Topological Quasi-vector Space and It's Characteristics

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ABSTRACT

This article discusses fundamental concepts and examples related to topological vector spaces. The primary goal of this article is to investigate topological quasi-vector spaces and their properties. This research endeavor begins with the discovery of various features of (topological) quasi-vector spaces that remain invariant under some kind of transformation. Any such transformation should be built in such a way that it can preserve the full quasi-vector spaces - structure. The features of quasi-vector space are used to characterize specific (topological) quasi-vector spaces that play an important role in this article and play a vital function in this research.

Keywords: Homomorphism, Linear Spaces, Order-morphism, Quasi-vector space, Topological vector space, Topology.

Introduction

Topological vector spaces, particularly Banach and Hilbert spaces, are crucial in various mathematical disciplines, including functional analysis, physics, engineering, and other scientific fields, providing a platform for studying functional behavior. Aseev (1986) defined quasi-linear spaces as classical linear and non-linear spaces with subsets and multi-valued mappings. On these spaces, he established norm, quasi-linear operators, and functional analysis, encouraging numerous writers to produce more findings. The set K_c(E of all convex compact subsets of a normed space E is a good example. This is studied with break analysis. Intervals are outstanding instruments dealing with global optimization problems and supplementing recognized plans (Medar, et al. 2005) Markow's technique and Aseev's treatment are key approaches for dealing with quasi-linear spaces. Aseev's treatment provides the best foundation for linear functional analysis, extending set-valued algebra and analysis. The primer of normed quasi-linear spaces with bounded operators pays to this field (Markov, 2000 & 2004). Topological vector space $\mathbb{K}$ corresponds to a vector space E that constructed with a topology that permits the two continuous mappings below, where $\mathbb{K}$ ×E and E×E are shaped using the products of the products of the various topologies:

(1) **Addition of vectors:** E × E → E as (x_1, x_2) →x_1 + x_2,

(2) **Multiplication of vectors**: $\mathbb{K}$ × E → E as [k, x] → k x,

If a topology occurs on E, it is said to as compatible with the vector space construction. With topology τ, a topological vector space E is embodied by the symbol (E, τ). Two topological vector spaces over the same field are isomorphic for a continuous linear one-to-one mapping from one to the other and its inverse mapping is again continuous.

A topological vector space's dimension (E, τ) is alike to E. The continuity of the mapping suggests that we may reconstruct the TVS (E, τ) provided we are aware of a basic system of neighborhoods of zero. Translations do not affect any topological vector space (Aseev, 1986).

An improper of neighbors and a ∈ E is the collection of the type a + v. Accordingly, in the majority of it's applications, defining a topological vector space's topology just requires accepting a base of neighborhoods of zero (Markov, 2000 & 2004).

A compact set's scalar manifold is also compact, as is the sum of two compact sets. If the collection C(X) of all non-empty compact subsets of X exists, for following findings are applicable (Markov, 2000 & 2004).

(1)

$$\text{(i) } A \subseteq B \Rightarrow \alpha A \subseteq \alpha B$$

$$\text{(ii) } (\alpha + \beta)\cdot A \subseteq (\alpha A + \beta A$$

The collection C(X) has been shown to possess structural beauty through the use of the aforementioned addition, scalar multiplication, and intrinsic set inclusion order, despite its simplicity. Jana and Mitra introduced "quasi-vector space," a novel structure with compatible semi-group and partial order structures and scalar multiplication. By embedding each vector space as its subspace, generalizes it. "Topological Quasi-vector Space" (Jana & Mitra, 2004) is the name given to this novel topological algebraic structure in which topology was applied and create to be consistent with its characteristic forms.

Research Problem

Topological quasi-vector spaces are an intriguing area of study in functional analysis and topology. These spaces generalize the concept of vector spaces and incorporate topological structures, enabling a more comprehensive analysis of continuity and convergence in infinite-dimensional settings. This article explores the fundamentals of topological quasi-vector spaces, their defining properties, and some significant characteristics that distinguish them from other topological and vector spaces.

Objective and Methodology

This article explores topological quasi-vector spaces and their properties, a topic of global and local relevance in functional analysis. It discusses various analytical methodologies, reads works on these spaces, and explores mathematical magazines and research papers.

Topological Vector Spaces

Topological vector spaces are having the following fundamental ideas and examples:

Topology in Linear Space

A linear vector space is a topological vector space. Therefore, start by learning the fundamental ideas pertaining to topological and linear spaces independently. Let $\mathbb{K}$ be an algebraic field and C be the complex number field.

$[E \times E] \rightarrow E$ defines $(u, v) \rightarrow u + v$,

and $[\mathbb{K} \times E] \rightarrow E$, states $(\lambda, v) \rightarrow \lambda v$

(2)

where $\mathbb{K}$ is algebraic field and C is complex.

Equation (2) Satisfied the below conditions:

(i) $u + v = v + u$ for all $u, v \in E$.

(ii) A unique element $0 \in E$ exists for $v + 0 = V$.

(iii) $v + (-v) = 0$, for $v \in v$

(iv) $\lambda (u + v) = \lambda u + \lambda v$, $\lambda (\mu v) = (\lambda \mu) v$,

$(\lambda + \mu) u = \lambda u + \mu u$ and $0 v = \lambda 0 = 0$, for all $u, v \in E$, $\lambda, \mu \in \mathbb{K}$,

Let $\mathbb{K}$ is scalars field and $\mathbb{K} \subset C$, given two vector spaces and let E and F.

$$A: E \rightarrow F \quad (3)$$

is a mapping if $A (\lambda u + \mu v) = \lambda A(u) + \mu A (v)$. A set Ker A:$\Rightarrow$ A^{-1} is called the kernel and set R is range is $A := A(E)$ (Bogachev & Smolynov, 2017).

A collection τ of subsets of X is a topology if

(i) X, $\emptyset$ are in X.

(ii) $V_1 \cap V_2$ are in τ for V_1, V_2 are in τ.

A pair (X, τ) is a topological space on X. A subset of a topological space X is open complement and closed (Bogachev & Smolynov, 2017).

(i) X, $\emptyset$ are in $\mathcal{F}$.

(ii) If F_1, F_2 are in $\mathcal{F}$, then $F_1 \cap F_2$ also in $\mathcal{F}$.

Metric space (M, d) is a pair and

$$d: M \times M \to [0, +\infty) \quad (4)$$

is a function to as a metric.

(i) d (a, b) = d (b, a) for a, b in M, d (a, b) = 0 for a = b,

(ii) d (a, c) ≤ d (a, b) + d (b, c) for a, b, c ∈ M.

Norm ||·|| on a metric space defined as

$$d\ (x, y) = \|x - y\| \quad (5)$$

The definition of (t. v, s) more precisely requires the following:

$$X \times X \to X$$

$$(X, Y) \to x + y$$

$$\mathbb{K} \times X \to X$$

$$(\lambda, x) \to \lambda x$$

Isomorphism X → Y is also a homeomorphism (bijective, linear, continuous, and inverse continuous).

Quasi- linear Space (QLS)

The $\varkappa_x$ is a topological space X characterizes the neighbors of a $x \in X$ and X be a topological vector space (TVS,) for G ⊂ X, $x \in X$.

If $G - x \in \varkappa_x$ and $x - G \in \varkappa_x$, then $G \in \varkappa_x$. TVS operate on this localization concept (Wilansky, 1978).

Set X is characterized a quasi-linear space (QLS) for x, y, z, $u \in X$, and scalars α, β satisfy the following conditions:

$x \leq x$,

$x \leq z$ if $x \leq y, y \leq z$

$x = y$ if $\leq$, $y \leq x$

$x + y = y + x$

$x + (y + z) = (x + y) + z$,

For, $0 \in X$, $x + 0 = x$

α , $(\beta . x) = (\alpha . \beta)$. x (6)

$\alpha(x + y) = \alpha . x + \alpha . y$,

1.x = x,

0. x = 0

$(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$,

$x + z \leq y + v$ i f $x \leq y, z \leq v$,

$\alpha \cdot x \leq \alpha \cdot y$ i f $x \leq y$

Inclusion relation "⊆," is an algebraic sum,

$A + B = \{a + b : a \in A, b \in B\}$ (7)

and

$\lambda A = \{\lambda a : a \in A\}$ (8)

$A + B = \{a + b : a \in A, b \in B\}$ (9)

∴ $K_C(E)$ $\{A \in K(E) : \text{convex}\}$.

0 is the minimal elements of X to its partial order "≤ " ; these elements are called as one order elements of X,

$x = 0$ if $x \leq 0$. (10)

Definition1. If + x' = 0, for x' ∈ x is an inverse of a $x \in$ X. It is unaccompanied for an inverse element. An element x is considered regular if it has an inverse; otherwise, it is singular.

Lemma 1. If an inverse element $x' \in X$ and the distributive criteria grip, the partial ordering of equal, and X is a linear. If $x' = -x$ gives $x - x = 0$ are the elements of x (Aseev, 1986).

Definition 2. If X is a quasi-linear space with the same partial ordering on X, and $Y \subseteq X$, then Y is devoted as subspace of X.

Theorem 1. If X, Y is a subspace of X,

$x, y \in Y, \alpha, \beta \in \mathbb{R}$, then $x +\ , y \in Y$.

Proof.

Assume X and Y is a subspace of X. Assuming that each element x in Y has an inverse element $x' \in Y$. Lemma 1describes that the partial ordering on Y is regular. In this case, the distributive requirements on Y hold for Y is a linear subspace of X.

Let QLS be X, and let Y be a subspace of X. Assuming each x in Y has an inverse element $x' \in Y$, the partial ordering on Y is regular by the equivalence according to Lemma 1. Y is a linear subspace of X, the distributive conditions on Y hold in this instance.

Definition 3. X_r and X_s are regular element in X and each singular element in X.

Theorem 2. If X_r, X_d in X then $X_s \cup \{0\}$ is a subspace.

Proof.

Let X_r is a subspace for $\lambda x' + y'$ is the reverse of $\lambda x + y$. Also let $X_s \cup \{0\}$ is a subspace of X for $x, y \in X_s \cup \{0\}$ and $\lambda \in \mathbb{R}$ then $x = y = 0$. If $x \neq 0$, and $(x + \lambda y) \notin X_s \cup (0)$ gives $(\ + \lambda y) + u = 0$ for some $u \in X$ and $x + (\lambda y + u) = 0$ and so $x' = \lambda y + u$ gives $x \in X_r$

Clearly, it gives, $y \in X_r$ for $y \neq 0$. There is inconsistency,

$x + \lambda y \in X_s \cup (0)$.

The proof for X_d alike (Markov,2000).

X_r, X_d and X_s U $\{0\}$ are subspaces of X,

Lemma 2. Every regular element is negligible for a quasi linear space.

Proof. We have to prove that $y \leq x \implies x \in X_r$ for y = x

If $y \leq x$ then adding x' in both sides

$y + x' \leq\ + x' = 0 \Longrightarrow y + x' \leq 0$.
(11)

If from minimality, $y + x' = 0$ gives the inverse element, y = x.

The standard way of study the topological vector space Z is basically two realities:

Translation $(x \mapsto a + x, a \in Z)$
(12)

Dilation $(x \longmapsto \alpha x, \alpha \neq 0)$

are homeomorphisms from Z onto itself. Using these homeomorphisms, it becomes sufficient to think an important system of neighborhood at the origin "θ" only, for, this system of neighborhood at θ can describe a workable neighborhood system at any point of Z. Then the whole analysis on Z becomes concentrated locally at θ. Sorry to say, such a well-established process of examining this space fails to develop a theory. (Markov, 2000 & 2004).

A translation $x \mapsto a + x$ although is continuous for any $a \in X$ (a topological qvs), fails to be a homeomorphism unless $a \in X_0$.

In fact, if $a \in X \setminus X_0$ then $x \mapsto a + x$ (13)

is neither injective nor subjective. Again, although enlargement is still a homeomorphism on topological qvs, it can only "resize" a set. It can't move an open set to some other place. These are the immediate hurdles one must face while trying to develop a theory for topological qvs.

Another barrier that distinguishes topological qvs theory from topological vector space theory is the non-distributive nature of scalar field addition over scalar multiplication. As a result, there is no way to build the notion of it in a topological vector space. A fundamental source of difficulty in the preceding difficulties is that a qvs is an additive semi-group rather than a group. As a result, it is extremely common to consider constructing the theory of topological qvs (Markov, 2000 & 2004).

But all these efforts were uselessly very soon; it is because, most of the theory of topological module be contingent on idempotent, whereas in a topological qvs there cannot be any idempotent element other than θ, as the following result shows:

Lemma 3. If $a = a + x$ in qvs, then $x = \theta$.

So in particular, $a = a + a \Longrightarrow a = \theta$, i. e a topological qvs cannot contain any idempotent other than θ. It therefore becomes almost obvious to find some new method to develop a theory for topological qvs. This study attempt in this direction begins with the finding of various properties of (topological) qvs which continue invariant under some kind of transformation. Any such transformation should be so designed that it possesses the capability of preserving the entire qvs-structure; in other words, such transformations should be capable sufficient to distinguish between two qvs in view of the qvs-structure (Medar,2005). The following is the definition of such a transformation:

Definition 4. A mapping $\varphi : X \to Y$ is an order-morphisms for X, Y and two quasi-vector spaces in the field K, then following axioms holds:

(i) $\varphi(x + y) = \varphi(x) + \varphi(y), \forall\ x, y \in X$

(ii) φ (α x) = α φ(x), ∀ x ∈ X, ∀ α ∈ K

(iii) x ≤ y (x, y ∈ X) ⇒ φ(x) ≤ φ(y)

(iv) p ≤ q (p, q ∈ φ(X)) ⇒ φ −1 (p) ⊆↓ φ −1 (q) and φ −1 (q) ⊆↑ φ −1 (p)

If moreover φ be bijective then it is called an order-isomorphism. X, Y are two topological qvs

φ: X → Y (14)

is a topological order- isomorphism. The concept of it between quasi-vector spaces produces an equivalence relation in the family of all qvs over a common field, distinguishing two qvs belonging to different equivalence classes (Aseev, 1986). It is therefore rational to name those properties of a qvs which remain invariant under order-isomorphism, as "qvs properties".

φ: X −→ Y (X, Y (15)

are two (qvs), φ(X_0) ⊆ Y_0 (16)

In fact, for any p ∈ X_0, φ(p) − φ(p) = φ (p − p) = φ(θ_x) = φ (0.p) = 0.φ(p) = θ_Y ⇒ φ(p) ∈ Y_0.

Also, if ∅ be an order-isomorphism then

φ|x_0 : x_0 −→ Y_0 (17)

is a vector space isomorphism and hence φ (x_0) = Y_0.

An important and very valuable method of construction of a new structure from some given structures of same type is the Cartesian product. For (topological) qvs also such a method works [1]. The details of this has been described below:

Lemma 4. Let X be a random family of quasi-vector spaces over the same field K symbolized by { X_λ : λ ∈ Λ}. Let X be the Cartesian product of these quasi-vector spaces

X : = Y $\prod_{\lambda\in\Lambda} X_\lambda$

x ∈ X iff x : Λ → $\bigcup_{\lambda\in\Lambda} X_\lambda$ is a map for x(λ) ∈ , X_λ ∀ λ ∈ Λ.

Then by axiom of choice, it is obvious that X is non-empty for Λ is non-empty and each X_λ contains at least the additive identity (Aseev,1986).

For each x ∈ X as x = (x_λ),

where x_λ = ∀ λ ∈x_λ, ∀ λ ∈Λ.

Partial order on X can be defined as

For x = (x_λ) , y = y_λ ∈ X , α ∈ K,

(i) x + y := ($x_\lambda + y_\lambda$),

(ii) α x := (αx_λ),

(iii) x ≤ y iff $x_\lambda \leq y_\lambda$ ∀ λ ∈ Λ.

The set of all one order elements is given by $X_0 = \prod_{\lambda\in\Lambda} [X_\lambda]_0$, and $[X_\lambda]_0$ being the set of all one order elements of X_λ .Again if each be X_λ a topological qvs, then $\prod_{\lambda\in\Lambda} X_\lambda$ becomes a topological qvs in order to the product topology. (Aseev,1986).

Result and Discussion

The properties of quasi-vector space are useful to characterize some (topological) quasi-vector spaces which have a important role in this study and which are in the central character in this research. The first among these belongs to the class of hyperspaces.

The main characteristics of topological qvs $\mathscr{b}(\mathcal{X})$ is comparable, powerfully comparable, compact primitive, reversible compact primitive and additive primitive; non-single primitive, non-zero primitive, non-convex, non-homogeneous and non-balanced qvs (Ganguly & Mitra,2010).

Theorem 3. (i) Assume that X is an additive primitive with strongly similar qvs and reversible primitive one and X has an order-isomorphism with $\mathscr{b}$ (X_0).

(ii) X is order-isomorphic with $\mathscr{b}$ (X_0), then topological qvs X is additive primitive, compact primitive, reversible compact primitive.

Proof. (i) Let φ : X → $\mathscr{p}(X_0)$

x ↦ P_x

Here, φ (x + y) = $P_{x+y} = P_x + P_y$,

Also, for any α ∈ 𝕂, P_{ax} = α P_x . So φ(αx) = α φ(x),

$$x \leq y \Rightarrow P_x \subseteq p_y \Rightarrow \varphi(x) \leq \varphi(y) \qquad (18)$$

X is strongly comparable to qvs,

(i) q(x) ≤ φ (y) ⇒ Px ⊆ P y ⇒ x ≤ y. (19)

For, x ≠ y ⇒ φ(x) ≠ φ(y) showing it is injective. Also, by reversible antiquity, φ converts surjective.

$$X \cong \mathscr{p}(X_0) \qquad (20)$$

(ii) If the map x $\rightarrow P_x$ gives an order-isomorphism between X and $\mathcal{P}(X_0)$ with the help of these definite qvs properties.

In the track of proof of the above theorem, it has been established that any non-trivial qvs cannot be simultaneously zero primitive and strongly comparable. So, all the zero primitive qvs discussed in this article are not strongly comparable and every strongly comparable qvs studied in this article is not zero primitive. In this state, it should also be noted that if a qvs X be strongly comparable then it must be comparable. In fact, if $P_x = P_y$ for any x, y ∈ X then by strong comparability, x = y. So, in a strongly comparable qvs distinct elements must have different primitive (Ganguly & Mitra,2010).

Though, in a comparable qvs distinct elements may have identical primitive; actually, the elements of a comparable qvs having identical primitive must have to be comparable to the partial order of the qvs (by definition). Thus, the property 'comparability' of a qvs is weaker than the property of 'strong comparability'.

Lemma 5. If X be a balanced topological qvs where any two members are comparable then

$$y = C_x(y)x \quad \forall\ x \neq \theta, \forall\ y \in X, \tag{21}$$

C_x is the comparing function.

Proof.

As X is a balanced topological qvs, we have

$$C_x(y)x \leq y . \tag{22}$$

Again, by definition of comparing function, for any

$$\epsilon > 0, (C_x(y) + \epsilon)\, x \nleq y. \tag{23}$$

Since any two members of this space are comparable, then

$$y \leq (C_x(y) + \epsilon)\, x \tag{24}$$

Randomness of $\varepsilon > 0$ shows that

$$y \leq (C_x(y)\, x . \text{ So } y = y \leq (C_x(y))x \tag{25}$$

Theorem 4. A topological qvs X is topologically order-isomorphic with [0, ∞) iff, it is homogeneous, convex and comparable.

Proof. The topological qvs [0, ∞) is a homogeneous, convex and comparable qvs. As these properties are qvs properties, so any qvs which is order-isomorphic with [0, ∞) also possesses these properties. Conversely, let X be a homogeneous, convex and comparable qvs. Then X becomes a balanced qvs. As every balanced qvs is zero primitive, so is X. Now X is a zero-primitive comparable qvs, means any two members of X are comparable (Ganguly &Mitra, 2010, 2011, 2004a, 2044b, & 2012).

Then by the lemma 5, for x ∈ X \ {θ}, the comparing function C_x is an injective function. In fact, if we take any two elements $\mathcal{Y}, \mathcal{Z}$ ∈ X ∀ C_x(y) = C_x(z) then

$$y = C_x(y)\, x = C_x(\mathcal{Z})\, x = \mathcal{Z}\,. \tag{26}$$

Now C_x is onto and we have to show that C_x is an order-morphism (or order-functional). To prove this it is sufficient to prove that for any $\mathcal{Y}, \mathcal{Z}$ ∈ X , $C_x(\mathcal{Y} + \mathcal{Z}) = C_x(y) + C_x(\mathcal{Z})$

$$\begin{aligned} C_x(\mathcal{Y} + \mathcal{Z}) &= C_x\,(C_x(\mathcal{Y})x + C_x\,(\mathcal{Z})x) \\ &= C_x\,(\mathcal{Y}) + C_x\,(\mathcal{Z}) \end{aligned} \tag{27}$$

So C_x becomes an order – isomorphism i.e. X ≅ [0,∞) algebraically.

This topological qvs is sole primitive, comparable, compact primitive, additive primitive and convex. But it is neither reversible compact primitive (Laxmikantham et al. 2006).

Theorem 5. A topological qvs X is topologically order-isomorphic with $\mathbb{R}^+ \times X_0$ iff X is a single primitive, comparable, convex qvs and the primitive function P : x → P_x from X to 2^{X_0} is incessant.

Proof. Let X is topologically order-isomorphic with $\mathbb{R}^+ \times X_0$. First of all, to show that the primitive function P_1 : x → Px of $\mathbb{R}^+ \times X_0$ is continuous. For any open set V in X_0,

$$U := \{0\} \times V \text{ is open in } [\,\mathbb{R}^+ \times X_0\,]_0 = \{0\} \times X_0. \tag{28}$$

Now Û := {x ∈ $\mathbb{R}^+ \times X_0$: Px ⊆ U} = $\mathbb{R}^+ \times$ V = ↑ U := {x ∈ $\mathbb{R}^+ \times X_0$: x ≥ u for some u ∈ U}, where $\mathbb{R}^+ \times$ V is open in $\mathbb{R}^+ \times X_0$.

The primitive function P_1 of $\mathbb{R}^+ \times X_0$ is continuous. As single primitiveness, comparability, convexity and continuity of primitive function are qvs properties so X also satisfies these properties (Knoles 1967).

Characteristics

Non-Hausdorff Topology: One of the key differences between topological quasi-vector spaces and topological vector spaces is that the former may have non-Hausdorff topology. In such spaces, points may not be separable by open sets, leading to more complex convergence and continuity behaviors.

Quasi-norms and Quasi-metrics: While traditional topological vector spaces are often studied using norms and metrics, topological quasi-vector spaces may use quasi-norms or quasi-metrics. A quasi-norm $\|\cdot\|$ on V satisfies similar properties to norms but allows for some relaxation in the triangle inequality, typically in the form

$\|x + y\| \leq C\,(\|x\| + \|y\|)$ for some constant $C \geq 1$.

Completeness and Completeness Types: Completeness in topological quasi-vector spaces can be more nuanced. A space is quasi-complete if every Cauchy net (or filter) converges. There are various types of completeness, such as sequential completeness, where every Cauchy sequence converges, and completeness with respect to quasi-norms.

Topological Dual Spaces: The dual space of a topological quasi-vector space V, denoted by V', consists of all continuous linear functionals on V. The topology on V' can vary, leading to different dual pairs and topologies such as the weak topology or the strong topology.

Locally Convex Structures: While not all topological quasi-vector spaces are locally convex, those that are enjoy additional properties. A locally convex topological quasi-vector space has a topology generated by a family of semi-norms, which allows for the application of techniques from convex analysis and optimization.

Examples and Applications

Function Spaces: Spaces of functions, such as L^P spaces and Sobolev spaces, often serve as examples of topological quasi-vector spaces, particularly when p≠2. These spaces have quasi-norms induced by integrals and derivatives, making them essential in the study of partial differential equations and functional analysis.

Distribution Spaces: The space of distributions, or generalized functions, extends the notion of function spaces to include objects like Dirac delta functions. These spaces are topological quasi-vector spaces used in the analysis of differential operators and the study of generalized solutions to differential equations.

Sequence Spaces: Spaces of sequences, such as $l^{p,}$ spaces, where sequences are summable to the p-th power, provide another rich class of examples. These spaces are pivotal in the study of series and Fourier analysis.

Conclusion

Topological quasi-vector spaces (Tqvs) over a field are assemblies of addition, scalar multiplication, and partial order with compatible topology.

Topological quasi-vector spaces represent a broad and versatile framework in mathematics, bridging vector spaces and topological spaces. Their non-Hausdorff nature, the use of quasi-norms, and the variety of completeness concepts offer a fertile ground for both theoretical exploration and practical applications. Understanding these spaces opens new avenues in functional analysis, providing tools to tackle complex problems in mathematical physics, differential equations, and beyond.

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