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Contraction Principles in Metric Space

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Abstract

This paper deals with contractions in metric. some historical accounts on contractions principles in metric space. Banach Contraction Principle is marvelous and widely applied fixed point theorem in nonlinear analysis. every contraction in a complete metric space has a unique fixed point.

Keywords: contraction principle, complete, metric space, unique fixed point

Introduction

In 1922, the following theorems of Banach Contraction Principle was first stated and proved by Banach for the contraction maps in the setting of complete metric spaces.

Theorem 2.2.1([16], [7]): (*Banach Contraction Principle*). Let (X,d) be a complete metric space, then each contraction map $T : X \to X$ has a unique fixed point.

Proof: Let h be a contraction constant of the mapping T.

We will explicitly construct a sequence converging to the fixed point.

Let x_0 be an arbitrary but fixed element in X.

Define a sequence of iterates $\{x_n\}$in X by

$x_n = T(x_{n-1}) \; (= T^n(x_0))$, for all $n \geq 1$.

Since T is a contraction,

we have $d(x_n, x_{n+1}) = d\big(T(x_{n-1}), T(x_n)\big) \leq hd(x_{n-1}, x_n)$,

for any $n \geq 1$.

Thus, we obtain $d(x_n, x_{n+1}) \leq h^n d(x_0, x_1)$, for all $n \geq 1$.

Hence, for any $m > n$, we have

$$d(x_n, x_m) \leq (h^n + h^{n+1} + \cdots + h^{m-1}) d(x_0, x_1)$$

$$\leq \frac{h^n}{1-h} d(x_0, x_1).$$

We deduce that $\{x_n\}$ is Cauchy sequence in a complete space X.

Let $x_n \to p \in X$. Now using the continuity of the map T,

We get $p = \lim_{n \to \infty} x_n = \lim_{n \to \infty} T(x_{n-1}) = T(p)$.

Finally, to show T has at most one fixed point in X,

Let p and q be fixed points of T.

Then, $d(p,q) = d(T(p),T(q)) \leq hd(p,q)$.

Since $h < 1$, we must have $p = q$.

This completes the proof.

Theorem 2.2.2: Let (X,d) be a complete metric space and let $T: X \to X$ be a contraction mapping, with Lipschitz constant $k < 1$.

Then, T has a unique fixed point ω in X, and for each $x \in X$,

We have $T^n(x) = \omega$

Moreover for each $x \in X$,

We have, $d(T^n(x),\omega) \leq \frac{k^n}{1-k} d(T(x),x)$.

An easy implication of the Banach Contraction Principle are the following theorem [7].

Theorem 2.2.3: Suppose (X,d) is a complete metric space and suppose $T: X \to X$ is a mapping for which T^N is a contraction mapping for some positive integer $N \geq 1$.

Then T has a unique fixed point.

The following theorem is related to complete metric space.

Theorem 2.2.4: Let (X,d) be a compact metric space with $T: X \to X$ satisfying $d\big(T(x),T(y)\big) < d(x,y)$For $x, y \in X$ and $x \neq y$.

Then T has a unique fixed point in X.

Theorem 2.2.5: Let (X,d) be a complete metric space and

Let $B(x_0, r) = \{x \in X : d(x,x_0) < r\}$, where $x_0 \in X$ and $x > 0$.

It may be the case that $T: X \to X$ is not a contraction on the whole space X, but rather a contraction on some neighborhood of a given point. In this case the result as follows [7]:

Theorem 2.2.6: Let (X,d) be a complete metric space and

Let $B_r(y) = \{x \in X : d(x,y) < r\}$, where $y \in X$ and $r > 0$.

Let $f : B_r(y) \to X$ be a contraction map with contraction constant $h < 1$.further, assume that $d(y,T(y)) < r(1-h)$.

Then, T has a unique fixed point in $B_r(y)$.

In 1930, Caccioppoli extended the Banach Contraction Principal as follows;

Theorem 2.2.7[32]: Let (X,d) be a complete metric space and let $T: X \to X$ be a mapping such that for each $n \geq 1$, there exists a constant c_n

such that

$d\big(T^n(x),T^n(y)\big) \leq c_n\, d(x,y), for\ all\ x, y \in X,$

where $\sum_{n=1}^{\infty} c_n < \infty$.

Then, T has a unique fixed point.

In 1968, Bryant V. W. extended Banach Contraction Principle as follows;

Theorem 2.2.8[28]: Let (X,d) be a complete metric space and let$T: X \to X$ be a mapping such that for some positive integer n,

T^n is contraction on X. Then, T has a unique fixed point.

Theorem 2.2.9: Let (X,d) be a complete metric space. A map $T: X \to X$ (not necessarily continuous).

Suppose the following condition holds:

$$\{for\,each\ \epsilon > 0\ there\,is\ a\ \delta(\epsilon) > 0\ \ such\,that\ if\ d(x, T(x)) < \delta(\epsilon),\ \ then\,T(B(x,\epsilon)) \subseteq B(x,\epsilon);\ here\,B(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}\}$$

If for some $u \in X$ we have

$$Lim_{n\to\infty}\, d\left(T^n(u), T^{n+1}(u)\right) = 0,$$

Then the sequence $\{T^n(u)\}$ converges to a fixed point of T.

In 1962E.Rakotch first generalization of Banach Contraction Principle as follows;

Theorem 2.2.10[178]: Let (X, d) be a complete metric space, and suppose that $T : X \to X$, satisfies

$$d(T(x), T(y)) \leq \eta(d(x,y))d(x,y),$$

for all $x, y \in X$,

Where η is a decreasing function on $R+\ to\ [0,1)$.

Then, T has a unique fixed point.

In 1969, Boyd D.W. and Wong J. S. W. more generalize theorem as follows;

Theorem 2.2.11[26]: Let (X, d) be a complete metric space, And suppose that $T : X \to X$ satisfies

$$d(T(x), T(y)) \leq \psi(d(x,y)),$$

For all $x, y \in X$,

Where $\psi : R \to [0, \infty)$ is *upper semi continuous* from the right,

That is, for any sequence

$$t_n \downarrow t \geq 0 \Rightarrow sup\psi(t_n) \leq \psi(t)$$

And satisfies $0 \leq \psi(t) < t$ for $t > 0$.

Then, T has a unique fixed point.

In 1969 Meir A. and Keeler E. extended Boyd and Wong theorem is as follows;

Theorem 2.2.12[140]: Let (X, d) be a complete metric space, and suppose that $T : X \to X$ satisfies the condition:

For each $\varepsilon > 0$, there exists $\delta > 0$

Such that for all $x, y \in X$,

$$\varepsilon \leq d(x,y) \leq \varepsilon + \delta$$

$$\Rightarrow d(T(x), T(y)) \leq \varepsilon.$$

Then, T has a unique fixed point.

In 1974, Ciric L. B. has generalized Banach Contraction Principle as follows;

Theorem 2.2.13: Let (X, d) be a complete metric space, and let $T : X \to X$ be a quasi-contraction, that is, for a fixed constant $h < 1$,

$$d(T(x), T(y)) \leq$$

$$h\, max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

For all $x, y \in X$.

Then, T has a unique fixed point.

In 1975 Matkowski J. extended by replacing ψ Theorem 2.2.11 as follows;

Theorem 2.2.14[134]: Let (X, d) be a complete metric space, and suppose that $T: X \to X$ Satisfies

$d(T(x), T(y)) \leq \psi(d(x, y))$,

for all $x, y \in X$,

Where $\psi: (0, \infty) \to (0, \infty)$ is monotone non decreasing and satisfies $\psi^n(t) = 0$ for all $t > 0$.

Then, T has a unique fixed point.

In 1976, Caristi J. proved a unique Fixed Point Results in complete metric space related to Banach Contraction Principle as follows;

Theorem 2.2.15[37]: Let (X, d) be a complete metric space. Then, each Caristi map $T: X \to X$ has a fixed point.

In 2001, Rhoades B. E. extended and improved in metric space of the generalization of Alber [6] in Hilbert space as follows;

Theorem 2.2.16[185]: Let (X, d) be a complete metric space, and suppose tha t $T: X \to X$ satisfies the following inequality

$d(T(x), T(y)) \leq d(x, y) - \psi(d(x, y))$, For all $x, y \in X$,

where $\psi: [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$.

Then, T has unique fixed point.

In 2003, Kirk W.A. obtained the asymptotic version of Boyd and Wong [26] as follows;

Theorem 2.2.17[124]: Let (X, d) be a complete metric space, and suppose that $T: X \to X$ Satisfies $d(T^n(x), T^n(y)) \leq \psi_n(d(x, y))$, for all $x, y \in X$,

where $\psi_n: [0, \infty) \to [0, \infty)$ are continuous and

$\psi_n \to \psi \in \Psi$ Uniformly.

Further, assume that some orbit of T is bounded.

Then, T has a unique fixed point.

In 2008, Dutta P. N. and Choudhury B. S. generalized Theorem 2.2.15 is as follows;

Theorem 2.2.18[61]: Let (X, d) be a complete metric space, and suppose that $T: X \to X$ satisfies the following inequality,

$\varphi\left(d(T(x), T(y))\right) \leq \varphi(d(x, y)) - \psi(d(x, y))$,

For all $x, y \in X$,

where both the functions $\varphi, \psi: [0, \infty) \to [0, \infty)$ are continuous and nondecreasing such that $\psi(t) = 0 = \varphi(t)$ If and only if $t = 0$.

Then, T has unique fixed point.

In 2011, Choudhurya B.S., Konarb P., Rhoades, B. E. and Metiya N. established more general result is as follows,

Theorem 2.2.19[43]: Let (X,d)be a complete metric space, and suppose that$T: X \to X$satisfies the following inequality

$\phi(d(T(x),T(y))) \leq \phi(m(x,y)) - \psi(max\{d(x,y),d(y,T(y))\})$, Where

$$m(x,y) = max\left\{d(x,y),d(x,T(x)),d(y,T(y)),\frac{1}{2}[d(x,T(y))+d(y,T(x))]\right\}$$

For all $x,y \in X$, and

$\phi,\psi: [0,\infty) \to [0,\infty)$ Are functions such that ϕ is alternating distance and ψ is continuous with $\psi(t) = 0$ If and only if $t = 0$.

Then, T has unique fixed point.

A direct consequence of the Theorem 2.2.19 is the as follows;

Corollary 2.2.1[7]: Let (X,d) be a complete metric space, and suppose that $T: X \to X$ satisfies the following inequality for all $x,y \in X$,

$$\phi\left(d(T^n(x),T^n(y))\right)$$
$$\leq \phi\ \left((max\left\{d(x,y),d(x,T^n(x)),d(y,T^n(y)),\frac{1}{2}[d(x,T^n(y))+d(y,T^n(x))]\right.\right.$$
$$\left.\left.-\ \psi(max\ \{d(x,y),d(y,T^n(y)\}\right\}\right)$$

Where n is a positive integer

And

$\phi,\psi: [0,\infty) \to [0,\infty)$

Are functions such that ϕ alternating distance and ψ is continuous with

$\psi(t) = 0$ If and only if $t = 0$.

Then, T has unique fixed point.

In 2008, T. Suzuki gave a new type of generalization of the Banach Contraction Principle as follows,

Theorem 2.2.20[203]: Let(X,d) be a complete metric space, and suppose that$T: X \to X$.

Define a non-increasing function $\psi: [0,1) \to (1/2,1]$ by

$$\psi(h) = \{1 \quad if \quad 0 \leq h \leq \frac{(\sqrt{5}-1)}{2}, \frac{1-h}{h^2} if \frac{(\sqrt{5}-1)}{2} < h < \frac{1}{\sqrt{2}} \frac{1}{1+h} if \frac{1}{\sqrt{2}}$$
$$\leq h < 1.,$$

Assume that there exists$h \in [0,1)$,

Such that

$$\psi(h)d(x,T(x)) \leq d(x,y)$$

$$\Rightarrow d(T(x),T(y)) \leq d(x,y),$$

For all$x,y \in X$.

Then, T has a unique fixed point.

Conclusion

Banach's Contraction Mapping Principle deserves a lot of credit for its simplistic approach in the field of analysis. Despite its simplicity it the most used fixed-point theorem in the long history of analysis, though given how convenient and easy to use the contractive condition on the mapping, it shouldn't be a surprise. Its requirements end with a mere complete metric space for its setting. It further solidifies its worth by providing a constructive algorithm.

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