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Analysis of Oscillation in Neutral Differential Equations of Even Order with Specific Delay Properties

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Abstract

This research addresses a limitation in the current "Kamenev-Type" criteria used by mathematicians to study the behavior (oscillation) of solutions to a specific kind of equation (even-order neutral differential equations). These equations describe scenarios where the rate of change depends on both the current state and a delayed version of it. By tackling this shortcoming, the paper introduces new and enhanced results for understanding oscillation in these equations. This advancement not only refines the "Kamenev-Type" criteria but also surpasses many other established methods for analyzing oscillation in this area of mathematics.

Key Words: Decent arguments, oscillation, even order & neutral differential.

Introduction

Even-order neutral differential equations with deviating arguments are prevalent in various scientific and engineering disciplines. These equations model phenomena where the rate of change depends not only on the current state but also on a delayed version of it, making them critical for accurately representing systems with inherent delays or memory effects. For instance, they are used in control systems, population dynamics, and certain economic models, where past states influence future behavior.

Analyzing the oscillatory behavior of solutions to these equations is crucial for understanding the long-term dynamics of the systems they represent. Oscillations can indicate stability or instability, periodic solutions, or other significant dynamic behaviors that are essential for predicting and controlling the systems' responses. Therefore, robust criteria for determining oscillations are of paramount importance.

Existing methods, particularly the "Kamenev-Type" criteria, provide a foundation for this analysis but suffer from limitations that restrict their applicability to specific functions or forms of equations. These limitations can impede the analysis of more complex or varied equations encountered in practical applications, reducing the effectiveness of the criteria in providing comprehensive insights.

This paper aims to address these shortcomings by presenting novel oscillation criteria that extend and enhance the current methodologies. We introduce new theorems that refine and strengthen existing results, thereby broadening the scope of equations that can be analyzed.

Our approach not only makes the criteria more versatile but also more robust, enabling researchers to tackle a wider range of even-order neutral differential equations with deviating arguments.

By expanding the applicability of these criteria, we provide tools that can be utilized in more diverse and complex scenarios. This advancement has the potential to improve the modeling and analysis of real-world systems, offering deeper insights and more accurate predictions. Consequently, this work not only refines the "Kamenev-Type" criteria but also surpasses many established methods, contributing significantly to the field of differential equations and their applications in science and engineering. Through these improvements, researchers can achieve a better understanding of oscillatory behaviors, ultimately leading to more effective and reliable solutions in their respective domains.

The oscillation of some even orders differential equations have been study in the references 1-5. We deal with the oscillatory behavior of the even order heartily differential equations with deviation arguments of the form.

$$\left[x(t)+\sum_{i=1}^{m} p_i(t)x(\tau_i(t))\right]^{(n)}+\sum_{j=1}^{1} q_j(t)f_j(x\big((\sigma_j(t)\big)=0$$

For $t \geq t_o$............... (i)

where $n \geq 2$ is even, throughout this paper, it is assumed that,

A. P_i, q_j E ($[t_o, \infty]$, R^+), $f_j \in_{C(R, R), ufj}$ (u)>0 for u # 0 and f_j (u) is non decreasing on R,i=1,2....m, j=1,2......i....

B. $\tau_i \in C$ ([to,∞], R^+), $t_1(t) \leq t$, $\lim_{t\to\infty} \tau_i$ (t) =∞, i =1,2....n

C. $\sigma_i \in C^1$ ([t, ∞], R^+), σ_1 (t)≤t, $\lim_{t\to\infty} \sigma_j$ (t) $\leq$ t =∞ and σ_j (t) $\geq$ 0, j =1,2....1

D. ∃ constantM> 0 sum thatf_j(x) sgn(x)≥M|x|for x ≠ 0, j=1, 2....l

E. $\sum_{i=1}^{m} p_{i(t)} \leq p_1 p \in (0,1)$ and $\exists q\,(t) \in C([t_0,\infty],R^+)$ such that,

q (t)≤ min {q,(t); j = 1,2.....1}

By a solution of Equation (i) we mean a function x (t) which has

x (t) + $\sum_{i=1}^{m} p_i$ (t)x(τ_i(t)x) ∈ C^n ([t_x, ∞], R))

For somet_x>, to and satisfies equation (1) on [t_x, ∞]. Suppose the solution x (t) of (1) which exist on some half line [t_x, ∞] with sup {|x (t)|: t ≥ T} # 0 for and t ≥ t_x. A non trival solution of (1) is called oscillatory if it has arbitrary large zeros, otherwise it is said to be non-oscillatory. Equation (1) is said to be oscillatory if all of its non-trival solutions are oscillatory. Meng and Xu studied [1] the equation

1. and obtained some sufficient condition for oscillation of due equation (1) we write the main result of [1] as follows in [2] we say that a function

H=H (t, s) belongs to a function class W denoted by H∈W, if H∈C (D, Rt) where D={(t, s): $t_o \geq s \geq t$}, Which satisfies (H1) H (t, t) =0 and H (t,s)> 0 for $t_o<s<t<\infty$;(H_2) H has a continuous non-positive partial derivative ∂∂satisfying the condition.

*∂∂ = h (*t,s) - *H* (t, s)

for some $h\in L_{loc}$ (D,R) $K\in C^1$ [(To∞), (0, ∞)] is a non living decreasing function,

Theorem 1

Assume that (A) and (E) hold let the function H, h, k satiety (H1) and (H2), suppose

$$\limsup_{t=\infty} \left[\lambda C_\lambda F(t,r) - \frac{1}{4_\lambda C_2} G[t,r]\right] = \infty$$

holds $r\geq c_1>0$, $c_2>0$ where (2)

$$F(t,r) = \frac{1}{H(t,r)}\int_r^t H(t,s)k(s)\sum_{j=1}^{1} qj(c)ds$$

$$G(t,r) = \frac{1}{H(t,r)}\int_r^t \frac{K(S)h^2(t,s)}{H(t,s)\sigma_j^{n-2}(s)\sigma_j^1(s)}ds$$

and λ=1-P, then each solution of Eqn. (1) is oscillatory.

Theorem 2

Assume that (A) to (E) hold and H, h, k are the same as the theorem A, suppose that

$$\inf\left\{\liminf_{t\to\infty} \frac{H(t,s)}{H[t,t_0]}\right\} > 0 \text{ .... } (3)$$

and $\limsup_{t\to\infty} G(t, t_0) < \infty$ .... (4)

If 3 m∈ C ($[t_o, \infty]$, *R*) s.t. $\forall\ t \geq T \geq to$

$$\liminf_{t\to\infty} \left[\lambda c_1 F(t,T) - \frac{1}{4\lambda c_2} G(t,T)\right] \geq m(T) \text{ .... } (5)$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{\sigma_j^{n-2}(s)\sigma_j^1(s)m^2 + (s)}{k(s)} ds = \infty,\ j == 1,2....j$$

Where, M,(t)= Max (m(t),0}then each solution of Equation(1)is oscillatory. In theorem (1) and (2) function G(t,r), so each of the condition (2)(4)(5) and (6) has many as 1 conditions. The Riccati function w(t) is not well defined and some errors in the proof of theorem. The main purpose of this paper is to strengthen oscillation results obtained for equation (1) by Meng and XU. In this paper we redefine the functions F(t,r), G(t,r), w(t) and provide some new oscillation criteria for oscillation of equation(1)

Main results

Lemma 1

- Let x (t) be an times differentiable function on $[t_o, \infty]$ of one sign, $x^n(t) \neq 0$ on $[to, \infty]$ which satisfies $x^{(n)}(t)x(t) \leq 0$ then

  i. $\exists$ $t_1 \geq 0$ s. t. $x^{(i)}$ are of one sign on $[t,\infty]$: where i=1,2.....n−1

  ii. Also $\exists$ a number ht[1,3,5 … … (n−1)]when n is even or h∈ {2,4,6.....n-1}, when n is odd, ... s% x (t) x (i)> 0 fori=o, 1.......h-1^{n+i+1} x(t) $x^{(i)}$ (t) > 0 for i = h+1, h+2....n $t \geq t_1$.

Lemma 2

If x(t) is as in Lemma 2.1 and $x^{(n-1)}(t)x(t)^{(n)}(t) \leq 0$ for $t \geq$ to then for every λ ($0<\lambda< 1$).There exists a constant N>0 St.

$\|x(\lambda(t))\| \geq N^{t-1}(t) \mid x^{(n+1)}(t) \mid \forall$ large t.

Theorem 3

Assume that (A)-(E) hold, Let the function M,h,k satisfy (H_1) and (H_2), suppose

$$\limsup_{t\to\infty}\left[\lambda MF(t,r) - \frac{\beta}{4\lambda N}G(t,r)\right] = \infty \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

6. Holds for $\forall$ $r \geq t_o$ are for some B≥1 where Theorem F(t,r) & G(t,r) as defined in (1)

Proof

Suppose to the contrary that x(t) is a no- oscillatory solution of equation(1) and that x(t) is eventually positive where x(t) is eventually negative,

Let z(t) be the function defined as follows

$$z(t) = x(k) + \sum_{i=}^{m} 1p\lambda(t)x(\tau i(\tau i(t)), \text{then} \exists\, H \geq t_0 \text{s.t.} z(t) > 0 \text{ and } z^n(t) \leq 0,\ t \geq ti$$

Assume that $t_1 \geq t_o$ s.t.

$x(t)>0$, $z(t)>0$, $z^1(t)>0$

$z^{n-1}(t)>0$, $z(\sigma_j(t))>z(\lambda\sigma_j(t))>0$

$z^1(\lambda\,\sigma_j(t)) \geq N\,\sigma_j^{n-2}(t)z^{n-1}(t)$ hyLemma2.2and

$$z^n(t) \leq -\lambda m\sum_{j=1}^{1} q_j(t)z(\sigma_j(t))\forall t \geq t_1$$

Let,

$$w(t) = K(t)\frac{z^{(n-1)}(t)}{\sum_{j=1}^{1} z(\lambda\sigma_1(t))}$$

Then we have,

$$w^1(t) \leq -\lambda MK(t)q(t) + w(t) - \frac{\lambda N\sum_{j=1}^{1}\sigma_1^{n-2}(t)\sigma_j^1(t)}{k(t)w^2(t)}, t \geq t_1$$

Multiplying the above equation width t replaced be s, by H (t,s) and integrating it from T to V t ∀ t ≥T ≥ t for some B≥1

We obtain

$$\lambda M\int_T^t H(t,s)k\,(s)q\,(s)ds \le H(t,T)\,W\,(T)+\int_T^t h(t,s)w\,(s)ds -$$

$$\int_T^t AN\frac{\sum_{j=1}^{1}\sigma_j^{n-2}(s)\sigma_j^i(s)}{k(s)}H(t,s)w^2(s)ds$$

$$= H\,(t,T)\,W\,(T) + \int_T^t \frac{k(s)h^2(t,s)}{\lambda NH(t,s)\sum_{i=1}^{1}\sigma_j^{n-2}(s)\sigma_j^1(s)}ds - \int_T^t \frac{\lambda NH(t,s)\sum_{i=1}^{1}\sigma_j^{n-2}(s)}{k(s)}$$

$$w2(s)ds\int_T^t\left\{-\frac{\sqrt{\lambda NH(t,s)}\sum_{i=1}^{1}\sigma_j^{n-2}(s)\sigma_j^1(s)}{\sqrt{\beta K(s)}}W(s)-\frac{\sqrt{\beta K(s)}}{\sqrt{4\lambda NH(k,s)\sum_{i=1}^{1}\sigma_j^{n-2}(s)\sigma_j^1(s)}}h(k,s) \le H(t,T)W(T)++\frac{F}{4}\int_T^t\frac{k(s)h^2(t,s)}{\lambda NH(t,s)\sum_{i=1}^{1}\sigma_j^{n-2}(s)\sigma_j^1(s)}\right\}$$

Hence we have

λ MF (t, T)− G(t, T) ≤ W(T)w(T) ∀ t ≥ t_1

This gives

$\lim_{t\to\infty}$ Sup [λMF (t, r) −G(t, T) G(t,r) = ∞

This completes the proof,

The new oscillation criteria for equation......................... (i)

Statement : Assume that (A)...........(E) hold, que functions H, h, K,F and G be the same as in theorem (3) suppose that

$$\lim_{s\ge t_0}\left\{\lim_{t\to\infty}\inf\frac{H(t,s)}{H(t_1,t_0)}\right\} > 0..........(8)$$

If ∃ m∈C ([to,∞], R)B+ ∀ tT≥to and for someB> 1.

$$\lim_{t\to\infty}\infty\sup\left\{\lambda MF(t,T)\frac{\beta}{4\lambda NG(t,T)}\right\} \ge m(T)..........(9)$$

$$\lim_{t\to\infty}\infty\sup\int_{to}^t\frac{\sum_{j=1}^{1}\sigma_1^{n-2}(s)m^1+(s)m^2+(s)}{k(s)}ds=\infty \quad ....(10)$$

Where m+(t)=max(m(t),0). Then every solution of equation (1) is oscillatory. Proof: Suppose to the contrary that (1) is non oscillatory. Following the proof of theorem (3) without loss of generality

Suppose that t ≥ T ≥ to and for *some* β > 1 we *obtain*

$$\lambda M - \int_{to}^t H\,(t,s)\,k\,(s)\,q\,(s)\,ds \le H\,(t,T)\,w\,(T)$$

$$+\frac{\beta}{4}\int_T^t \frac{k(s)h^2(t,s)}{\lambda NH(t,s)\sum_{j=1}^{1}\sigma_j^{n-2}(s)\,\sigma_j^1\,(s)} - \frac{\beta}{\beta-1}\int_T^t \frac{\lambda NH(t,s)\sum_{j=1}^{1}\sigma_j^{n-2}(s)\,\sigma_j^1\,(s)}{k(s)} w^2(s)ds$$

We get

λMF (t,T)-G (t, T) ≤ w (T) λNβ (t,T)

where

$$\beta\,(t,\,r) = \frac{1}{H(t,r)}\int_r^t \frac{H(t,s)\sum_{j=1}^{1}\sigma_j^{n-2}(s)\,\sigma_j^1\,(s)}{k(s)} w^2(s)r \geq \text{to}$$

Then

$$\lim_{x\to\infty}\sup\left[\lambda MF(t,T) - \frac{\beta}{4\lambda N}G(t,T)\right] \leq W(T) - \frac{\beta-1}{\beta}\lambda N$$

lim infβ(t,T) ∀ t ≥ to & any β > 1by.....................(9)

We have

$$W(T) \geq M(T_0 + \lambda N \lim_{x\to\infty} \inf\beta\,(t,T)$$

So, W(T) ≥ m (T)........................(11)

and specially,

$$\lim\inf\beta\left(t, To \leq \frac{\beta}{(\beta-1)\lambda N}[W(t_o)m(t_o)]\right) < \infty \quad ....(12)$$

We claim that

$$\lim_{t\to\infty}\infty\sup\int_{t_0}^{t}\frac{\sum_{j=1}^{1}\sigma_j^{n-2}(s)\sigma_j^1(s)}{k(s)}w^2(s)ds < \infty \quad ....(13)$$

Suppose on contrary that

$$\lim_{t\to\infty}\infty\sup\int_{t_0}^{t}\frac{\sum_{j=1}^{1}\sigma_j^{n-2}(s)\sigma_j^1(s)}{k(s)}w^2(s)ds \quad ....(14)$$

By(8) ∃ a positive ς satisfying *inf* $\lim_{t\to\infty}\inf\frac{H(t,s)}{H(t,t_o)} \geq \zeta$ > 0.............. (15)

Let *η* be any arbitrary positive number form (14)

∃ $t_1 > t_o$ s. t.

$$\int_{t_0}^{t}\frac{\sum_{j=1}^{1}\sigma_j^{n-2}(s)\sigma_j^1(s)}{k(s)}w^2(s)ds \geq \eta/(\zeta)\forall t \geq h$$

Then for t> t_1 we have B (t_1, t_o)

$$= \frac{1}{H(t,t_o)}\int_{t_1}^{t} H(t,s)d\left[\int_{t_0}^{t}\frac{\sum_{j=1}^{1}\sigma_j^{n-2}(s)\sigma_j^1(v)}{k(v)}w^2(v)dv\right]ds \geq \frac{\eta H(t,t_1)}{\zeta H(t,t_{01})}$$

By (15) $\exists\ t_2 \geq t_1$ s.t. $\forall t > t_2 \geq \int$
which implies B $(t,t_o) \geq \eta\ \forall\ t > t_2$ since η is arbitrary we have

$$\lim_{t\to\infty} B(t,t_o) = \lim_{t\to\infty} B(t, t_o) = \infty$$

which contradicts (12), thus, (13) hold then by (11) and (13) we get,

$$\lim_{t\to\infty}\operatorname{Sup}\int_{t_0}^{t}\frac{\sum_{j=1}^{1}\sigma_j^{n-2}(s)\sigma_j^{1}(s)}{k(s)}w^2(s)ds \leq \lim_{t\to\infty}\sup\int_{t_0}^{t}\frac{\sum_{j=1}^{1}\sigma_j^{n-2}(s)\sigma_j^{1}(s)}{k(s)}w^2(s)$$

which contradicts (10)
This completes the proof.

Remarks

Let β=1, in theorem 3, reduces to theorem(1): We obtain the some result in statement 1 in which we quit the assumption (4) in theorem (2).
Therefore Theorem (3) and statement(1) are generalization and improvements of the results obtained (1)
Remark 2 with an appropriate choices of the function H, h, and K, one can derive a number of oscillation criteria for equation (1) from our theorems.
Let K(t) = 1 $\alpha > 0$ is a constant $H(t, s)=(t-s)^{\alpha}$ $h(t, s)=-\alpha(t, s)^{\alpha-1}$, $t \geq s \geq to$
and we have

$$\lim_{t\to\infty}\frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty}\frac{H(t,s)^{\alpha}}{H(t,t_0)^{\alpha}} = 1$$

for $s \geq t_o$

Results and Discussion

This paper improves upon existing methods for analyzing how solutions to a particular kind of equation (even-order neutral differential equations) change over time (oscillate). These equations describe situations where the rate of change depends not just on the current state but also on a delayed version of it. The limitations of prior "Kamenev-Type" criteria are addressed. The paper introduces new theorems (Theorem 3 and Statement 1) with improved oscillation criteria compared to previous work. These new criteria allow for a wider range of functions to be considered. Additionally, the concept of Riccati functions, which were problematic before, is redefined here. Lemmas 1 and 2 establish key properties of solutions that are essential for proving the new theorems.

Overall, this research offers a significant advancement in analyzing oscillation behavior for these equations. The new theorems provide more general and powerful criteria, allowing researchers to apply them to a broader class of equations. The refined definitions and lemmas ensure a more robust foundation for the analysis. Future research could involve applying these new criteria to real-world problems and exploring further generalizations of the criteria.

Conclusion

In conclusion, this paper successfully addressed limitations in existing methods for analyzing the oscillatory behavior of even-order neutral differential equations. We introduced novel oscillation criteria through Theorem 3 and Statement 1, which surpass previous results by Meng and Xu [1, 2] in terms of generality. These advancements allow researchers to consider a wider range of functions within the analysis. Furthermore, the paper rectified the problematic definition of Riccati functions and established crucial properties of solutions through Lemmas 1 and 2. These refinements provide a more robust foundation for future studies. The improved criteria open doors for applying oscillation analysis to a broader range of real-world problems, while further research directions include exploring generalizations of these criteria and potential connections to other oscillation theories.

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