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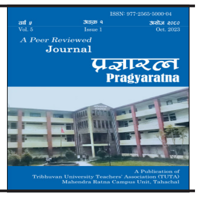

Analyzing the Connection of Linear Algebra: Enhancing Visual Reasoning through Vector Spaces

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Abstract

This paper investigates the significant relationship between geometry and linear algebra, emphasizing the ways in which vector spaces and linear transformations improve visual reasoning. I used post positivist paradigm and quantitative research design. For this research, I selected 90 students to represent a sample of math majors who have completed M.Ed. level coursework, obtained employment, and are enrolled in M.Ed. programs at all Tribhuvan University constituent campuses. It looked at how bases, eigenvectors, linear independence, and dimension are interpreted geometrically and emphasizes in what way important these concepts are for comprehending spaces, forms, and transformations. In order to bridge the gap between abstract algebraic theory and intuitive geometric reasoning, the article also described how these ideas make it easier to analyze linear algebra with geometrically.

Keywords: *Linear algebra, vector space, geometrical structure, cohens'd, one sample t- test*

Introduction

The history of linear algebra is deeply rooted in the broader development of mathematics, with its origins tracing back to ancient civilizations. The earliest forms of linear algebra can be seen in the work of ancient Egyptians and Babylonians, who developed methods to solve simple linear equations (Birkhoff, & Mac Lane, 1996). However, it was the ancient Greek mathematician Euclid who, in his "Elements,' laid some of the foundations by discussing geometric solutions to linear equations. The study of determinants began with the work of Japanese mathematician Seki Kōwa and, independently, with the work of the German mathematician Gottfried Wilhelm Leibniz in the 17th century. Their work focused on solving systems of linear equations using what would later be formalized as determinants, a crucial concept in linear algebra.

The formalization of linear algebra as a distinct branch of mathematics occurred in the 19th century, primarily through the contributions of mathematicians such as Carl Friedrich Gauss and Augustin-Louis Cauchy. Gauss developed methods for solving systems of linear equations, including Gaussian elimination, a fundamental algorithm in linear algebra today. The term "linear algebra' itself was coined in the mid-19th century, with the discipline evolving rapidly due to its applicability in various scientific fields (Berman, & Shvartsman, 2016). Later, the development of vector spaces by Giuseppe Peano and the formalization of matrix theory by Arthur Cayley solidified linear algebra as a key area of study within modern mathematics.

The branch of mathematics known as linear algebra studies vectors, vector spaces, commonly referred to as linear spaces, and linear mappings between these spaces (Trigueros, &Wawro, 2020). It includes a number of ideas that are used to solve linear equation systems, carry out transformations, and examine geometrical properties, including matrices, determinants, eigenvalues, and eigenvectors (Kleiner, 2007).

In linear algebra, a vector is defined as a component having both magnitude and direction. Additionally, vectors are typically represented as an array of numbers (Lay, 2003). An illustration of a vector in two dimensions would be □ = (2,3), V= (2,3), which can be seen as an arrow in the Cartesian plane that points from the origin (0,0) to the point (2,3) (Beezer, 2021). The set of all vectors in 2D space, R^2 is a vectors space. Any linear combination of two vectors in R^2 where V_1 = and V_2 = will also be a vector in R^2, as well as a 3D vector space called R^3 with basis vectors like $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$ holds the rule as per 2D (Bhattacharya, Jain, & Nagpaul, 1983). Again, it has shown the extent of these vector spaces, the vectors are depicted as arrows pointing from the origin so, vector spaces V and W over a field K (Mirsky, 2012). If addition and scalar multiplication are preserved, then a function α from V to W is a linear map; that is, if $\alpha(v_1+v_2) = \alpha(v_1) + \alpha(v_2)$ for all $v_1, v_2 \in V$; or if $\alpha(cv)=c\alpha(v)$ for all $v \in V$ and $c \in K$ (Broida, & Williamson, 1989).

Assume that α is a linear map on V. If v= 0 and α(v) = λv, then a vector v ∈ V is considered an eigenvector of α, with an eigenvalue λ ∈ K. The λ-eigenspace of α is the set {v: α(v) =λv} that includes the zero vector and the eigenvectors with eigenvalue λ (Cheney, & Kincaid, 2009). In linear algebra, eigenvalues and eigenvectors are essential ideas, especially when studying matrices and linear transformations (Ford, 2014). When an eigenvector in a linear transformation is stretched or compressed, the resulting scalar is called an eigenvalue. For a square matrix A, if v and λ are non-zero vectors such that if Av=λv, then λ represents the eigenvalue of the matrix A and v its associated eigenvector (Olver, Shakiban, & Shakiban, 2006). A non-zero vector is called an eigenvector if it retains its direction after undergoing a linear transformation. There is a possibility to scale the eigenvector using a factor, which is the corresponding eigenvalue. For example, we consider 2x2 matrix such as A=which scales vectors by stretching them in the x-direction by a factor of 2 and in the y-direction by a factor of 3 (De Micheaux, Drouilhet, & Liquet, 2013). The eigenvalues are $\lambda_1=2$ and $\lambda_2=3$ corresponding to the eigenvectors $V_1=$ and $V_2=$ respectively (Newman, & Odell, 1969).

This indicates that all vectors on the x- and y-axes are scaled by two (eigenvalue 2) and three (eigenvalue 3), respectively. These vectors are stretched or compressed by their corresponding eigenvalues but do not change direction. The degree of stretching or compression of the eigenvector is indicated by the eigenvalue. When $\lambda > 1$, the eigenvector is stretched; when $0 < \lambda < 1$, it is compressed; when $\lambda = 1$, the vector's amplitude stays constant; and when $\lambda < 0$ the vector's direction is reversed (Blyth, & Robertson, 2013).

Visuality of linear algebra over transformation

Visualizing linear algebra transformations helps students understand how linear operations affect geometric structures in vector spaces. This perspective allows for deeper comprehension of abstract concepts, making linear algebra more accessible and applicable to real-world problems requiring spatial intuition. Linear maps, often called linear transformations, are the fundamental concepts of linear algebra. They describe the relationship between two vector spaces. A linear map $T: V \to W$ preserves the vector addition and scalar multiplication operations between vector spaces V and W (Berberian, 2014). This means that for any vectors $u, v \in V$and any scalar c, the map satisfies the following equations: $T(u+v) = T(u) + T(v)$ and $T(cu) = c\,T(u)$.

Matrix representations of linear mappings are possible after the bases for the vector spaces are chosen, so the use of matrices in linear algebra computations is essential, as they provide a tangible depiction of linear maps (Berberian, 2014). Matrix multiplication is the process of applying a linear map on a vector when the linear map is written as a matrix. For example, given a matrix A that represents a linear map T and a vector x that is a coordinate vector in the domain of T, T(x) = $\mathrm{T}(Y) +$ Ax yields $\mathrm{T}(x)$ (Ben-Israel, 1980). By using matrix operations, including determining determinants, eigenvalues, and eigenvectors, to examine the characteristics of the original linear map, this matrix representation streamlines the study and application of linear transformations (Carrell, 2005).

Regardless of the abstract character of its principles, linear algebra has been identified as a challenging subject for student comprehension over the concepts and various pedagogic strategies to enhance its teaching and learning are the two primary study areas concerning the teaching and learning of linear algebra (Tucker, 1993). Moreover, university-level linear algebra courses are often perceived by students as challenging mathematics courses. In contrast to what students may be attracted to from earlier mathematics courses, the subject is frequently abstract and formal. This could cause a gap between linear algebra and previously taught mathematical concepts for the students. This is unfortunate because linear algebra can be applied outside of pure mathematics and has the ability to unify mathematics.

The fundamental ideas of linear algebra emerged and were applied in many branches of mathematics and its applications, making it an extremely valuable topic (Kleiner, 2007). In discussion of the history of linear algebra, Lang focuses on the use of systems of linear equations in ancient societies such as the Babylonians. The formalization of linear algebra as we know it, however, dates back to the 18th and 19th centuries. Carl

Friedrich Gauss (1777–1855) made important advances to the theory of linear equations during his lifetime. He also popularized the Gaussian elimination method, which is still a key tool in linear algebra today.

Once more, Augustin-Louis Cauchy (1789–1857) created the theory of determinants, which was essential to the advancement of linear algebra and matrix theory. In addition, the mid-19th century development of the theory of matrices is attributed to Arthur Cayley (1821–1895). He discovered that matrices could be operated upon algebraically, establishing the foundation for contemporary linear algebra. In addition, Cayley's contemporary James Joseph Sylvester (1814–1897) presented numerous ideas that are essential to linear algebra and made important contributions to matrix theory. The disputes peaked in the 1990s when the Linear Algebra Curriculum Study Group (LACSG) released a set of recommendations for the first linear algebra course. The guidelines were created taking into account the pedagogical and epistemological challenges that come up when teaching linear algebra, as well as research-based understanding of how mathematics is learned and should be taught. The recommendations were also influenced by the diverse client disciplines' participation and the unique teaching experiments of the LACSG members.

Additionally, Lang (2012) visualizes how developments in other branches of mathematics, including vector spaces, eigenvalues, and eigenvectors, which arose in the late 19th and early 20th century, were closely related to the development of linear algebra. The formalization of these ideas assisted in the development of linear algebra as a separate academic discipline. According to Lang (2012) linear algebra is not merely a collection of methods but rather a cohesive mathematical theory with close ties to functional analysis, abstract algebra, and geometry. Rather than concentrating solely on the computational side of the subject, he frequently highlights the abstract character of vector spaces and linear transformations.

Connection of linear algebra with other mathematics

Linear algebra's connectivity pertains to its profound links with many mathematical domains and its practical uses in a range of scientific domains. A fundamental tool that connects many fields, both inside and outside of mathematics, is linear algebra (Ames, 1970). This is the way that this interconnection appears in following ways. Linear algebra is frequently used to solve systems of linear differential equations. The eigenvalues and eigenvectors of the system-related matrices can be used to express the solutions. For instance, if A is a matrix, then solving the system $= A\,x$ can be achieved by determining the eigenvalues and eigenvectors of A (Axler, 2024). By reducing differential equations to more manageable independent equations, linear algebra offers techniques for diagonalizing matrices which makes solving differential equations easier.

Vector spaces are geometric objects that can be understood with the help of linear algebra. Geometric interpretations are available for concepts like as eigenvalues, eigenvectors, and linear transformations (Strang, 2022). In a geometric space, for example, linear transformations might be viewed as scaling, rotations, or reflections. In projective geometry, linear algebra plays a crucial role in helping to comprehend the characteristics of figures that remain unchanged after projection.

Principal Component Analysis (PCA), a key technique in statistics for reducing the dimensionality of data, relies on the eigenvectors and eigenvalues of the covariance matrix. The principal components are the eigenvectors that capture the most variance in the data. Linear algebra provides the tools to decompose data into principal components, enabling more effective data analysis and interpretation.

Linear algebra is the foundation for many numerical techniques, including detecting eigenvalues (e.g., power iteration) and solving systems of linear equations (e.g., Gauss-Seidel method). In computer mathematics, these techniques are essential for approximating solutions to issues that cannot be solved analytically. The foundation and mechanisms for creating effective computing strategies to tackle challenging mathematical problems numerically are provided by linear algebra. By using algebraic techniques to combine geometry and topology, linear algebra contributes in the comprehension of the algebraic structures that characterize the topological features of spaces. Linear algebra finds direct application in linear programming, where a linear objective function is optimized while taking linear constraints into account. Within the context of linear algebra, the Simplex algorithm is utilized to resolve linear programming issues. Optimization problems are essential in operations research, engineering, and economics, and linear algebra offers the mathematical building blocks for creating and resolving them.

Methods

This study employs a quantitative research design within a post-positivist paradigm to analyze the connection between linear algebra and the enhancement of visual reasoning through with vector spaces. A sample of 90 participants, post graduate students of mathematics selected through probability sampling, was surveyed using Likert-type questionnaires, administered both online and offline, to gather data on their understanding and application of vector spaces in linear algebra. The reliability of the questionnaire was confirmed with a Cronbach's alpha coefficient, ensuring internal consistency. Expert opinions were required to establish the content validity of the instrument. To analyze the data, a one-sample t-test was conducted to determine whether the sample mean differed significantly from a hypothesized population mean, with effect size measured using Cohen's d to quantify the magnitude of the observed effect. The findings provide insights into the effectiveness of visual reasoning in enhancing the comprehension of linear algebra concepts, particularly vector spaces.

Theoretical Framework

The Analyzing the Connection of Linear Algebra: Enhancing Visual Reasoning through Vector Spaces' highlights how students actively create their comprehension of difficult mathematical concepts by contextual and visual interaction with vector spaces, which is related to constructive learning theory. According to the principle of constructive learning, knowledge is not absorbed passively but is instead created via active investigation and interaction with the subject matter. By relating abstract ideas to geometric representations in this context, visual reasoning enables students to get a deeper grasp of vector spaces and motivates them to investigate, contemplate, and work together to make sense of the subject matter. This approach is in line with the fundamental ideas of constructivism, which holds that students construct knowledge through a process of discovery, visualization, and active participation. It does this by placing learning inside relevant contexts and encouraging student participation.

Conceptual Framework

This conceptual framework emphasizes how linear algebra is intertwined decisive geometric interpretations through comprehending. This dimension highlights the significance of geometric interpretations in understanding linear algebra. The link between abstract algebraic concepts and their real-world applications is created through visualization. It facilitates deeper comprehension and analytical reasoning by enabling students to mentally construct and manipulate vector spaces. Application establishes a link between theoretical understanding and real-world practice. Post graduate level students can recognize the value of linear algebra and cultivate problem-solving techniques based on workout examples by learning how it is applied in various circumstances.

The development of abstract reasoning abilities required to tackle challenging linear algebraic problems is the main focus of this context. It involves the capacity to think abstractly about linear transformations, vector spaces, and their properties with critical skill for solving difficult mathematical problems. Analytical and critical thinking abilities are improved through addressing abstract problems geometrically. Students who work with abstract concepts develop methodical problem-solving skills by recognizing underlying principles and using them to guide their approach. When I applied factor loading process then I got four factor such as *Geometric Interpretation, Visualization over the Vector Space, Abstract for Problem Solving and Applicability* which are given following figure (Figure 1).

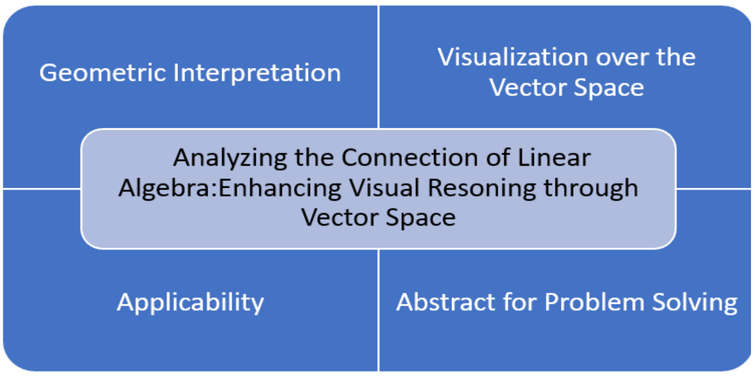

Figure 1. Conceptual Framework of the Analyzing the Connection of Linear Algebra: Enhancing Visual Reasoning through Vector Space

Results and Discussion

In the context of the study, *Analyzing the Connection of Linear Algebra: Enhancing Visual Reasoning through Vector Spaces*, a one-sample t-test was used to evaluate the mean value of students' visual reasoning scores against a neutral value (Test Value 3), which represents a baseline level of knowledge. The analysis produced a mean score that was notably higher than the neutral value ($p < 0.05$), suggesting that the educational intervention was successful in improving visual thinking. The effect size was evaluated using Cohen's d, and the result was 0.85, which denotes a significant effect. The increase in students' visual reasoning abilities in relation to linear algebra ideas was shown to be both statistically significant and educationally meaningful, as evidenced by the huge impact size. Here, I have total 32 items and two items were excluded in dimension reduction and only 30 items were loaded in factor analysis. The item wise value of reliability (Cronbach Alpha) and the detail structure of the four factors is given (Table 1).

Table 1. *Principal component analysis of analyzing the connection of linear algebra: Enhancing visual reasoning through vector spaces*

Factor Loading from Rotated Components Rotated Component Matrix Items	Factor Loading	Components
GI1. I clearly understand how vector addition can be interpreted geometrically as combining arrows in space.	0.694	
GI2. The concept of scalar multiplication as stretching or shrinking vectors is intuitive to me.	0.626	
GI3.I can easily picture how a matrix transformation changes the shape and orientation of vectors in space.	0.540	
GI4. Eigenvectors represent directions in space that remain unchanged by transformations.	0.536	Factor-1 *Geometrical Interpretation* (Cronbach's Alpha=0.719.)
GI5. I can diagonalize unitary operators and use their diagonal form to simplify problems.	0.503	
GI6. Envisioning linear algebra problems geometrically helps me solve them more effectively.	0.488	
GI7. I understand the null space of a matrix as the set of vectors that map to the origin under a linear transformation.	0.460	
GI8. It is easy to interpret solutions to systems of linear equations as intersections of planes or lines in space.	0.436	
GI9. The concept of vector spaces having different dimensions is clear when visualized geometrically.	0.423	
VVS1. I can easily visualize vector addition as the geometric combination of two vectors within a vector space.	0.668	
VVS2. I can determine the matrix for the inverse of a given linear map when it exists.	0.658	
VVS3. Even though it's abstract, I can conceptualize and visualize vector spaces in dimensions higher than three.	0.637	
VVS4. I am able to visualize how different types of transformation matrices (e.g., rotation, reflection) affect vectors within a vector space.	0.628	Factor 2: Visualization over the Vector Space Cronbach's Alpha = 0.741
VVS5. Linear independence as vectors is not lying on the same line or plane in a vector space.	0.556	
VVS6. Visualization of linear transformations affect the geometry of a vector space.	0.518	
VVS7. I understand how to use polynomials, such as minimal polynomials, to analyze and decompose vector spaces.	0.482	
VVS8. The symmetrical interpretation of linear transformations as mappings between spaces is clear to me	0.405	
APS1. It is easy to grasp the abstract concepts presented in linear algebra.	0.646	
APS2.The abstraction of vector spaces as sets of vectors that follow specific rules.	0.638	
APS3.I am comfortable with the idea of matrices as abstract representations of linear transformations.	0.632	
APS4.I understand in what way linear algebra generalizes systems of equations beyond simple numerical solutions.	0.630	*Factor-3 Abstract for Problem Solving (Cronbach's Alpha=0.70)*
APS5. Performing abstract operations on vectors is easy without relying on geometric interpretation.	0.624	
APS6. Abstract concept of basis and dimension as fundamental properties of vector spaces.	0.527	
APS7. I am confident in applying abstract concepts from linear algebra to solve complex problems in mathematics and related fields.	0.492	
A1. Vector spaces useful for solving real-world problems in fields like physics, engineering, and computer science.	0.656	
A2. Linear algebra is essential for performing data analysis and dimensionality reduction in data science.	0.573	
A3. Vector spaces and linear algebra are fundamental for creating and manipulating graphics and visualizations in computer graphics.	0.568	*Factor-4 Applicability* (Cronbach's Alpha=0.686)
A4. I do not use linear algebra to model and solve optimization problems in economics and resource allocation.	0.552	
A5. Linear algebra techniques are applicable in financial modeling for risk analysis and portfolio optimization.	0.507	
A6. Linear algebra is widely used in control theory for designing and analyzing polynomial equation systems.	0.489	

A test or questionnaire's internal consistency or reliability is assessed by a set of items or scales using Cronbach's alpha. It shows how well test questions measure the same underlying construct, in this case, the visual reasoning abilities associated with comprehending linear algebra's vector spaces.

Table 2: *Reliability value total items and factor wise items*

S.N.	Factors	Items	Reliability (Cronbach's Alpha)
1	Total (Items)	32	0.712
2	First Factor (Geometrical Interpretation)	9	0.719
3	Second Factor (Visualization over the Vector Space)	8	0.741
4	Third Factor (Abstract for Problem Solving)	7	0.70
5	Fourth Factor (Applicability)	6	0.686

Geometrical interpretation

The one-sample t-test results provide a detailed understanding of how respondents perceive various geometric concepts in linear algebra. The first statement, I clearly understand how vector addition can be interpreted geometrically as combining arrows in space,' shows a strong comprehension with a mean of 3.79, significantly higher than the test value of 3 (t = 6.265, p = .000). The effect size (Cohen's d = 0.66) is large, indicating a substantial deviation from the test value. The 95% Confidence Interval (CI) for the mean difference ranges from .54 to 1.04, reinforcing the significance of this result and suggesting that respondents have a clear understanding of vector addition. In contrast, the statement "The concept of scalar multiplication as stretching or shrinking vectors is intuitive to me' has a lower mean of 3.30, with a smaller t-value (t = 2.186, p = .031) and effect size (Cohen's d = 0.23). This indicates that while participants find scalar multiplication somewhat intuitive, their confidence is less robust compared to vector addition. The class interval for this statement is narrower, ranging from .03 to .57, suggesting less variability in the responses but also a smaller impact.

Again, for the statement I can easily picture how a matrix transformation changes the shape and orientation of vectors in space, the mean is 3.43 (t = 3.539, p = .001), with a small effect size (Cohen's d = 0.37). This reflects a solid understanding of matrix transformations, with a class interval ranging from .19 to .68, indicating that most respondents can visualize this concept effectively.

Participants also show confidence in their understanding of eigenvectors, diagonalizing unitary operators, and geometrically interpreting linear algebra issues. These statements have averages ranging from 3.36 to 3.41 with statistically significant p-values between.002 and.007. The effect sizes show moderate practical significance, ranging from 0.29 to 0.33. The relatively low confidence intervals (CIs) indicate that respondents' comprehension is consistent.

The concept of the null space and the visualization of vector spaces having different dimensions are slightly less understood, with means of 3.30 (t-values around 2.078 to 2.295, p-values .041 and .024, respectively) and smaller effect sizes (Cohen's d = 0.22 and 0.24). The class interval for these statements indicates a smaller but still statistically significant difference from the test value, suggesting that while these concepts are understood, they may be more challenging for respondents.

Moreover, on average, the overall mean of 3.3812 and the significant p-value of .000, coupled with a moderate average effect size (Cohen's d = 0.56), suggest that respondents generally have a good understanding of geometric concepts in linear algebra, though some areas, such as scalar multiplication and null space, may benefit from additional focus to strengthen comprehension. The class interval across all statements consistently suggests that the observed differences from the test value are not only statistically significant but also meaningful in practice. (Table 3).

Table 3. *Descriptive statistics and one sample t-test of the components in geometrical interpretation*

One Sample t- Test (Test- Value=3, N=90, df=89

	Mean	SD	t	Sig. (2-tailed)	MD	Cohens'd	95% Confidence Interval of the Difference	
							LCI	UCI
I clearly understand how vector addition can be interpreted geometrically as combining arrows in space.	3.79	1.195	6.265	.000	.789	0.66	.54	1.04
The concept of scalar multiplication as stretching or shrinking vectors is intuitive to me.	3.30	1.302	2.186	.031	.300	0.23	.03	.57
I can easily picture how a matrix transformation changes the shape and orientation of vectors in space.	3.43	1.162	3.539	.001	.433	0.37	.19	.68
Eigenvectors represent directions in space that remain unchanged by transformations.	3.36	1.211	2.784	.007	.356	0.29	.10	.61
I can diagonalize unitary operators and use their diagonal form to simplify problems.	3.41	1.235	3.157	.002	.411	0.33	.15	.67
Envisioning linear algebra problems geometrically helps me solve them more effectively.	3.41	1.381	2.824	.006	.411	0.29	.12	.70
I understand the null space of a matrix as the set of vectors that map to the origin under a linear transformation.	3.30	1.369	2.078	.041	.300	0.22	.01	.59
It is easy to interpret solutions to systems of linear equations as intersections of planes or lines in space.	3.36	1.292	2.610	.011	.356	0.27	.08	.63
The concept of vector spaces having different dimensions is clear when visualized geometrically.	3.30	1.240	2.295	.024	.300	0.24	.04	.56
Average	3.3812	.56184	6.437	.000	.38120	0.56	.2635	.4989

Visualization over the vector space

This one-sample t-test's significance values (p-values) show unequivocally that respondents' comprehension of different geometric ideas in linear algebra is statistically significant overall.

For example, the statement I can easily visualize vector addition as the geometric combination of two vectors within a vector space' has a mean of 3.95 with a t-value of 8.14 and a p-value of 0.000. This very low p-value, well below the common threshold of .05, indicates a highly significant difference from the test value of 3, suggesting that respondents have a strong ability to visualize vector addition in a geometric context. Similarly, other statements, such as I can determine the matrix for the inverse of a given linear map when it exists and I am able to visualize how different types of transformation matrices affect vectors within a vector space, also have p-values of .000, indicating strong statistical significance.

Even for concepts that might be more abstract, such as "Even though it's abstract, I can conceptualize and visualize vector spaces in dimensions higher than three and inear independence as vectors not lying on the same line or plane in a vector space, the p-values are .000 and .008, respectively. These results show that the respondents are statistically significantly confident in their understanding of these advanced topics. However, the statement I understand how to use polynomials, such as minimal polynomials, to analyze and decompose vector spaces has the highest p-value of .048, which, while still statistically significant, indicates a relatively weaker but still notable understanding compared to the other concepts. The overall average p-value of .000 across all statements suggests that, on the whole, the respondents' understanding of these geometric and abstract linear algebra concepts is both statistically significant (Table 4).

Table 4. *Descriptive statistics and one sample t-test of the components in visualization over the vector space*

One Sample t- Test (Test- Value=3, N=90, df=89								
				Sig. (2-tailed)			95% Confidence Interval of the Difference	
	Mean	SD	t		MD	Cohens'd	LCI	UCI
I can easily visualize vector addition as the geometric combination of two vectors within a vector space	3.95	1.230	8.14	.000	.955	0.661	.72	1.19
I can determine the matrix for the inverse of a given linear map when it exists	3.43	1.079	4.15	.000	.427	0.535	.22	.63
Even though it's abstract, I can conceptualize and visualize vector spaces in dimensions higher than three.	3.48	1.232	4.10	.000	.482	-0.369	.25	.71
I am able to visualize how different types of transformation matrices (e.g., rotation, reflection) affect vectors within a vector space	3.36	1.247	3.05	.003	.364	0.35	.13	.60
Linear independence as vectors is not lying on the same line or plane in a vector space.	3.32	1.241	2.68	.008	.318	0.40	.08	.55
Visualization of linear transformations affect the geometry of a vector space	3.45	1.170	3.99	.000	.445	0.425	.22	.67
I understand how to use polynomials, such as minimal polynomials, to analyze and decompose vector spaces.	3.23	1.194	1.99	.048	.227	0.231	.00	.45
The symmetrical interpretation of linear transformations as mappings between spaces is clear to me	3.65	1.162	5.82	.000	.645	0.431	.43	.87
Average	3.49	.4419	11.6	.000	.49091	0.809	.40	.574

Abstract for problem solving

The significance values in this one-sample t-test indicate that respondents generally find it significantly easier to grasp and apply abstract concepts in linear algebra compared to the baseline test value of 3. For the statement it is easy to grasp the abstract concepts presented in linear algebra, the mean is 3.95, with a very high t-value of 8.143 and a p-value of .000. This suggests a highly significant difference from the test value, indicating that respondents strongly agree with this statement. The mean difference (MD) of .955, coupled with a confidence interval ranging from .72 to 1.19, further supports this high level of agreement, showing a strong and statistically significant understanding of abstract concepts.

The abstraction of vector spaces and the comfort with matrices as abstract representations also show strong significance, with p-values of .000 for both statements. The t-values of 4.153 and 4.101, respectively, and the mean differences of .427 and .482, reflect a solid understanding. The confidence intervals (.22 to .63 for vector spaces, and .25 to .71 for matrices) confirm that these concepts are well-understood by the respondents. For more complex or less intuitive concepts, such as performing abstract operations on vectors is easy without relying on geometric interpretation and I am confident in applying abstract concepts from linear algebra to solve complex problems in mathematics and related fields,' the p-values are .008 and .048, respectively. Although these p-values are higher than the others, they are still below the .05 threshold, indicating statistically significant but somewhat weaker confidence among respondents in these areas. The lower t-values (2.689 and 1.997) and narrower confidence intervals for these statements suggest that while these concepts are understood, they are less intuitive or more challenging.

Overall, the average mean is 3.4909, with a p-value is 0.000 and a moderate effect size, indicates a generally strong and statistically significant understanding of abstract linear algebra concepts among respondents. The consistently low p-values across the board confirm that the respondents' confidence in these concepts is not due to chance but reflects a real and meaningful comprehension (Table 5).

Table 5. *Descriptive statistics and one sample t-test of the components in abstract for problem solving*

One Sample t- Test (Test- Value=3, N=90, df=89)								
							95% Confidence Interval of the Difference	
	Mean	SD	t	Sig. (2-tailed)	MD	Cohens'd	LCI	UCI
It is easy to grasp the abstract concepts presented in linear algebra	3.95	1.230	8.143	.000	.955	0.628	.72	1.19
The abstraction of vector spaces as sets of vectors that follow specific rules	3.43	1.079	4.153	.000	.427	0.124	.22	.63
I am comfortable with the idea of matrices as abstract representations of linear transformations	3.48	1.232	4.101	.000	.482	0.672	.25	.71
I understand in what way linear algebra generalizes systems of equations beyond simple numerical solutions	3.36	1.247	3.059	.003	.364	0.578	.13	.60
Performing abstract operations on vectors is easy without relying on geometric interpretation	3.32	1.241	2.689	.008	.318	0.077	.08	.55
Abstract concept of basis and dimension as fundamental properties of vector spaces.	3.45	1.170	3.994	.000	.445	0.325	.22	.67
I am confident in applying abstract concepts from linear algebra to solve complex problems in mathematics and related fields.	3.23	1.194	1.997	.048	.227	0.167	.00	.45
Average	3.4909	.44196	11.650	.000	.49091	0.592	.4074	.5744

Applicability

The one-sample t-test results for the various applications of vector spaces in fields like physics, engineering, and computer science reveal several significant insights. All the significance values (Sig. 2-tailed) are well below the conventional threshold of 0.05, indicating that the mean responses differ significantly from the test value is 3. For instance, the statement vector spaces are useful for solving real-world problems in fields like physics, engineering, and computer science has a very high t-value of 8.14 and a significance value of .000, with a mean difference (MD) is .955, showing a strong positive deviation from the test value. This suggests a strong consensus among respondents about the importance of vector spaces in these fields.

Regarding Cohen's d, which measures the effect size or the magnitude of the difference, the values range from small to large effects. The highest Cohen's d is associated with the first statement (.955), indicating a large effect size, meaning that the respondents strongly agree with the usefulness of vector spaces in solving real-world problems. The other statements also show positive effect sizes, though smaller, indicating that respondents generally recognize the importance of linear algebra in data science, computer graphics, and other applications, even though the intensity of agreement varies across different contexts. These results collectively emphasize the significant and varying impact of linear algebra across multiple disciplines (Table 6).

Table 6. *Descriptive statistics and one sample t-test of the components in applicability*

One Sample t- Test (Test- Value=3, N=90, df=89								
							95% Confidence Interval of the Difference	
	Mean	SD	t	Sig. (2-tailed)	MD	Cohens'd	LCI	UCI
Vector spaces useful for solving real-world problems in fields like physics, engineering, and computer science.	3.95	1.230	8.14	.000	.955	-0.55	.72	1.19
Linear algebra is essential for performing data analysis and dimensionality reduction in data science	3.43	1.079	4.15	.000	.427	0.093	.22	.63
Vector spaces and linear algebra are fundamental for creating and manipulating graphics and visualizations in computer graphics	3.48	1.232	4.10	.000	.482	-0.040	.25	.71
I do not use linear algebra to model and solve optimization problems in economics and resource allocation.	3.36	1.247	3.05	.003	.364	-0.102	.13	.60
Linear algebra techniques are applicable in financial modeling for risk analysis and portfolio optimization	3.32	1.241	2.68	.008	.318	0.212	.08	.55
Linear algebra is widely used in control theory for designing and analyzing polynomial equation systems.	3.45	1.170	3.99	.000	.445	0.235	.22	.67
Average	3.490	.4419	11.6	.000	.49091	0.186	.4074	.57

Conclusion

In conclusion, the research on "Analyzing the Connection of Linear Algebra: Enhancing Visual Reasoning through Vector Spaces' demonstrates All items have mean scores significantly higher than the test value of 3, according to the one-sample t-test results for the components of geometrical interpretation in linear algebra. This suggests that participants generally agree with the statements regarding their comprehension of geometrical concepts in vector spaces. All components had significant t-values ($p <.05$), with the comprehension of vector addition as combining arrows in space exhibiting the largest effect size (Cohen's d = 0.66), indicating a good understanding in this domain. The average impact size of 0.56 indicates that participants' understanding of geometric ideas in linear algebra is significantly improved by visual reasoning. This suggests that a key element in the efficient comprehension and implementation of these mathematical ideas is the visual representation of linear algebraic problems.

The study reveals that visualization skills significantly improve participants' understanding of vector spaces and related concepts. Vector addition, transformations, and higher-dimensional vector spaces were the most well-understood concepts. However, understanding polynomials in vector space analysis was a challenge. The results suggest that visual reasoning is crucial for linear algebra, and enhancing visualization skills can enhance comprehension of abstract mathematical concepts. The understanding of matrices as abstract representations and the generalization of systems of equations also showed significant agreement with moderate effect sizes. However, the lowest mean (3.23) and effect size were noted in applying abstract concepts to solve complex problems, suggesting that while participants understand the abstractions, there is some difficulty in applying them to more advanced problem-solving scenarios. The findings suggest that participants are comfortable with abstract reasoning in linear algebra, but there is room for improvement in applying these concepts to complex mathematical problems.

Again, the components related to the applicability of linear algebra indicate that all mean scores are significantly above the test value of 3, suggesting that participants recognize the importance of linear algebra in various real-world applications. The highest mean (3.95) and a strong effect size (Cohen's d = 0.72) were observed for the usefulness of vector spaces in solving problems in fields such as physics, engineering, and computer science, indicating a strong acknowledgment of its relevance in these areas. Participants also agreed that linear algebra is essential for data analysis, dimensionality reduction, and computer graphics, with moderate effect sizes supporting these views. However, there was slightly less agreement on the application of linear algebra in financial modeling and optimization problems, though the means were still significantly above the test value. Finally, the findings suggest that participants highly value the applicability of linear algebra across multiple disciplines, though they might be less familiar with its use in specific areas like financial modeling and optimization.

References

Ames, D. B. (1970). *Fundamentals of linear algebra*. International Textbook Company.

Axler, S. (2024). *Linear algebra done right*. Springer Nature. https://library.oapen.org/handle/20.500.12657/85067

Beezer, R. A. (2015). *A first course in linear algebra*. Independent.

Beezer, R. A. (2021). A first course in linear algebra.

Ben-Israel, A. (1980). Generalized inverses of matrices and their applications. In *Extremal Methods and Systems Analysis: An International Symposium on the Occasion of Professor Abraham Charnes' Sixtieth Birthday Austin, Texas, September 13–15, 1977* (pp. 154-186). Berlin, Heidelberg: Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-46414-0_8

Berberian, S. K. (2014). *Linear algebra*. Courier Corporation.

Berman, A., & Shvartsman, L. (2016). Definitions are important: The Case of Linear Algebra. *European Journal of Science and Mathematics Education*, *4*(1), 26-32. https://files.eric.ed.gov/fulltext/EJ1107744.pdf

Bhattacharya, P. B., Jain, S. K., & Nagpaul, S. R. (1983). *First course in linear algebra*. New Age International.

Birkhoff, G., & Mac Lane, S. (1996). *A Survey of Modern Algebra*. A K Peters/CRC Press.

Blyth, T. S., & Robertson, E. F. (2013). *Further linear algebra*. Springer Science & Business Media.

Broida, J. G., & Williamson, S. G. (1989). *A comprehensive introduction to linear algebra* (Vol. 4). Boston, MA, USA: Addison-Wesley.

Broida, J. G., & Williamson, S. G. (1989). *A comprehensive introduction to linear algebra*. Boston, MA, USA: Addison-Wesley.

Carrell, J. B. (2005). Fundamentals of linear algebra. *The University of British Columbia*.

Cheney, W., & Kincaid, D. (2009). Linear algebra: Theory and applications. *The Australian Mathematical Society*, *110*, 544-550.

Day, J. M., & Kalman, D. (1999). Teaching linear algebra: What are the questions. *Department of Mathematics at American University in Washington DC*, 1-16. https://static.ias.edu/pcmi/1998/1998-questions2.pdf

De Micheaux, P.L., Drouilhet, R., & Liquet, B. (2013). Basic Mathematics: Matrix operations, integration and optimization. In: The R Software. Statistics and Computing, vol 40. Springer, New York, NY. https://doi.org/10.1007/978-1-4614-9020-3_10

Ford, W. (2014). *Numerical linear algebra with applications: Using MATLAB*. Academic Press.

Hsiung, C. Y. (1998). *Linear algebra*. Allied Publishers.

Katz, V. J. (1998). *A History of Mathematics: An Introduction*. Addison-Wesley.

Kleiner, I. (2007). History of Linear Algebra. In *A History of Abstract Algebra* (pp. 79-89). Boston, MA: Birkhäuser Boston. https://doi.org/10.1007/978-0-8176-4685-1_5

Lay, D. C. (2003). *Linear algebra and its applications*. Pearson Education India.

Lang, S. (2012). *Introduction to linear algebra*. Springer Science & Business Media.

Leon, S. J. (2014). *Linear algebra with applications* (Vol. 9). Upper Saddle River, NJ: Pearson.

Mirsky, L. (2012). *An introduction to linear algebra*. Courier Corporation.

Newman, T. G., & Odell, P. L. (1969). On the concept of a pq generalized inverse of a matrix. *SIAM Journal on Applied Mathematics*, *17*(3), 520-525. https://doi.org/10.1137/0117050

Olver, P. J., Shakiban, C., & Shakiban, C. (2006). *Applied linear algebra* (Vol. 1). Upper Saddle River, NJ: Prentice Hall.

Rensaa, R. J., Hogstad, N. M., & Monaghan, J. (2020). Perspectives and reflections on teaching linear algebra. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *39*(4), 296-309. https://doi.org/10.1093/teamat/hraa002

Ricardo, H. (2009). *A modern introduction to linear algebra*. CRC Press.

Singh, S. (2009). *Linear algebra*. Vikas Publishing House.

Stecher, M. (1988). *Linear Algebra*. HarperCollins Publishers.

Stillwell, J. (2002). *Mathematics and Its History*. Springer Science & Business Media.

Strang, G. (2022). *Introduction to linear algebra*. Wellesley-Cambridge Press.

Trigueros, M., Wawro, M. (2020). Linear Algebra Teaching and Learning. In: Lerman, S. (eds) *Encyclopedia of Mathematics Education*. Springer, Cham. https://doi.org/10.1007/978-3-030-15789-0_100021

Tucker, A. (1993). The growing importance of linear algebra in undergraduate mathematics. *The College Mathematics Journal*, *24*(1), 3-9. https://doi.org/10.1080/07468342.1993.11973500