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A Survey on Models of Flows over Time with Flow Dependent Transit Times

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Abstract

Ford and Fulkerson introduce the maximum dynamic flow problems with fixed transit times on the arcs and developed the first well known algorithm that sends maximum flow from the source to the sink by augmenting along s-t paths and prove the maximum amount of flow is equal to the total capacity of the arcs in minimum cut. In flows over time with fixed transit time on the arc, the time it takes to traverse an arc does not depend on the current flow situation on the arc. But in real situation, the flow units travelling on the same arc at the same time do not necessarily experience the same pace, i.e., flow units are in general not entering and leaving an arc in the same order. In this paper, we discuss the time expanded graph for load-dependent and inflow-dependent transit times. We also discuss the discrete and continuous time flow, earliest arrival flow and quickest transhipment problem.

Keywords: *Flow-dependent transit times, Discrete and continuous flow over time, Earliest arrival flow, Time expanded graph, Relaxation property.*

Introduction

We use the term flow-dependent transit time to express that transit times depend on the flow in one way or another. The travel time in congested road takes more than an empty road. This is called flow-dependent transit times. A model of flow-dependent transit times on arcs must take density, speed and flow rate evolving along an arc into consideration. This means that most world real applications are of dynamic nature.

The transit time of an arc specifies the amount of time it takes for flow to travel from the tail to the head of that arc. A flow over time in a network specifies a flow rate entering an arc for each point in time. Transit times of an arc directly depend on the number of vehicles that travel through the arc at that moment in time. This is also known as flow-dependent transit times. Flow-dependent transit times can be divided into two ways: load-dependent transit times and inflow-dependent transit times. In load-dependent transit times, total amount of flow on the arc called load of arc, is used as input of the transit time function (Köhler &Skutella, 2002). In inflow-dependent transit times, transit time on arc solely depends on the current rate of inflow into that arc. So, the transit times are considered as functions of the rate of inflow. The flow units travelling on the same arc at the same time do not necessarily experience the same pace, i.e., flow units are in general not entering and leaving an arc in the same order. We will present the models of load-dependent and inflow-dependent transit times in detail later.

In flows over time with fixed transit time on the arc, the time it takes to traverse an arc does not depend on the current flow situation on the arc. Flow variation over time is an important feature in network flow problems arises in various application such as road or air traffic control, production systems, communication networks (e.g., the internet), and financial flows. In such situation, the amount of time needed to traverse an arc of the underlying network increases as the arc becomes more congested. This characteristic is obviously not captured by static flow model.

Time dependent travel time is the travelling time between two locations which depend on the time of departure throughout the day. The travelling time between locations is always considered constant. But, the time it takes to travel from one location to another can vary a lot during the day due to traffic congestion. Thus, we identify the need of considering time-dependent data for routing component.

In inflow-dependent transit time model only the rate of inflow into an arc is explicitly bounded; the flow rates evolving along an arc can be arbitrarily large. This model, in general, does not obey the first in, first out (FIFO) property (i.e., no overtaking) on an arc. In model of Inflow-dependent transit time, the transit time experienced by an infinitesimal unit of flow on an arc is determined when entering this arc and only depends on the inflow rate at that moment in time.

Ford and Fulkerson (1958, 1962) introduced the maximum dynamic flow problems (MDFP) and developed the first well known algorithm that sends maximum flow from the source to the sink by augmenting along s-t paths and proved the maximum amount of flow is equal to the total capacity of the arcs in minimum cut. Hoppe (1995), Hoppe and Tardos (1995) show that there is a non-trivial generalization of the result of Ford and Fulkerson to the case of multiple sources and sinks.

Fleischer and Tardos (1998) presented natural transformation for many discrete dynamic flows based on chain decomposable flows to transform into continuous dynamic flows and proved its optimality. Hamacher and TJandra (2002) formulated a continuous linear program for general continuous network with cost minimization for given bounded measurable functions of cost, upper bounds, rates of demand or supply and levels of storage in each node.

One of the first model for time dependent flows with flow-dependent transit times has been defined by Merchant and Nemhauser (1978). They proposed a non-linear and non-convex program with discretized time steps. In their model, the outflow out of an arc in each time period solely depends on the amount of flow on that arc at the beginning of the time period. Carey (1987), introduced a slight revision of the model of Merchant and Nemhauser yielding a convex problem instead of a non-convex one. In the model of Köhler and Skutella (2002) the pace of flow on an arc depends on its current load, i.e., the entire amount of flow which is currently travelling along that arc. Carey and Subrahmanian (2000) introduced a generalized time-expanded network for flow-dependent transit times.

An important application of flow over time problem is evacuation planning problem. In continuous time setting, different dynamic network flow problems are solved for evacuation planning problem. Dhamala and Pyakurel (2015, 2018) studied the continuous time dynamic flow and introduced the continuous contraflow models. They have presented efficient algorithms to solve maximum dynamic, quickest and earliest arrival contraflow problems with natural transformation of Fleisher and Tardos (1998) by inversing the direction of arcs at time zero. Dhamala, Pyakurel and Dempe (2019) introduce efficient algorithms for evacuation planning problems in inflow-dependent transit times as well as load-dependent transit times with contraflow approach.

This paper is organized as follows: In section two, we define necessary basic definitions and Notations. We introduce different models with load dependent and in-flow dependent transit times in section three. In section four, we introduced time expanded graph as fan graph and related bow graph. Finally, concluding remarks is given in last section.

Basic Definitions and Notation

Let G = (V, A) be a directed network with node set V and arc set A. Let s and t be source and sink node of V respectively, and D be the positive demand value. Letu_a be positive capacity of an arc $a \in$ A which is interpreted as an upper bound on the rate of flow entering a, i, e., a capacity per unit time, Let τ_a be a positive transit time of an arc a which determines the amount of time it takes for flow to travel from the tail to the head node of that arc.

For any intermediate node v of V, at any timeθ, if inflow may exceed the outflow, then the flow can be stored in node v, it is called flow conservation constraints. For a flow over time with finite time horizon, for any node v V$\in$ {s, t}, the inflow into node v until time T is equal to the totaloutflow out of node v until time T. The cost c(x) of flow x is the total travel time spentin the network. For any arc a, it can be written as c(x) = $\sum_{a\in A} x_a\tau_a(x)$.

An s-t flow over time (also known as dynamic s-t flow or time dependent s-t flow) f on network G with time horizon T is given by function $f_a: [0, \mathrm{T}) \rightarrow \boldsymbol{R}^+$, for each $a \in$ A, where $f_a(\theta)$defines the rate of flow (per unit time)entering arc a at time θ. This flow arrives at the head node of a at time $\theta + \tau_a(f_a(\theta))$. The relaxation is defined on an expanded graph with fixed transit times on the arcs. In relaxed model, the relaxation relies on an expanded graph with fixed transit times on the arcs.

Models of Flows

Here we discuss the different network flow models in graph G as follows:

Model of Discrete Flow over Time

We assume that all transit times are integral values. A discrete flow over time f in G assigned to every arc $a \in A$ is a function $f_a(\theta)$: A $\times \boldsymbol{Z}^+ \rightarrow \boldsymbol{R}^+$. Similarly, We say that the flow over time f has time horizon T if no flow is entering an arc a after the time T – 1– τ_a i.e., $f_a(\theta) = 0$ for all $\theta \geq$ T $- \tau_a, a \in$ A. Flow conservation constraints for discrete flow model is

$$\sum_{a\in\delta^+(v)} \sum_{\theta=0}^{\zeta} f_a(\theta) - \sum_{a\in\delta^-(v)} \sum_{\theta=\tau_a}^{\zeta} f_a(\theta - \tau_a) \leq 0. \quad (1)$$

for all $\zeta \leq$ T $- 1$ and v $\in$ V – {s, t}. If $f_a(\theta) \leq u_a$ for all $\theta \in \boldsymbol{Z}^+$ and $a \in$ A then the flow f is said to be feasible. The flow over time f satisfies the supplies and demand if

$$\sum_{a\in\delta^+(v)} \sum_{\theta=0}^{T-1} f_a(\theta) - \sum_{a\in\delta^-(v)} \sum_{\theta=\tau_a}^{T-1} f_a(\theta - \tau_a) = D. \quad (2)$$

for every v $\in$ {s, t}. The cost of discrete flow over time f is defined as

$$C(f) = \sum_{a\in A} C_a \sum_{\theta=0}^{T-1} f_a(\theta) \quad (3)$$

Model of Continuous Flow over Time

The transit time of an arc $a\in$A is interpreted as the time it takes for flow to traverse arc a. More precisely, flow whichis entering arc a at time θ, arrives at head (a)at time θ + τ_a. The flow f is called feasible, if the capacity $u_{\boldsymbol{a}}$is an upper bound on the rate of flowentering arc a at any moment in time, i. e., $f_a(\theta) \leq u_a$, for all $a \in \boldsymbol{R}^+$and $a \in$ A.

The continuous flow over time f in network G is a Lebesgue measurable function f_a: A $\times \boldsymbol{R}^+ \rightarrow \boldsymbol{R}^+$, for every $a \in$A. Here $f_a(\theta)$is the rate of flow per unit time that enters arc $a$at time θ. Clearly,$f_{\boldsymbol{a}}(\theta) = 0$ for $\theta < 0$. We say that the flow over time f has time horizon T if no flow is entering an arc $a$afterthe time T $- \tau_a$ i. e. $f_a(\theta) = 0$ for all $\theta \geq$ T $- \tau_a, a \in$ A..

If the flow is allowed to storage at intermediate nodes, then it is called flow conservation. It means that flow enters a node and holds back for some time before it is sent onward. Mathematically, flow conservation is modelled as

$$\sum_{a\in\delta^+(v)} \int_0^{\zeta} f_a(\theta)\, d\theta - \sum_{a\in\delta^-(v)} \int_{\tau_a}^{\zeta} f_a(\theta - \tau_a)\, d\theta \leq 0. \quad (1)$$

for all $\zeta \in [0, T)$ and $v \in V - \{s, t\}$.

Inequality (1) becomes equality for $v \in V - (s, t)$ at time $\zeta = T$. the flow f is said to be feasible if $f_a(\theta) \leq u_a$ for all $\theta \in \boldsymbol{R}^+$ and $a \in A$.

The flow over time f satisfies the supply and demands if

$$\sum_{a\in\delta^+(v)} \int_0^T f_a(\theta)\, d\theta - \sum_{a\in\delta^-(v)} \int_{\tau_a}^T f_a(\theta - \tau_a)\, d\theta = D. \qquad (2)$$

for every $v \in \{s, t\}$. Similarly, the value of s-t flow over time f is given by

$$|f| = \sum_{a\in\delta^+(v)} \int_0^T f_a(\theta)\, d\theta - \sum_{a\in\delta^-(v)} \int_{\tau_a}^T f_a(\theta - \tau_a)\, d\theta. \qquad (3)$$ until time T.

Here |f| is the total amount of flow leaving the source nodes until time T and that, because of flow conservation, this value is equal to the total amount of flow arriving in the sink node t until time T. The cost s-t flow over time is defined as

$$C(f) = \sum_{a\in A} C_a \int_0^T f_a(\theta)\, d\theta. \qquad (4)$$

Example -1: Discrete and continuous flows over time are closely related and we will see many results that are true for both kinds of flow. In discrete flow, packages of flow are sent through an edge that arrive at the same time but not in continuous flow. Consider the graph G with V (G) = {s, t}, E (G) = {(s, t)}, u{s, t} = 1, and $\tau\{s, t\} = 2$. Then, with a time horizon of T = 3, we could send a flow of size 2 in the discrete model (by sending two flow packages of size 1 at time 0 and 1) but only a flow of size 1 in continuous model (by starting to send flow at time 0 but stopping to send flow at time 1).

Model of Earliest Arrival Flow

An earliest arrival flow in the network is to arrive maximal flow from the source s to the sink t for every step of discrete time $0 \leq \theta \leq T$. In other word, an earliest arrival flow problem is to determine an s–t flow over time which simultaneously maximizes the amount of flow arriving at the sink before time θ, for all $\theta \in [0, T)$. The earliest arrival flow computed by the successive shortest path algorithm has the property that it simultaneously maximizes the amount of flow departing from the sourcesafter time θ, for all $\theta \in [0, T)$. Such a flow is called a latest departure flow. Flows overtime featuring both properties (earliest arrival and latest departure flow) are called universally maximal.The model of earliest arrival flow can be explained as follows:

Consider the network N $(V, A, u_a, \tau_a, s, t, T)$ where the symbols have their usual meaning. Let 0, 1, 2, ... , T – 1, T be the given discrete time periods. Let $f_a(\theta)$ denotes the amount of flow that enteringarc $a \in A$ at time θ. If D (T) is the net flow leaving v = s or entering v = t, for each $v \in V$, during the T time periods 0 to 1, 1 to 2, 2 to 3, ... , T - 1 to T.

Then the earliest arrival flow model is

Max D(θ) for each θ, θ = 0 , 1, 2, ..., T. such that

$$\sum_{a\in\delta^+(v)}\sum_{\theta=0}^{T} f_a(\theta) - \sum_{a\in\delta^-(v)}\sum_{\theta=\tau_a}^{T} f_a(\theta-\tau_a) = D(\theta), \text{ for } v = s \quad (1)$$

$$\sum_{a\in\delta^+(v)}\sum_{\theta=0}^{T} f_a(\theta) - \sum_{a\in\delta^-(v)}\sum_{\theta=\tau_a}^{T} f_a(\theta-\tau_a) = 0, \quad \text{for } v \in V\text{-}\{s, t\} \quad (2)$$

$$\sum_{a\in\delta^+(v)}\sum_{\theta=0}^{T} f_a(\theta) - \sum_{a\in\delta^-(v)}\sum_{\theta=\tau_a}^{T} f_a(\theta-\tau_a) = -D(\theta), \text{ for } v = t \quad (3)$$

$0 \le f_a(\theta) \le u_a$ for all θ ∈ {0, 1, 2, . . . , T} and $a \in A$. Here $\delta^+(v)$ and $\delta^-(v)$ denote the set of arcs leaving and entering node v, respectively. A polynomial time approximation scheme for computing universally maximal flows overtime was founded by Hoppe and Tardos (1995). The algorithm sends a 1-ϵ fraction of the maximal flow that can reach sink t by time θ, $\theta \in [0, T)$, and it sends a 1 – ϵ fraction of the maximal flow that can leave the source s after time θ, $\theta \in [0, T)$. The algorithm combines capacity scaling with the successive shortest path algorithm.

Here we mention some consequences of model for earliest arrival flows:

Theorem -1: All earliest arrival flows are maximal dynamic flows.

Proof: A flow which is maximal for each discrete time step $0 \le \theta \le T$ is obviously maximal for time interval 0, 1, 2, … , T. So that all earliest arrival flows are maximal dynamic flows.

Corollary -1: Maximal dynamic flows are not necessarily earliest arrival flows.

Model of Quickest Transhipment Problem

In quickest flow problem we send a given amount of flow from the source to the sink in the shortest possible time. Quickest transhipment problem is the generalization of quickest flow problem with given vector of supplies and demands at sources and sinks respectively. The problem is to find a flow over time that satisfies all supplies and demands within minimal time.

Hoppe and Tardos's strongly polynomial time algorithm produce a solution that does not make use of storage at intermediate nodes. Their approach relies on chain-decomposable flows which generalize the class of temporally repeated flows. These flows are represented by a set of paths, but, unlike temporally repeated flows, these paths may use backward arcs. The downside of this algorithm is that it requires a submodular function minimization oracle as a subroutine and is therefore not of practical use.

The model of quickest transhipment flow can be explained as follows:

A dynamic network $\mathcal{N} = (G, u, \tau, S)$ consists of a directed graph G = (V, A) with a non – negative capacity u_{yz} and integral transit time τ_{yz} associated with each edge yz ∈A, and a set of terminals (i. e. sources or sinks) S⊆V. In dynamic transhipment problem, there are multiple sources and multiple sinks and there is given a time bound T, and supplies v_x, where $v_x \ge 0$ for every source x∈S+, and $v_x \le 0$ for every sink x ∈ S-. If dynamic flow f with time horizon T exists and $|f|_x = v_x$ for every x then it is feasible flow. If we minimize the time T in problem so that the result ingdynamic transhipment problem is feasible then it is called quickest transhipment problem.

Theorem -2: The dynamic transhipment problem $(\mathcal{N}, \nu, \mathrm{T})$ is feasible if and onlyif $o(\mathrm{A}) \geq \nu(\mathrm{A})$ for every subset A⊆S, where $o(\mathrm{A})$ denote the total supply of A and ν (A) denote the maximum amount of flow that the sources in A can send to the sinks S\A in time T.

Model of Load-dependent Transit Times

Köhler and Skutella (2002) investigate the model of flows over time with load-dependent transit times. The load of an arc is the total amount of flow on the arc. The underlying assumption of the model is that the speed on an arc is a function of the load. Let l_a be the load on arc a and $\tau_a(x_a)$ be the transit times on arc a for the static flow x_a rate then the relation is

$$l_a = x_a \tau_a(x_a) \qquad \text{(i)}$$

If $\tau_a$is monotonically increasing and convex, then, in a static flow, the flow rate $x_a$is a strictlyincreasing and concave function of the load l_a. In this case, the transit time can also beinterpreted as an increasing function $\tau_a$of the load l_a, i.e.

$$\tau_a(x_a) = \widehat{\tau_a}\,(l_a) \qquad \text{(ii)}$$

If we interpret the static flow value $x_a$as flow rate over time on arc $a$is proportional to the inverse of $\widehat{\tau_a}(l_a)$For a time – dependent flow f, $l_a(\theta)$denotes the total amount of flow (i. e., load) on arc $a$at time θ. Similarly, $\widehat{\tau_a}\,(l_a(\theta))$ represent the flow over time with load-dependent transit times

In flow-dependent transit times model, at each point in time, the uniform speed on an arc depends only on the amount of flow or load which is currently on that arc. For the case of steady state flows which do not vary over time, the constant load of an arc can be determined by the constant flow rate on the arc, i.e., by the number of flow units traversing the arc per time unit. Therefore, the transit time of an arc is a function of its flow rate in this case.

A Model for Load-dependent Transit Times

The in-flow rate is measured at the tail and the outflow rate is measured at the head of an arc. The transit time on an arc depends on its current load, which is the amount of flow (i.e., number of cars) currently on that arc.

A flow over time on an arc a with time horizon T can be explained by its flow rate $f_a: (0, \mathrm{T}] \rightarrow \boldsymbol{R}^+$. An s − t flow over time with load − dependent transit times, which is given by flow rate functions $(f_a)_{a \in A}$, , satisfying the following constraints:

(i) for all $a \in A$ and $\theta \in (0, \mathrm{T}]$ such that $0 \leq f_a(\theta) \leq u_a$ (capacity constraints)

(ii) the total amount of flow that has arrived in v until time θ is an upper bound in the total amount of flow that has left v until time θ, for each node v ≠ s and every point in time $\theta \in (0, \mathrm{T}]$ (flow conservation constraints)

(iii) for all v $\in V \setminus \{s, t\}$, equality holds in (ii) for θ = T and; moreover l_a(T) =0 for all $a \in A$ (i.e., all flow must have arrived at the sink at time T).

Here we mention some consequences of model for load-dependent transit times:

Lemma -1: [Köhler and Skutella, 2002] If there exists a flow over time f with load-dependent transit times which sends D units of flow from s to t within time T, then there exists a static flow x of value at least D/T for the static flow problem stated above.

Theorem -3: [Köhler and skutella, 2002] Suppose that there exists a flow over time with (LDTT) that sends Q_0 units of s-t flow within time period T. Then a flow over time satisfying the same demand within time horizon at least 2T exists in the class of temporally repeated flows. Also, this solution can be computed polynomially within time (2 + ϵ) T for any given ϵ> 0.

Model of Inflow-dependent Transit Times

Flow over time with inflow-dependent transit times is an extension of the flows over time with fixed transit times. In flows over time with fixed transit times, transit times are fixed so that flow on arc a progress at constant speed. In inflow − dependent transit times, transit times experienced by an infinitesimal unit of flow on an arc is determined when entering this arc and only depends on the inflow rate at that moment of time. In the flows over time with inflow − dependent transit times, flow entering arc $a \in$ A at time θ arrives at head (a)at time $\theta + \tau_a(f_a(\theta))$,, where, $\tau_a : [0, u_a] \to \boldsymbol{R}^+$ is transit time function. In particular, the time of an arc only depends on the current flow rate. Since in time – dependent flow, we require that all arcs must be empty from time T, so for all arcs $a \in$ A and $\theta \in \boldsymbol{R}^+$ we have $\theta + \tau_a(f_a(\theta)) < T$ whenever $f_a(\theta) > 0$.In this case, flow conservation model is of the form

$$\sum_{a\in\delta^+(v)} \int_{0\le\theta<\zeta} f_a(\theta)\ d\theta - \sum_{a\in\delta^-(v)} \int_{\theta\ge0:\ \theta+\tau_a(f_a(\theta))\le\zeta} f_a(\theta)\ d\theta \le 0. \qquad (1)$$

for all $\zeta \in [0, T]$ and v $\in$ V− {s, t} and equality hold for all v $\in$ V− {s, t} at time ζ = T

The flow over time f satisfies the supply and demands if

$$\sum_{a\in\delta^+(v)} \int_{0\le\theta<\zeta} f_a(\theta)\ d\theta - \sum_{a\in\delta^-(v)} \int_{\theta\ge0:\ \theta+\tau_a(f_a(\theta))\le\zeta} f_a(\theta)\ d\theta = D. \qquad (2)$$

for v ∈{s, t}.The value of s-t flow over time f is given by

$$|f| = \sum_{a\in\delta^+(s)} \int_0^T f_a(\theta)\ d\theta - \sum_{a\in\delta^-(s)} \int_{\tau_a}^T f_a(\theta)\ d\theta. \qquad (3)$$

with flow-dependent transit time $(\tau_a)_{a \in A}$ and underlying path decomposition $(x_p)_{p \in P}$, the value of a temporally repeated flow f is given by

$$|f| = \sum_{p \in P} (T - \tau_p(x))x_p = T|x| - \sum_{a \in A} \tau_a(x_a)x_a. \qquad (4)$$

Example -2: For every arc a ∈ A, let T.$\tau_a : [0, u_a] \to \mathbf{R}^+$ and $\tau'_a : [0, u_a] \to \mathbf{R}^+$ denote transit time functions on arc a such that τ_a (x) ≤ τ'_a (x), for all x ∈ $[0, u_a]$. Then, a flow over time with inflow − dependent transit times $(\tau'_a)_{a\in A}$ and time horizon T naturally defines a flow over time with inflow − dependent transit times $(\tau_a)_{a\in A}$ and time horizon T.

Theorem -4: [Köhler and skutella 2002] Suppose that there exists a flow over time sending Q_0 flow units from s to d within time T for non-decreasing piecewise constant transit time functions. Then a temporarily repeated flow with (IFDTT) can be computed in strongly polynomial time that sends the same amount of flow from s to d within time horizon at most 2T.

Model of Static Flow Problem

In static maximum flow problem with bounded convex cost, the cost of flow x_a on arc a is $x_a\tau_a(x_a)$ and total cost must not exceed demand D. The static flow problem can be written as follows:

$$\text{Max} \quad \sum_{a\in\delta^-(\mathrm{t})} x_a - \sum_{a\in\delta^+(\mathrm{t})} x_a$$

$$\text{such that } \sum_{a\in\delta^-(\mathrm{v})} x_a - \sum_{a\in\delta^+(\mathrm{v})} x_a = 0 \text{ for all v } \in \mathrm{V} \setminus \{\mathrm{s}, \mathrm{t}\},$$

$$\sum_{a\in A} x_a\tau_a(x_a) \le \mathrm{D},$$

$$0 \le x_a \le u_a \text{ for all } a \in A.$$

Here $\delta^+(\mathrm{v})$ and $\delta^-(\mathrm{v})$ denote the set of arcs leaving and entering node v, respectively.

Corollary -2: A static flow f is maximal if and only if there is no flow augmenting path with respect to f.

Time Expanded Graph

Given graph G = (V, A) with integral transit times on the arcs and an integral time horizon T, the T-time expanded graph of G, denoted by G(T), is obtained by creating T copies of V, labelled V(0) through V(T-1), with the θth copy of node v denoted v$(\theta), \theta \in \{0, 1, \dots, \mathrm{T}-1\}$. For every arc a = (v, w) ∈ A and $0 \le \theta \le \mathrm{T} - \tau_a$ there is an arc $a(\theta)$ from v(θ) to w $(\theta + \tau_a)$ with same capacity as arc a. It storage of flow at node v ∈ V is allowed, we include an infinite capacity holdover arc from v(θ) to v$(\theta$ + 1), for all $0 \le \theta \le$ T -1, which models the possibility to hold flow at node v.

Theorem -5: The dynamic flow in the given network is equivalent to the static flow in the corresponding time-expanded network.

Model of Generalized Time Expanded Graph

Carey and Subrahmanian (2000) consider a generalized time-expanded graph for flow-dependent transit times. They assume that each arc a = (v, w)has a piecewise linear transit time function τ_a given by break points $0 = x_0 < x_1 < \dots < x_l$, where $\tau_a(x_i) = \mathrm{i}, \mathrm{i} = 0, \dots, l$. They introduce a copy $(\mathrm{v}(\theta), \mathrm{w}(\theta + i))$ with capacity x_i, for each point in time θ and each transit time $i = 0, \ldots, T-1-\theta$. They consider static network flow formulation in this generalized time − expanded graph with additional bundle constraints linking the flow of the arcs $\left(v(\theta), w(\theta+i)\right)$, i $0, \ldots, T-1-\theta$. They derive necessary conditions which guarantee that at most two neighbouring arcs $(v(\theta), w(\theta+i))$ and $(v(\theta), w(\theta+i+1))$ carry flow. The generalized time expanded graphs can be explained as follows:

(a) The Fan graph

The fan graph G^F is generalized time expanded graph in which transit times indirectly depend on the flow rate. The fan graph G^F is defined on the set of nodes $\{v_\theta: v \in V, \theta = 0, 1, \ldots, T-1\}$ The fan consists of capacitated horizontal arcs and uncapacitated arcs pointing upwards. The capacities of the horizontal arcs try to control the distribution of flow according to the transit time function [Köhler & Skutella, 2002]. In fan graph τ_a^s represent the step function character of transit time function τ_a of an arc a where τ_a is a piecewise constant, non − decreasing and left continuous function with only integral values. Fan graph is shown in figure − 1. In figure 1(a)the flow is at most x_1, x_2 and x_3, for example suppose 2, 4 and 6 respectively, with transit time τ_a as 1, 3 and 6 respectively of an arc a. Figure 1(b) shows the fan at θ = 0 consisting of capacitated horizontal arcs and uncapacitated arcs pointing upward. Figure 1(c) shows the fan graph of arc a as a time expanded graph

(b) The Bow graph:

The fan graph is defined as the time-expansion of the bow graph. Bow graph captures in- and out-flow dependent transit times. This can be achieved by introducing regulating arcs also at the head node of each arc. Flows over time bow graph model do not constitute a relaxation of flows over time with inflow-dependent transit times in G. Fan graph may become very large but the bow graph is the time expansion of smaller graph. The bow graph $G^B = (V^B, A^B)$ arises from the original graph by expanding each arc $a \in$ A according to its transit time function.

In a bow graph [Köhler & Skutella, 2002], every arc $a \in A^B$ has capacity u_a and a constant transit time $\tau_a \in \boldsymbol{R}^+$. For arc $a \in$ A, suppose $0 = x_0 < x_1 < \ldots < x_k = u_a$ are break points and corresponding transit times are $\tau_1 < \tau_2 < \cdots < \tau_k$. Flow entering rate $x \in (x_{i-1}, x_i]$ needs τ_i time units to traverse arc a. In bow graph, arc a is divided into two arcs:

bow arcs denoted by $b_1, \ldots, b_k$ and regulating arcs denoted by $r_1, \ldots, r_k$. The bow arcs b_i are uncapacited and they represent all possible transit times τ_i of arc a for i = 1, … , k. The capacity of regulating arcs r_i is set to x_i, i = 1, … , k and their transit times are zero which limit the amount of flow entering the bow arcs. The set of bow arcs and regulating arcs associated to an arc $a \in$ A is denoted by A_a^B and which refer to A_a^B as the expansion of arc a. Figure 2(b)represents the expansion of an arc a according to transit time function using bow graph.

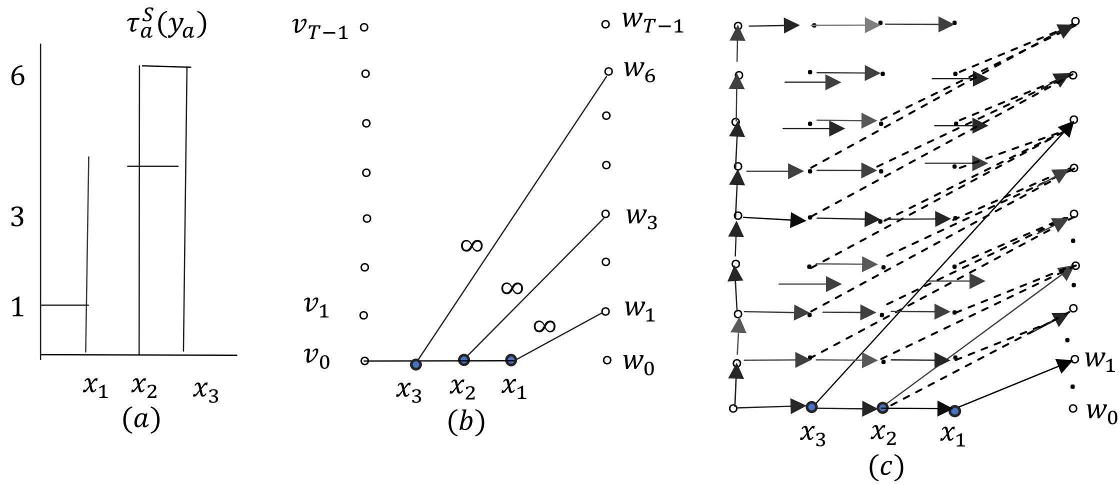

Figure-1: Definition of fan graph, expansion of a single arc a

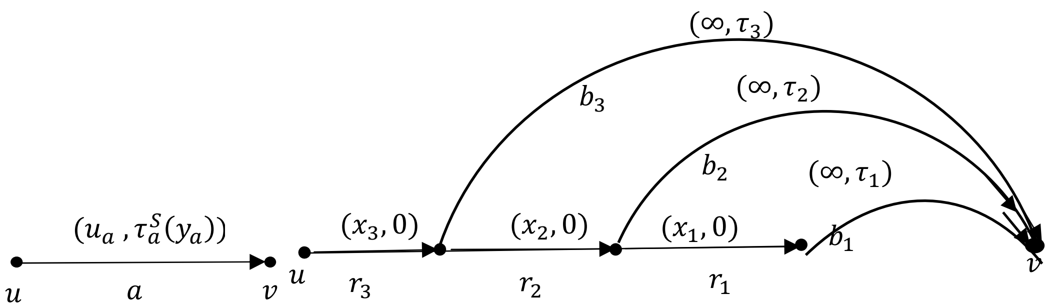

Figure-2 : Definition of the bow graph, expansion of a single arc

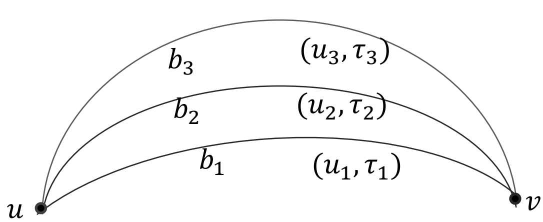

Figure-3: An expansion of single arc a = (u, v) according to modified bow graph

Relaxation Property of the New Model of Modified bow graphs

Any flow over time f with inflow-dependent transit times $(\tau_a^S)_{a \in A}$ in G with time horizon T and cost C can be interpreted as a flow over time f^B (with constant transit times)in G^B with same time horizon T and same cost C as follows: if, in the original graph G, flowis entering arc $a \in A$ at time θ with flow rate $f_a(\theta)$, then, in the bow graph, this flow is sent onto the bow arc $a \in A_a^B$ representing transit time $\tau_a^s(f_a(\theta))$.

The bow graph, denoted by $G^B = (V^B, A^B)$, is defined on the same vertex set as G, i. e., $V^B = \mathrm{V}$, and is obtained by creating several copies of an arc, one for possible transit time on the arc. Thus arc a is replaced by creating m parallel bow arcs $b_1, b_2, \dots, b_m$. The transit time of bow arc b_j is τ_j and capacity u_j for j = 1, 2, … , m. We denote the set of bow arcs corresponding to arc a by A_a^B and refer to A_a^B as the expansion of arc a. The cost coefficient of every arc e A_a^B are identical to those of arc a, i. e., $c_e = c_a$. For every arc e $\in A_a^B$, let a (e) denote the original arc a, which is shown in figure-3.

The main difference between new modified bow graph and previously defined bow graph is as follows: In modified model we omit the regulating arcs which, in previous model, limit the amount of flow entering the bow arcs. In particular, all bow arcs representing the same original arc share capacity. In the modified model, capacities are directly assigned to the bow arcs. They no longer share capacities; moreover we include arc costs in the new model.

Conclusion

In this paper, we have studied different models of flow over time and flow dependent transit times with load-dependent and inflow-dependent transit times. Ford and Fulkerson's time expanded graph is related to flows over time with constant transit times but the fan graph is a generalization of the time-expanded graph which preserves the property of flow over time with flow-dependent transit times by standard network flow techniques. The fan graph is only a relaxation for the setting of inflow-dependent transit times where flow units entering and simultaneously might travel the arc at different paces. Since 2+ ϵ approximation algorithm can be used for the quickest s-t flow problem in the setting of both load-dependent and inflow-dependent transit times.

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