta>0,\ x>0$$

ii. Generalized Weibull Extension (GWE):

The probability density functions of Generalized Weibull Extension (GWE). This model is very flexible in modeling various types of lifetime distribution. It will be denoted by $GWE(\alpha, \beta, \lambda)$, [Sarhan, A.M. and Apaloo, J.(2013)].

$$f_{GWE}(x) = \alpha\,\beta(\lambda x)^{\beta-1}\exp\left\{(\lambda x)^{\beta}+\frac{1}{\lambda}\left(1-\exp\left((\lambda x)^{\beta}\right)\right)\right\}$$
$$\left[1-\exp\left\{\frac{1}{\lambda}\left(1-\exp\left((\lambda x)^{\beta}\right)\right)\right\}\right]^{\alpha-1} \quad ;x>0$$

iii. Logestic Inverse Exponential Distribution (LIE):

Logistic Inverse Exponential Distribution is two parameters distribution given by Chaudhary, A. K., & Kumar, V. (2020

$$f(x;\alpha,\lambda) = \frac{\alpha\lambda}{x^{2}}\,\frac{exp\{\lambda/x\}\left[exp\{\lambda/x\}-1\right]^{\alpha-1}}{\left[1+\left[exp\{\lambda/x\}-1\right]^{\alpha}\right]^{2}}$$

iv. Generalized Exponential (GE) distribution:

The PDF of the generalized exponential distribution (Gupta & Kundu, 1999) is given by

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}\left\{1-e^{-\lambda x}\right\}^{\alpha-1} \quad \alpha>0, \lambda>0, x>0$$

Here we have calculated Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC) of the model LGIE. These values are also obtained for the models taken in consideration. Proposed model has lesser values of AIC, BIC, CAIC and HQIC indicating that the LGIE model fits the real data set more adequately than thw models considered.

Table 4

Log-likelihood values (LL), AIC, BIC, CAIC and HQIC of the LGIE

Model	-LL	AIC	BIC	CAIC	HQIC
LGIE	111.4192	228.8385	235.0711	229.2748	227.055
LEW	111.4528	228.9056	235.1382	229.342	227.1221
GWE	111.7103	229.4206	235.6532	229.857	227.6371
LIE	113.0267	230.0534	234.2085	230.2677	228.8644
GE	114.9471	233.8942	238.0493	234.1085	232.7052

Figure 5 represents the plots of histogram vs pdf of fitted distribution and the plots of Empirical distribution function vs estimated distribution function of LGIE distribution and the distributions taken in considerations.

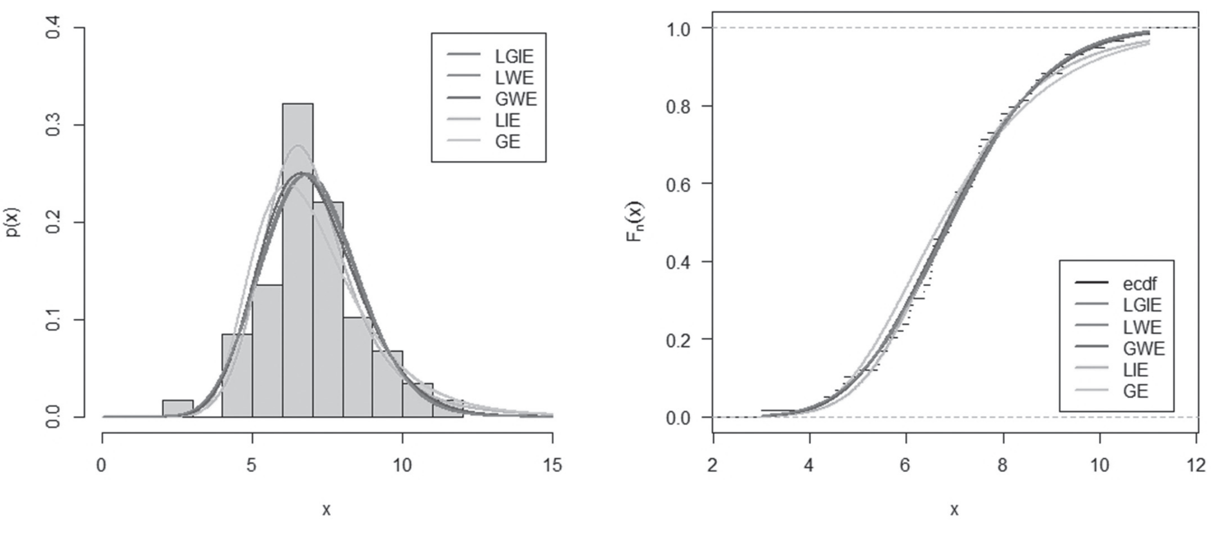

Figure 5: The Histogram pdf of fitted model (left) and Empirical distribution function with estimated distribution function (right).

We have also calculated the values of Kolmogrov-Smirnov(KS),Anderson-darling(AD),and Cramer-Von Mises(CVME) values and corresponding p values when parameters are estimated using MLE. It is noted that in most of the cases LGIE has minimum value and higher p-values than the competing distributions.

Table 5

KS, AD, CVME values along with corresponding p-values

Model	KS(p-value)	AD(p-value)	CVME(p-value)
LGIE	0.064869 (0.9513)	0.19732 (0.9912)	0.033958(0.9626)
LEW	0.062573(0.9638)	0.20219(0.9898)	0.034714(0.9595)
GWE	0.069986(0.9152)	0.22629(0.9815)	0.037877(0.9452)
LIE	0.062743(0.963)	0.29027(0.9451)	0.030439(0.9756)
GE	0.11128(0.4272)	0.79575(0.4835)	0.133630(0.4446)

Conclusions

Here, we have formulated a new model called lindley generalized inverted exponential distribution containing three parameters. Here we have presented some important statistical as well as mathematical properties of the model. We also derived the expression for hazard function, reliability function, skewness and kurtosis etc. Three important method of estimation are used for the estimation of the parameters of the model .The probability density curve of LGIE have shown that its shape is increasing-decreasing and right skewed. The curve is flexible for modeling for a real-life data also. Hazard function shows that it monotonically increasing or reverse j-shaped or constant depending on the values of parameters of the model.. real set data and different model validation criteria show that the proposed model fits data better than the other models taken in consideration.

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