Perpendicular Trisectors and Base Trisectors in Triangles and Quadrilaterals

Prakash Pant, Hem Lal Dhungana, Sudip Rokaya
Central Campus, Mid-Western University
hem.dhungana@mu.edu.np
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ABSTRACT

The four centers of a triangle- namely centroid, circumcenter, orthocenter, and incenter are well known. These are characterized by some famous lines known as median, perpendicular bisector, altitude, and angle bisector. Bisectors, whether side bisector or angle bisector, are commonly seen. Trisectors, both angle trisectors and base trisectors, also have interesting properties, including Morley's theorem (property of angle trisectors). The properties of other trisectors have been overlooked and this paper aims to highlight them. The research objective of this paper is to review the properties of base trisectors, perpendicular trisectors and n-section in Quadrilateral. We explore the properties and proofs and finally get a generalization to the quadrilateral trisection theorem. The major findings of this research work are the discovery of similar triangles in trisectors, leading us to propose i) base trisectors theorem, ii) perpendicular trisectors theorem, and iii) quadrilateral n-section theorem.

Keywords: Morley's Theorem, Trisection, Trisectors.

Introduction

Beginning of a Journey

Geometry has always offered much to explore. While going through the book 'Geometry Revisited' (H.S.M., 1989 & Greitzer, S. L., 2016) we came across this mathematically profound theorem which is known as Morley's Theorem. It is a theorem of angle trisectors of a triangle. The numerous proofs of this theorem exist; including some nice proofs on a paper (Chepmell, 1922) have further studied the Hyperbolic Morley's Triangle and Morley's Tetrahedran (Cundy, 1984). The other types of trisections discussed in this paper have never appeared in any paper. Well, the properties of bisectors are well known and have been taken from Lemmas in Olympiad Geometry (Greitzer, 2016). A Beautiful Journey through Olympiad Geometry (Evan, 2016) and Euclidean Geometry in Mathematical Olympiads (Sam, Cosmin, & Stefan, 2016).

All-around Bisections

For the median, i) It divides the triangle into two equal halves ii) Three Medians of a triangle are concurrent at an interior point in the triangle which we call centroid (usually denoted by G). ii) Centroid divides median in the ratio 2:1 iii) Centroid of a plane object is the center of mass of the object and so on.

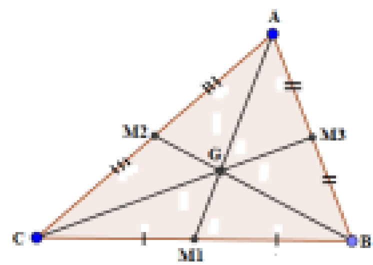

Figure 1: Medians

For altitude, i) Three altitudes of a triangle are concurrent at a point which we call orthocenter (H). ii) Reflection of orthocenter around a side of a triangle lies in the circumcircle of the triangle iii) Orthocenter is also the radical center for three non-coaxial circles having cevians for diameters in a triangle.

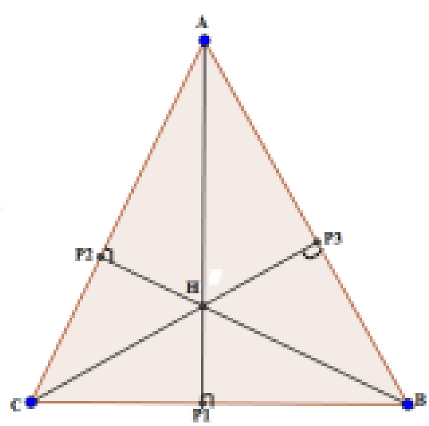

Figure 2: Altitudes

For perpendicular bisectors, i) Three perpendicular bisectors of sides are concurrent at a point which is equidistant from all three vertices and hence being the center of the circumscribed circle (O). ii) Diameter of the circumcircle times sine of an angle of a triangle gives the measure of the opposite side of the triangle. iii) The orthocenter and the circumcenter in a triangle are isogonal conjugates. iv) And, not to forget that orthocenter, centroid and circumcenter lie in the same straight line.

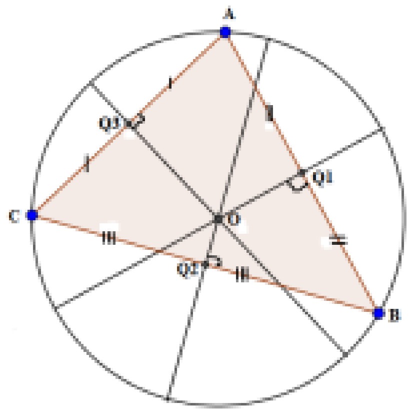

Figure 3: Perpendicular Bisectors

For angle bisector, i) Three angle bisectors of a triangle are concurrent at a point that is equidistant from each of the sides of the triangle and hence being the center of the inscribed circle (I). ii) Product of the radius of the inscribed circle and the semi-perimeter is the area of the triangle. iii) And, the famous Steiner Lehmus theorem, if two angle bisectors are equal, the triangle is isosceles.

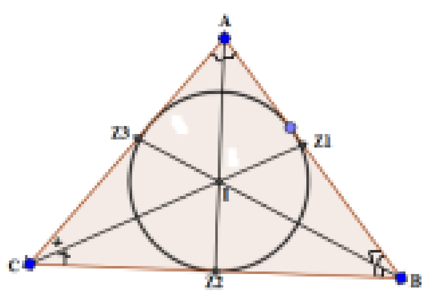

Figure 4: Internal Angle Bisectors

There are exterior angle bisectors too which define the ex-radius and gives us this interesting formula: i) $r \times r_a \times r_b \times r_c = \Delta^2$. All of these properties were no doubt different but with a common idea i.e. bisection. The notion that bisection gives us great ideas was well established until we met up with Morley's Trisectors theorem from where we started believing TRISECTION can give us even better results than this.

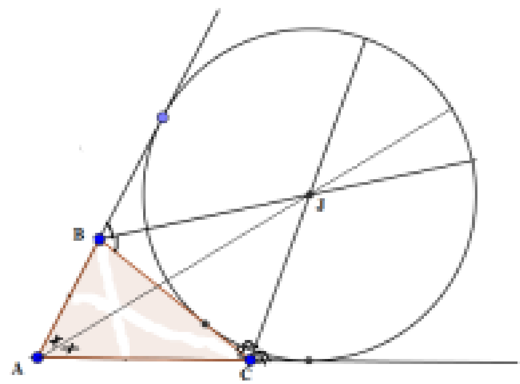

Figure 5: External Angle Bisectors

On Search of Trisections

So, we headed on researching trisectors and trisections in a triangle. During the research, we found, “Yes, there are important properties in not just angle trisectors as given by Morley but also in base trisectors, and perpendicular trisectors.” Furthermore, an important trisection property was discovered in quadrilaterals which could be generalized for nth-section too.

Research Objective

According to the problem stated above, the following are the research objective of the paper: (i) to review the properties of base trisectors, perpendicular trisectors and n-section in Quadrilateral.

Methodology

This research paper is heavily based on paperwork and Geo-Gebra. The primary source of information for the geometrical theorems used in this research work has been a popular book- *Geometry Revisited*. Paper works have been the major way to discover trisection properties and Geo-Gebra to verify them. Figures were made with Geo-Gebra.

Let us begin with the well-known Trisection theorem ” Morley’s Theorem”.

Morley's Theorem

Statement of Morley's Theorem

"The points of intersection of the adjacent trisectors of the angles of a triangle are the vertices of an equilateral triangle."

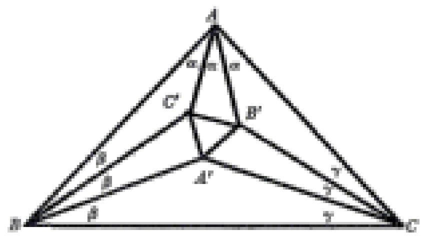

Figure 6: Morley's Theorem

Here, in figure 6, pairs of adjacent trisectors are (AC', BC'), (BA', CA'), (CB', AB'). And the intersection of these three pairs forms the equilateral triangle △ A′B′C′.

Moving towards Base Trisectors and Perpendicular Trisectors from Morley's Angle Trisectors

Well, Morley's Theorem was a motivation for us to move toward the other trisection theorems. It gave us a feeling that "Maybe there are more of such coincidences for us to find in trisections". And yes, with some paperwork and Geo-Gebra, we did notice and observe some notable properties in them which are presented below.

Base Trisectors Theorem

Introduction to First Neighbor Points and Second Neighbor Points

First Neighbor Points: In figure 7, the pair of points that are closer to a specified vertex in adjacent pairs of arms are first neighbor points. For eg:- (B_2, C_1),(C_2, A_1), (A_2, B_1) are three pairs of first neighbor points.

Second Neighbor Points: In the figure 7, pair of points that are farther from a specified vertex in adjacent pairs of arms are second neighbor points. For eg: (B_1, C_2), (C_1, A_2),(A_1, B_2) are three pairs of second neighbor points.

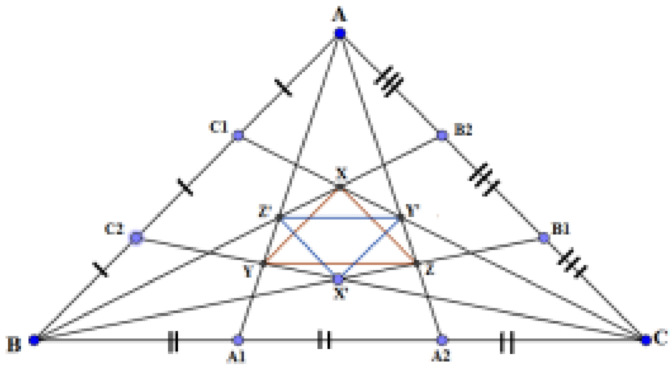

Figure 7: Base Trisectors Theorem

Statement of Base Trisectors Theorem

"The points of intersection of the base trisectors standing on first neighbor points and passing through the vertex of the triangle are the vertices of a triangle similar to the original triangle." . . ... (i)

as well as

"The points of intersection of the base trisectors standing on second neighbor points and passing through the vertex of the triangle are the vertices of a triangle similar to the original triangle." . . . . (ii)

According to (i), $\triangle XYZ \sim \triangle ABC$ and

According to (ii), $\triangle X'Y'Z' \sim \triangle ABC$

Proof of Base Trisectors Theorem

Proof:

Look at $\triangle XB_2C_1$ and $\triangle XBC$. As $\frac{AC_1}{C_1B} = \frac{AB_2}{B_2C} = \frac{1}{2}$, thus by thales theorem, $B_2C_1 \parallel BC \Rightarrow \triangle XB_2C_1 \sim \triangle XBC$

Then,

$$\frac{C_1X}{CX} = \frac{B_2X}{BX} = \frac{B_2C_1}{BC} = \frac{1}{2}$$

Similarly, from other vertices,

$$\frac{C_1Y}{CY} = \frac{A_1Y}{AY} = \frac{A_1C_2}{AC} = \frac{1}{3}$$

And

$$\frac{C_1Y}{CY} = \frac{A_1Y}{AY} = \frac{A_1C_2}{AC} = \frac{1}{3}$$

Thus, By Thales theorem, $XY \parallel AB$, $YZ \parallel BC$ and $XZ \parallel AC$. So, $\triangle XYZ \sim \triangle ABC$. Similar, reasoning holds for $\triangle X'Y'Z' \sim \triangle ABC$. Noting the importance of Morley's Theorem in higher Euclidean Geometry, we believe this theorem should also be of great importance.

Perpendicular Trisectors Theorem

Statement of Perpendicular Trisectors Theorem

In the figure 8,*" The points of intersection of the perpendicular trisectors standing on first neighbor points in a triangle are the vertices of a triangle similar to the original triangle." ...... (vi)*

Also,*" The points of intersection of the perpendicular trisectors standing on second neighbor points in a triangle are the vertices of a triangle similar to the original triangle." .... (vii)*

Along with this,*" Also, the circumcenter of the newly formed triangles is*

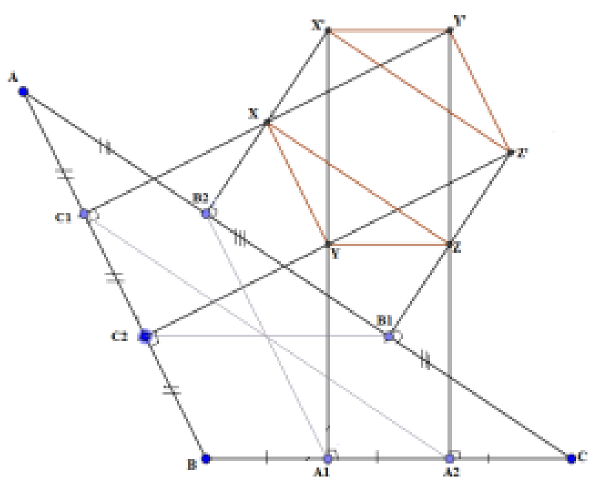

Figure 8: Proof (vi) and (vii): Perpendicular Trisectors Theorem

Same as the circumcenter of the original triangle. "....... (viii)

According to (vi), $\triangle XYZ \sim \triangle ABC$ and

According to (vii), $\triangle X'Y'Z' \sim \triangle ABC$.

Proof of Perpendicular Trisectors Theorem

Proof for (vi)& (vii):

The proof of Perpendicular Trisectors Theorem is kind of visual yet simple.

Take $\triangle A_2BC_1$ and $\triangle CA_1B_2$. Then, Since $BC_1 \parallel A_1B_2$ and $C_1A_2 \parallel B_2C$, $\triangle C_1BA_2 \sim \triangle B_2A_1C$. And, as $BA_2 = A_1C$, $\triangle C_1BA_2 \cong \triangle B_2A_1C$.

Next, it can be clearly seen that triangle A_1B_2C is just 1 unit translation to the right of A_2C_1B. And, thus point Y which is the circumcenter of triangle A_2C_1B also shifts by 1 unit right to give point Z which is the circumcenter of triangle A_1B_2C. So, $YZ = A_1A_2 = \frac{BC}{3}$ and $YZ \parallel BC$.

Similarly, XY $\parallel$ AB and XZ $\parallel$ AC. So, $\triangle XYZ \sim \triangle ABC$ reduced by factor of $\frac{1}{3}$.

Similar proof holds for $\triangle X'Y'Z' \sim \triangle ABC$.

Proof for (viii):

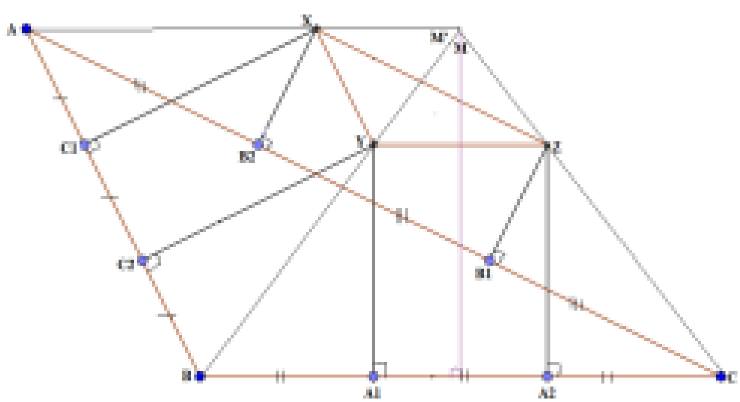

Figure 9: Proof (viii): Perpendicular Trisectors Theorem

Now, we will try to prove that circumcenter of the newly formed Trisectors triangle XYZ is same as the circumcenter of the original triangle ABC.

First let us join CZ and BY and extend them to meet at M. Since $YZ \parallel BC$ and $YZ = A_1A_2 = \frac{1}{3}BC$, so $\frac{MZ}{ZC} = \frac{MY}{YB} = \frac{1}{2}$. Next, let BY and AX meet at M'. Then $\frac{M\prime Y}{YB} = \frac{M\prime X}{XA} = \frac{1}{2}$. But MY/YB

=> M=M' (since both M and M' are lying on the same line BY).

Thus, we have now proved that extended line AX, BY, CZ all meet at M. Now, it remains to prove that M is not any other point but O.

First, we note that, triangle BYA_1 and CZA_2 are congruent from which BY = CZ which implies YM = ZM. Thus, we see that both $\triangle MYZ$ and $\triangle MBC$ are isosceles triangles. This implies that M is the point from where the perpendicular bisector of both BC and YZ will pass through.

Furthermore, it can be shown in a similar way that M is the point from where 12

the perpendicular bisector of both XY and AB will pass through. Thus, M is nothing but the intersection of perpendicular bisectors of both triangles $\triangle XYZ$ and $\triangle ABC$. Thus, being the circumcenter of triangle $\triangle XYZ$ and $\triangle ABC$.

Thus, $\triangle XYZ$ and $\triangle ABC$ have the same circumcenter M and now we can safely name it O.

It can well be tried in a similar fashion to prove that triangle X'Y'Z' also has the same circumcenter.

A complete Picture of Perpendicular Trisectors Theorem

This section offers a complete picture of the Perpendicular Trisectors theo rem.

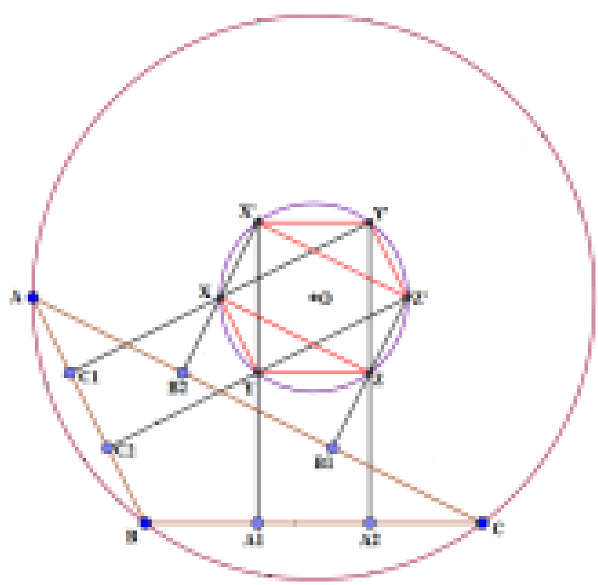

Figure 10: A complete picture of Perpendicular Trisectors Theorem

In the figure 10, as mentioned earlier, $\triangle ABC$, $\triangle XYZ$ and $\triangle X'Y'Z'$ are similar and circumcenter of all triangles $\triangle ABC$, $\triangle XYZ$ and $\triangle X'Y'Z'$ is O.

These are three worthy trisection equivalents of Angle Bisector, Median and perpendicular bisector. After all of these triangular results, the upcoming section presents a trisection theorem on quadrilateral.

Quadrilateral Trisectors Theorem

Statement of Quadrilateral Trisectors Theorem *“Lines joining opposite points of trisection trisect each other ”........(i)*

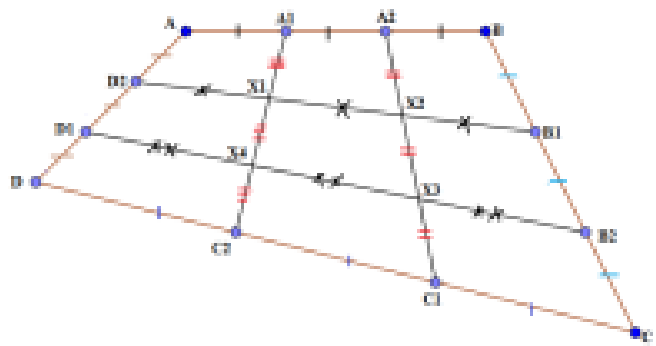

Figure 11: Quadrilateral Trisectors Theorem

which you can relate to the other theorem:

“Lines joining opposite points of bisection bisect each other” or more com monly stated as *”The lines joining adjacent midpoints of a quadrilateral form a parallelogram.”*

According to (i) and figure 11,

if $AA_1 = A_1A_2 = A_2B$, $BB_1 = B_1B_2 = B_2C$, $CC_1 = C_1C_2 = C_2D$, $DD_1 = D_1D_2 = D_2A$,

it implies $D_2X_1 = X_1X_2 = X_2B_1$, $D_1X_4 = X_4X_3 = X_3B_2$ and $A_1X_1 = X_1X_4 = X_4C_2$, $A_2X_2 = X_2X_3 = X_3C_1$

Quadrilateral Trisection Theorem is also one of the important properties

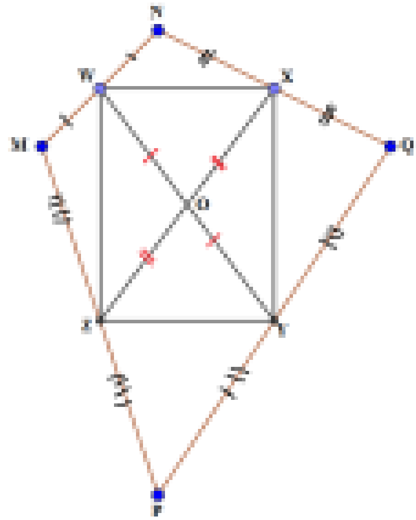

Figure 12: Quadrilateral Bisectors Theorem

in trisection that exists. Not just for bisection and trisection, this one holds for any number of divisions made on a side of the quadrilateral.

Let's say for the one as given in the Figure 13.

We can prove this by induction on one piece of the quadrilateral.

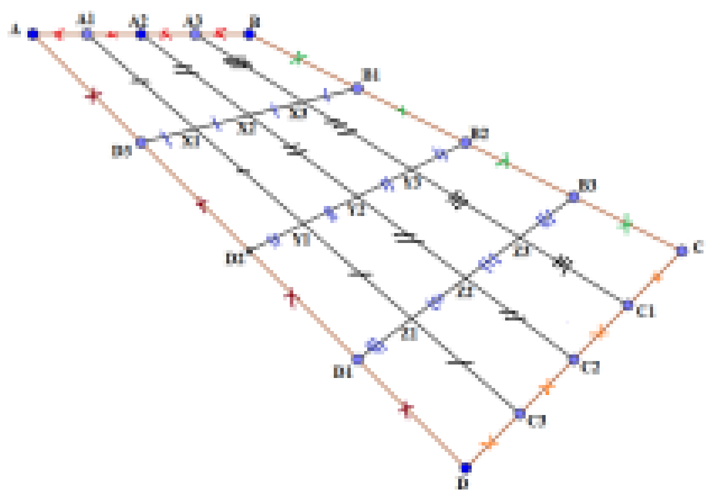

Figure 13: Generalization of the Theorem

Applications

Trisection Theorems can have applications in complex mathematical problem solving as well as in Math Olympiad problems. It can also have effective applications in geometric engineering during real-life modeling of the surroundings. It also helps to further research and human creativity in the field of geometry.

Conclusions, Suggestions and Limitations

Properties of bisection are well known to students of Mathematics. But, tri section is still a bit unknown. This paper has intended to unleash all the potentials of Trisectors and the equivalent theorems. Through the discussion and proofs, we believe we have successfully accomplished our aim of making the Trisectors theorem as accessible as the bisector theorem.

These theorems, if they became standard theorems in the mathematical community, could further progress in solving complex geometry problems and encourage deeper mathematical understanding. We thus argue that these are useful and could be made a requirement for basic Geometry. It could be a powerful way of exploring new coincidences and results in mathematics.

The trisection theorems, which ideally would be well known, have been once again reviewed and stated as theorems. This paper has also offered a simple extension to quadrilaterals and suggested application of the theorems to complex geometry problems. However, this paper does have some limitations. It is limited to the trisection theorems for triangles only; trisection was not explored for quad-sections and penta-sections or quadrilateral and pentagons. It was limited to the researchers within a school.

Further Recommendations

We have progressed from bisection to trisection in this research, and the equivalent theorems can be researched for quad-sections and penta-sections too. Eventually, one could generalize equivalent theorems for n-section and propose a powerful geometry theorem. These things will certainly help us to broaden the understanding of our geometry.

Another kind of progress is moving from triangle to Quadrilateral for bisection theorems and then to pentagon, Hexagon and generalizing the bi section theorem for n-gon. We could also generalize the trisections theorems, quad-section theorems, and many more for n-gons. Achieving this feat can completely revolutionize how we see geometry.

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Author Contribution

Sudip Rokaya and Prakash Pant were involved in the major paperwork and observations. For the proof., Assis. Prof. Hem Lal Dhungana, (Mid-West University, Surkhet, Nepal) wrote the final drafts. All authors were involved in writing the main manuscript. All authors reviewed the manuscript.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper. The research was conducted in an unbiased and objective manner, and the authors have no personal, financial, or professional relationships that could potentially influence the results or conclusions presented in the paper.

Data Availability Statement

As this paper is a mathematical study, there are no experimental or empirical data sets involved. Therefore, there is no data availability statement applicable to this research. The study is purely theoretical and analytical in nature, focusing on mathematical concepts and proofs related to perpendicular and base trisectors in triangles and quadrilaterals. The results and findings are based on mathematical derivations and logical reasoning.

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