Received Date: Sept. 2023 Revised: Nov. 2023 Accepted: Dec. 2023

The Study Of Max-Min Composition In Fuzzy Relation

Yogendra Prasad Shah
Assistant Professor, Department of Mathematics
Patan Multiple Campus, Tribhuvan University
Email: yog.9841@gmail.com
Doi: https://doi.org/10.3126/ppj.v3i2.66201

Abstract

One of the foundational concepts in both pure and applied mathematics is the notion of relations. Fuzzy relations extend this concept much like fuzzy sets generalize the fundamental idea of sets. This study provides a comprehensive overview of operations, compositions, and the various types inherent in fuzzy relations.

Keywords: Fuzzy relation, min-max composition, Fuzzy ordering relations etc.

Introduction

In the year 1965, L. A. Zadeh unveiled the revolutionary concept of Fuzzy Theory. Fuzzy set theory, a profound augmentation of classical set theory, was born. A relation, a mathematical articulation of situations where specific elements within sets share connections, unfolded as a pivotal construct. Relations, serving as intricate descriptors, play an indispensable role in structuring information. Consequently, this document meticulously investigates the operations, compositions, and classifications of these relations, delving into the nuances that underpin their fundamental nature.

Definition:

Fuzzy Relation:

An n-ary Fuzzy relation, denoted as R, is a fuzzy set on U1 × U2 × U3 × … × Un, with U1, U2, …, Un as domains. A 2-ary fuzzy relation is also termed a binary fuzzy relation. Similarly, a 3-ary is also called a ternary fuzzy relation. Thus, a binary fuzzy relation (BFR) is a of the form

$$R = \Sigma \frac{R(u,v)}{(u,v)} \text{ where } (u, v) \text{ varies over } u \times v.$$

We say that R is from U to V and is indicated by R: U→ V.

Example: Let U= {a, b, c} and V= {x, y}. Then a BFR on U × V is given by

R	X	Y
a	0.6	1.0
b	0.3	0.5
c	0.4	0.2

The tabular or matrix representation of R is crucial, especially in handling FFRs. R is equivalently expressed as a matrix when domain elements are implicit and don't require explicit specification, enhancing convenience in analysis.

Operations on Fuzzy Relations

Consider domains $U_1, U_2, \dots, U_n$ and define U as their Cartesian product, i.e., $U = U_1 \times U_2 \times \dots \times U_n$. By the definition of the Cartesian product, an element $u \in U$ takes the form $u = (u_1, \dots, u_n)$, constituting an n-tuple, showcasing the inherent structure within the composite domain U.

Therefore, PF(U) is equivalent to PF($U_1 \times U_2 \times \dots \times$ Un). This equality underscores that each n-ary fuzzy relation (FR) on $U_1 \times U_2 \times U_3 \times \dots \times U_n$ is inherently a fuzzy set on U, and conversely. Consequently, the standard operations that apply to fuzzy sets are equally applicable to FRs. To summarize swiftly, let $U = U_1 \times U_2 \times \dots \times U_n$.

Equality: Given R, S in PF(U), we assert that R equals S if and only if their values are identical for every u in U, denoted as R(u) = S(u) for all elements within the domain.

Containment: For R, S in PF(U), we declare that R is a subset of S if and only if R(u) is less than or equal to S(u) for all elements u in U.

Union: For R, S in PF(U), the union of R and S, denoted by $R \cup S$, is defined as $(R \cup S)(u) = [R(u), S(u)]$ for all elements u in U.Intersection

For R, S in PF(U), the intersection of R and S, denoted by $R \cap S$, is defined as $(R \cap S)(u) = \min[R(u), S(u)]$ for every element u in U.

Complementation: For R in PF(U), the complement of R, denoted as R', is defined as $R'(u) = 1 - R(u)$ for every element u in U.

α-cuts of the Fuzzy Relation

Several properties of fuzzy sets seamlessly extend to fuzzy relations, and among them is the concept of α-cuts along with its associated properties.

Let R be a fuzzy relation on $U \times V$, and for a given α where $0 < \alpha \leq 1$, the α-cut of R is denoted by Rα and defined as follows:

Let R be a FR on Ux V and α be such that 0<α≤ 1.Then, the α cut of R, denoted by R_α, is defined by

$R_\alpha = \{(u, v) \mid R(u, v) \geq \alpha\}$

It's worth noting that Rα represents a crisp set on U × V, making it a crisp (binary) relation on U × V. The α-cut of R adheres to the decomposition theorem or resolution form property, which is as follows:

Let R be a fuzzy relation on U x V. Then R= $\sum (\alpha_\alpha)$ where $\sum$ is taken over all α.

The following example illustrates the above point.

***Example*:**

Let R be a fuzzy relation on U×V given by the matrix

R = $\begin{bmatrix} 0.7 & 0.4 \\ 0.4 & 0.0 \end{bmatrix}$ Then $R_{0.4} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $R_{0.7} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Now, $(0.4 \times R_{0.4}) \cup$

$(0.7 \times R_{0.7}) = \begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0 \end{bmatrix} \cup \begin{bmatrix} 0.7 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.4 \\ 0.4 & 0 \end{bmatrix} = R$

This verifies the above theorem.

Composition of fuzzy Relations

The composition of fuzzy relations assumes a pivotal role, and it can be defined in various ways. In the current context, focus is directed towards two specific methods: max-min composition and max-product composition. However, the discussion will delve into and elaborate upon one of these methods.

Max-min composition of two fuzzy relations: Let R be a BFR on U × V and S a BFR on V × W. Then, the max-min composition of R and S (that is composition of R followed by S is a BFR on $U \times W$, denoted by $S \ o \ R$ and is given by

(S∘R) (U, W)= max [min{R (U, V), S (V, W)}] here the maximum is taken over all v in V.

More generally, let R be in pF (U x V) and S be in pf (V x W), where U = $U_1 \times U_2 \times U_3 \times \ldots \times U_k$; V = $V_1 \times V_2 \times \ldots \times V_m$; W = $W_1 \times W_2 \times \ldots \times W_n$

Then, the max-min composition of R and S, denoted by $S\ oR$ is a fuzzy relation on $U \times W$ and is given by

$$(S\ oR)(u,w) = \max[\min\{R(u,v), S(v,w)\}]$$

Where

u	=	$(u_1 \times u_2 \times u_3 \times \ldots \times u_k)$
v	=	$(v_1 \times v_2 \times v_3 \times \ldots \times v_m)$
w	=	$(w_1 \times w_2 \times w_3 \times \ldots \times w_n)$

The maximum is taken over all v in V.

Notably, R represents a (k+m)-ary fuzzy relation, while S is an (m+n)-ary fuzzy relation, with V serving as the common domain, also known as the linking domain, for both R and S. Termed the compatibility condition for composition, this requirement ensures the harmonious integration of fuzzy relations R and S.

And finally, SoR is a (k +n)-ary fuzzy relation.

Examples

Consider the fuzzy relations R on U×V and S onV×W, Where

U = {a, b, c}
V = {x, y, z}

And W = {&, *},
given in matrix form by

$$R = \begin{bmatrix} 1.0 & 0.4 & 0.5 \\ 0.3 & 0.0 & 0.7 \\ 0.6 & 0.8 & 0.2 \end{bmatrix}$$

And

$$S = \begin{bmatrix} 0.7 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.4 \end{bmatrix}$$

Then $S\ oR$ can be defined and it is fuzzy relation on $U \times W$.

Now,

$$(SoR)(a,\&) = \max[min\{R(a,v), S(v,\&)\}]\ \ for\ ever\ v\ in\ V$$
$$= [\min\{R(a,x), S(x,\&)\}, \min\{R(a,z), S(z,\&)\}]$$

$= \max[\min(1, 0.7), \min(0.4, 0.2), \min(0.5, 0.8)]$
$= \max[0.7, 0.2, 0.5]$
$= 0.7$

Similar computation for (a, *), (b, &), (b, *), (c, &) and (c, *) yield the following matrix of $S\ o\ R$:

$$S\ oR = \begin{bmatrix} 0.7 & 0.4 \\ 0.7 & 0.4 \\ 0.6 & 0.8 \end{bmatrix}$$

We now discuss a special case of composition which plays a fundamental role in fuzzy logic.

This special case deals with the composition of a fuzzy set and a fuzzy relation, as explained in the following definition.
Let A be a fuzzy set on U and R be a fuzzy relation on U×V, where,
$V = V_1 \times V_2 \times \ldots \times V_n$ Then,
Then, the composition of A followed by R also called the image of A under R, denoted by

$(RoA)(V) = \max[min[a(u), R(V)]]\, where\ 'max'\ is\ taken\ over\ all\ u\ in\ U.$

Certainly, RoA stands as an n-ary fuzzy relation on V. In the scenario where n equals 1, it transforms into a fuzzy set on V. Analogously, following a comparable approach, the max-product composition of A and R can be defined.

Properties of Min –Max composition

Associativity: The max-min composition is associative, i.e.

$(R_1oR_2)oR_3 = R_1o(R_2oR_3)$

Reflectivity:

Let R be a fuzzy relation on X× X.

Then

i. R is said to be a reflexive if $\mu_R(x, x) = 1, for\ all\ x \in X$

ii. ii. R is said to be ϵ reflexive if $\mu_R(x, x) \geq 1, for\ all\ x \in X$

iii. R is said to be weakly reflexive if $\mu_R(x, y) \leq \mu_R(x, x), \mu_R(y, x) \leq \mu_R(x, x)\ for\ all\ x, y \in X$

Symmetry:

- A fuzzy relation R is termed symmetric if, for any elements x and y, the membership degree of R in relation to (x, y) is equal to the membership degree of R in relation to (y, x), i.e., R(x, y) = R(y, x).
- A relation R is called anti symmetric if for $x \neq y$,
  Either $\mu_R\ (x, y) \neq \mu_R(y, x)$

Or $\mu_R(x,y) = \mu_R(y,x) = 0,\ for\ all\ x, y \in X$

- A relation is called perfectly anti symmetric if for $x \neq y$, wherever $\mu_R(x,y) > 0$, then $\mu_R(y,x) = 0$, for all x, y ∈ X
- If $R_1$is reflexive and R_2 is an arbitrary fuzzy relation derived from R

  $R_2 \subset R_1 o R_2$
- If R is reflexive, then $R \subset R_1\ o\ R_2$
- If R_1 and R_2 are symmetric fuzzy relation, then the composition $R_1 o R_2$ is symmetric if $R_1 o R_2 = R_2\ o\ R_1$.

If R is symmetric fuzzy relation, then each power of R is termed max-min transitive

- if $R\ oR \subseteq R$
- If R is symmetric and transitive fuzzy relation, then $\mu_R(x,y) \subseteq \mu_R(x,x),$ for all x, y ∈ X.
- If R is reflexive and transitive fuzzy relation, then $R\ o\ R = R$
- If R_1 and R_2 are transitive fuzzy relations and $R_1 o R_2 = R_2 o R_1$, then composition $R_1 o\ R_2$ is transitive.

If a relation is perfectly antisymmetric, then relation is called perfect fuzzy relation. It is also called fuzzy partial order relation.

Definition: A total fuzzy order relation or a fuzzy linear ordering is a fuzzy order relation such that for all x, y ∈ X, $x \neq y$, either

$\mu_R(x,y) > 0$

$or\ \mu_R(y,x) > 0.$

Conclusion

Operations involving fuzzy relations hold significant importance in the development of fuzzy models within the realm of process modeling. Relations, being fundamental associations, constitute the cornerstone of various methodological approaches in both science and engineering.

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