Prāgyik Prabāha Vol. 12, No. 1: 34-41, August, 2024 Tribhuvan University Teachers' Association, Tribhuvan Multiple Campus, Tribhuvan University, Nepal
DOI: https://doi.org/10.3126/pp.v12i1.69970

Applications of the Banach-Stone Theorem on Algebra

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Article History: Received 15 March 2024; Reviewed 27 June 2024; Revised 15 July 2024; Accepted 25 August 2024

Abstract

For two compact Hausdorff spaces X and Y, C(X) and C(Y) are isomorphic if and only if X and Y are homeomorphic. This class result was given by Banach in 1932 and generalized by Stone in 1937. This opened the new way of research towards algebra isomerphisms. The algebraic version of Banach-Stone theorem asserts that C(X) and C(Y) are algebraic isomorphism if and only if X and Y are homeomerphic. In this paper we study the structure on the group of isometric isomorphism from C(X) to itself as an application of Banach-Stone theorem.

Keywords: *Linear Isomorphism, Semi direct product, Normal Subgroup, Homeomorphism, Automorphism.*

Introduction

In 1932 Banach [1] introduced the problem of how the topological properties of the metric spaces *X* and *Y* are characterized from the linear structure of *C(X)* and *C(Y)*. To this problem he gave the classical result which had further generalized by Stone [8] in 1937, combinely called the Banach- Stone Theorem. According to this classical result for any two compact topological spaces *X* and *Y, C(X)* and *C(Y)* are linear isometric if and only if *X* and *Y* are homeomorphic. This theorem includes the algebraic isomorphism and Riesz isomerphism too [2].

The isometry between two normed spaces *X* and *Y* is a bijection $\phi : X \to Y$ such that ||x ||ϕ(x) − ϕ(y)|| = ||ϕ(x) || − ||ϕ(y) || for all x, y ∈X. If ϕ is linear we call it linear isometry. From the result of Mazur and Ulam 1932, ϕ is linear isometry if $\phi(0) = 0$.

The algebraic version of Banach-Stone theorem was first introduced by Gelfand and Kolmogorov [5] in 1939.

This states that let *X* and *Y* be a compact spaces. Then *C(X)* and C(X) are isomerphic as algebras if and only if X and Y are homeomerphic.

To connect the algebraic properties of C(X) with the topology of *X*, various algebraic structures have been taken on *C(X)* like lattice structure. The first result in this field was due to Kaplansky [6]. It characterized the topology on compact space *X* by the lattice structure of C(X).

In second section we review the Banach-Stone theorem and its converse result. Then in this section we study the semi direct product on *C(X)* as the application of the Banach-Stone theorem. Finally we conclude the article by proving the result $Isom(C(X)) \cong Um(X) \rtimes_{\sigma} Homeo(X)$.

Theorem 1. [4] (Banach-Stone Theorem) *Let X and Y be compact Hausdorff spaces and let T be a surjective isometry* $C(X) \to C(Y)$. *Then there exists a homeomerphism* $\tau : Y \to X$ *and a function* $h \in C(Y)$ *such that* $|h(y)| = 1$ *for all* $y \in Y$ *and*

$$T(f)(y) = h(y)f(\tau(y)), \ \forall f \in C(X) \text{ and } y \in Y$$

The converse part is also true. Let h and τ as stated, and define the map $T : C(X) \to C(Y)$

By $\forall f \in C(X)$ and $f \in Y$, $T(f)(y) = h(y)f(\tau(y))$.

Since τ is homeomerphism and the assumption |h(y)| = 1 for all y ∈ Y we have that

$$\begin{aligned} \|T(f)\| &= Sup_{y \in Y} |T(f)(y)| \\ &= Sup_{y \in Y} |h(y)f(\tau(y))| \\ &= Sup_{y \in Y} |h(y)||f(\tau(y))| \\ &= Sup_{y \in Y} |f(x)| = \|f\| \end{aligned}$$

Thus T is isometric. It is also subjective. For any $g \in C(Y)$ we define

$$f = \frac{1}{h \circ \tau^{-1}} (g \circ \tau^{-1})$$

Since h, τ^{-1} and g are continuous, f is also continuous being the product of continuous maps.

Since $h(y) \neq 0$, for all $y \in Y$. Let $y \in Y$, then we have

$$\begin{aligned} T(f)(y) &= h(y)\left(\frac{1}{h \circ \tau^{-1}}\right).(g \circ \tau^{-1})(\tau(y))) \\ &= h(y)\left(\frac{1}{h \circ \tau^{-1}}\right)(\tau(y).(g \circ \tau^{-1})(\tau(y))) \\ &= \frac{h(y)}{h(y)}.g(y) \\ &= g(y). \end{aligned}$$

Here we study the special application of Banach-Stone theorem in the group theory. We use the converse of the Banach-Stone theorem also. For this all we need the followings from the group theory.

Direct Product

Let G be a group under the operation $*$ and H be a group under the operation $\circ$. Define an operation on G × H by

$$(g, h) \cdot (g', h') = (g * g', h \circ h').$$

Then $G \times H$ is a group. If e_1 is the identity element of G and e_2 is that of H then (e_1, e_2) is the identity element of $G \times H$ and (g^{-1}, h^{-1}) is the inverse element of (g, h). Such obtained group is called the direct product of G and H. The following theorem is the characterization of the direct product of the groups.

Theorem 2. [9] *If G is a group containing normal subgroups H and K with* $H \cap K = \{1\}$ *and* $HK = G$, *then*

$$G \cong H \times K.$$

Proof. We first show that for arbitrary $g \in G$, $g = hk$, where $h \in H$ and $k \in K$ is unique. Let

$h_1k_1 = h_2k_2$ then $h_2h_1^{-1} = k_1k_2^{-1}$. Clearly, $h_2h_1^{-1} \in H$ and $k_1k_2^{-1} \in K$. Since $H \cap K = \{1\}$,

$h_2h_1^{-1} = k_1k_2^{-1} = 1$

$\Rightarrow h_1 = h_2$ and $k_1 = k_2$.

We may now define a function $\varphi : G \to H \times K$ by $\varphi(g) = (h, k)$, where $g = hk$, $h \in H$ and $k \in K$. We see whether φ is homomorphism, let $g' = h'k'$, so that $gg' = hkh'k'$.

Since K is a normal subgroup of G, for $k \in K$, $h \in H (hkh^{-1})k^{-1} \in K$.

Similarly, since H is a normal subgroup of G, for $k \in K$, $h \in H h(kh^{-1}k^{-1}) \in H$.

But $H \cap K = \{1\}$ so that $hkh^{-1}k^{-1} = 1$ and $hk = kh$.

So can write

$$\varphi(hkh'k') = \varphi(hh'kk')$$
$$= (hh', kk')$$
$$= \varphi(g)\varphi(g').$$

Finally, we show that φ is bijective. If $(h, k) \in H \times K$, then the element defined by $g = hk$ satisfies $\varphi(g) = (h, k)$, hence φ is surjective.

If $\varphi(g) = (1, 1)$, then $g = 1$, so that $\ker \varphi = 1$ and is injective.

Therefore, φ is isomorphism.

Semi direct product

If we suppose H a normal subgroup and K need not be normal with the conditions:

(1) $H \cap K = \{1\}$;

(2) $HK = G$,

Then the obtained product group is called the semi direct product.

Example 1.

Suppose $H \cong \mathbb{Z}/3$ and $K \cong \mathbb{Z}/2$. There is the possibility of G as,

(1) If K is a normal in G then we already know $G \cong H \times K$.

(2) If K is not normal in G for example G might be the symmetric group S_3, with

$$H = \{(1),(123),(132)\} \text{ and } K = \{(1),(12)\}.$$

This example shows that there may be more than one semi direct product for a given H and K. Thus H and K are not the enough to recover the structure of the new group G. For this we need a homomorphism from K to the group of automorphisms of H;

$$\varphi : K \to Auto(H).$$

For $k \in K$ we define the automorphism φ_k on H given by conjugation:

$$\varphi_k(\mathrm{h}) = \mathrm{kh}k^{-1} .$$

The following theorem shows that the φ is homomorphism.

Theorem 3. [7] *Let K and H be groups then the map : K → Auto(H) a homomorphism.*

Proof. Here we show that $\varphi_{k_1k_2} = \varphi_{k_1}\varphi_{k_2}$ for any $k_1, k_2 \in K$.

$\varphi_{k_1}\varphi_{k_2}(h) = \varphi_{k_1}(k_2hk_2^{-1}) = k_1k_2hk_2^{-1}k_1^{-1}$ for any $h_1, k_1 \in K$.

For any $h \in H$,

$\varphi_{k_2}(\mathrm{h}) = \varphi_{k_1}(k_2hk_2^{-1}) = k_1k_2hk_2^{-1}k_1^{-1}$.

Again,

$\varphi_{k_1k_2}(\mathrm{h}) = k_1k_2h(k_1k_2)^{-1} = k_1k_2hk_2^{-1}k_1^{-1}$

This shows that, $\varphi_{k_1k_2} = \varphi_{k_1}\varphi_{k_2}$.

Which completes the proof.

Theorem 4. [7] *Given groups H and K and a homomorphism K → Auto(H) there is a semi direct product group G based on this information. We can construct it as follows*:

The underlying set of G is the set of pairs (h, k) with h ∈H and k ∈K. The multiplication on this set is given by the rule

$(h, k)(h', k') = (h\varphi_k(h'), kk')$,

the identity element is (1, 1) and inverse is given by

$(h, k)^{-1} = (\varphi_{k^{-1}(h^{-1})}, k^{-1})$.

Proof. Here we show that the multiplication is associative, existence of identity and inverse law hold.

For associativity,

$((h, k)(h', k'))(h'', k'') = (h\varphi_k(h'), kk')(h'', k'')$

$= (h\varphi_k(h')\varphi_{kk'}(h''), kk'k'')$

$= (h\varphi_k(h')\varphi_k(\varphi_{k'}(h''), kk'k'')$

$= (h\varphi_k(h'\varphi_{k'}(h'')), kk'k'')$

$= (h, k)(h'\varphi_{k'}(h''), k'k'')$

$= (h, k)((h', k')(h'', k''))$.

For identity:

$(h, k)(1, 1) = (h\varphi_k(1), k) = (h, k)$,

$(1, 1)(h, k) = (1\varphi_1(h), k) = (h, k)$.

For inverse:

$(h, k)(\varphi_{k^{-1}}(k^{-1}), k^{-1} = (h\varphi_k(\varphi_{k^{-1}}(h^{-1})), kk^{-1})$

$= (hh^{-1}, kk^{-1})$

$= (1, 1)$.

Thus this operation forms a group.

Now it remains to show that it is the desired semi direct product of *H* and *K*. We have the injective maps *H* → *G* given by *h*↦ *(h, 1)* and *K* → *G* given by *k*↦ *(1, k)*. These both maps are homomorphism since

$(h, 1)(h', 1) = (h\varphi_1(h'), 1) = (hh', 1)$

and

$(1, k)(1, k') = (1\varphi_k(1), kk') = (1, kk')$.

From this homomorphism we can say that H and K are the subgroup of G. Then

$H \cap G = \{(1, 1)\}$ and $HK = G$

since

$(h, 1)(1, k) = (h, k)$.

Finally we show that H is normal in G and that the action of K on H by conjugation in G is given by the original homomorphism φ. Both follow from the calculation,

$(1, k)(h, 1)(1, k)^{-1} = (1, k)(h, 1)(1, k^{-1})$

$= (\varphi_k(h), k)(1, k^{-1})$

$= (\varphi_k(h), 1)$.

The semi product of H and K is denoted by $G = H \rtimes K$.

Now we turn our attention to the group structure on $C(X)$. Let X be a compact Hausdorff Space. The following sets

$Isom(C(X)) = \{T : C(X) \to C(X) : T \text{ is isomerphic isomerphism}\}$,

$Homeo(X) = \{\tau : X \to X : \tau \text{ is homeomerphism}\}$,

And

$Um(X) = \{h \in (X) : |h(X)| = 1 \ \forall x \in X\}$

are groups with respect to the operation of composite map and the point wise multiplication. That is $(Isom((X)), \circ)$, $(Homeo(X), \circ)$ and $(Um(X), \circ)$ are groups.

Here our main aim is to study the semi product of $Um(X)$ and $Homeo(X)$. For this we need the following theorem to get a desired result.

Theorem 5. *The map* $\sigma : Homeo(X) \to Auto(Um(X)) \quad \tau \mapsto \tau^*$, *is a homepmerphism where* τ^* *is defined as*

$\tau^* : Um(X) \to Um(X)$, $h \mapsto h \circ \tau^{-1}$.

Proof.

Let $\tau_1, \tau_2 \in Homeo(X)$ and $h \in Um(X)$. Then we have that

$\sigma(\tau_1 \circ \tau_2)(h) = (\tau_1 \circ \tau_2) * (h) = h \circ (\tau_1 * \tau_2)^{-1}$

$= (h \circ \tau_2{}^{-1}) \circ \tau_1{}^{-1} = \tau_1 * (h \circ \tau_2{}^{-1})$

$= (\tau_1^* \circ \tau_2^*)(h)$

$= (\sigma(\tau_1) \circ \sigma(\tau_2))(h)$.

Thus $\sigma(\tau_1 \circ \tau_2 = \sigma(\tau_1) \circ \sigma(\tau_2)$.

This shows that σ is homomorphism.

From **Theorem 4** *$Um(X) \rtimes_\sigma$ Homeo(X) is* a group with the operation

$$(h_1, \tau_1)\,(h_2, \tau_2) = ((h_1.\sigma(\tau_1)(h_2), (\tau_1 \circ \tau_2)).$$

Lastly we prove our main result using the Banach-Stone theorem and its converse.

Theorem 6.*The map*

$$\psi : Isom(C(X)) \rightarrow Um(X) \rtimes_\sigma Homeo(X)$$

T↦ (h, τ^{-1}) is a group isomorphism

where

T : C(X) → C(X) defined as T(f) = h · (f ∘ τ).

Proof. From the Banach-Stone theorem there exists unique function *h ∈ Um(X)* and *τ ∈ Homeo(X)* for every *T ∈Isom(C(X))* such that

$$\forall f \in C(X) \text{ and } x \in X; \quad T(f)(x) = h(x)\, f(\tau(x)).$$

So *ψ* is well defined and injective. Again from the converse of the Banach-Stone theorem *ψ* is onto. Now it remains to show that *ψ* is homomorphism.

For this,

let $T_1, T_2 \in$ *Isom(C(X))*. Assume that $\psi(T_1) = (h_1, \tau^{-1})$ and $\psi(T_2) = (h_2, \tau^{-1})$ for certain maps $\tau_1, \tau_2 \in$ *Homeo(X)* and $h_1, h_2 \in$ *Um(X)*. Now from the group operation on *Um(X)* $*$ *Homeo(X)*

We have,

$$\psi(T_1)\psi(T_1) = (h_1, \tau^{-1})(h_2, \tau^{-1}) = (h_1.\sigma(\tau^{-1})(h_2), \tau_1^{-1} \circ \tau_2^{-1})$$

$$= (h_1 \cdot h_2 \circ \tau_1, (\tau_2 * \tau_1)^{-1}).$$

On the other hand, we have for an arbitrary map *f ∈C(X)*

$$(T_1 \circ T_2)(f) = T_1(T_2((f))$$

$$= T_1(h_2.f \circ \tau_2) = h_1.(h_2.f \circ \tau_2) \circ \tau_1$$

$$= h_1.h_2 \circ \tau_1.f \circ \tau_2 \circ \tau_1$$

Then we conclude that $\psi\,(T_1 \circ T_2) = (h_1 \cdot h_2 \circ \tau_2,\ (\tau_2 \circ \tau_1)^{-1})$.

Therefore $Isom(C(X)) \cong Um(X) \rtimes_\sigma Homeo(X)$.

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