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Pfaffian Differential Equation and It's Application in Thermodynamics and Dynamics

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Abstract

This article provides a basic overview of the Pfaff function and equation. It also explains the concept or idea and method for solving the Pfaffian differential equation. The Pfaff differential equation has been shown to be useful in thermodynamics and dynamics.

Keywords: Pfaffian equation, equation of motion, Euler-Lagrange equation, canonical form, transformation, and partial differential equation.

Introduction

The pfaffian differential equation, published in 1814 in a memoir by the German mathematician Johann Friedrich Pfaff, contributed to the solution of the Pfaffian differential equation [1].

The expression

$$\sum_{k=1}^{n} F_k(x_1, x_2, \cdots, x_n)\, dx_k \qquad \text{(i)}$$

$x_1, x_2, \cdots, x_n$ is known as Pfaffian Differential form in n variables, where F_k (k = 1,2,$\cdots$n) are functions of some or all of the n independent variables. Likewise, the relationship

$$\sum_{k=1}^{n} F_k\, dx_k \qquad \text{(ii)}$$

is called a Pfaffian Differential Equation.

Pfaffian function is ones whose derivatives are the original function again and again.

For example, if f(x) = e^x then $\frac{df(x)}{dx} = e^x$ and again we get e^x. Hyperbolic functions, trigonometric functions on bounded intervals are the Pfaffian functions.

Pf DEs for two variables

In the case of Pf DE in two variables, Eq. (i) can be written in the form for various well-known functions [2]

$$P(x,y)\, dx + Q(x,y)\, dy \qquad \text{(iii)}$$

for the functions P(x, y) and Q(x, y). Equation (iii) can be expressed as follows:

$$\frac{dy}{dx} = -\frac{P}{Q} = f(x, y) \qquad \text{(iv)}$$

Pf DEs in three variables

The three factors Pf DE is written as takes the form

$$P\,dx + Q\,dy + R\,dz = 0 \qquad \text{(v)}$$

P, Q, and R are functions of x, y, and z, respectively. As

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{(vi)}$$

Indicates a family of space curves, we always identify a system of simultaneous differential equations with equation (v).

The major goal of this work is to present the use of the Pfaffian Differential Equation in thermodynamics and dynamics.

Law of thermodynamics and Pfaffian differential equation

In an adiabatic process, the first law of thermodynamics states that mechanical work done an amount of W is the function of the thermo dynamical variables

$$x_1, x_2, \cdots, x_n, \text{ and } x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)} \qquad \text{(vii)}$$

during the final and initial states of the system, not of their immediate value [2]. As a result, this may be written as

$$W = W\,(x_1, x_2, \cdots, x_n\,;\, x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)})$$

If a substance goes from the initial state $(x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)})$ to an immediate state $(x_1^{(1)}, x_2^{(1)}, \cdots x_n^{(1)})$ and then to a final state $(x_1, x_2, \cdots, x_n)$.

Taking the functional equation

$$W(x_1, x_2, \cdots, x_n\,;\, x_1^{(1)}, x_2^{(1)}, \cdots x_n^{(1)}) + W\,(x_1^{(1)}, x_2^{(1)}, \cdots x_n^{(1)}\,;\, x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)})$$
$$= W\,(x_1, x_2, \cdots, x_n\,;\, x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)}) \qquad \text{(viii)}$$

for the determination of the function W and this shows that there exists a function U(($x_1, x_2, \cdots, x_n$) called the internal energy of the system with

$$W\,(x_1, x_2, \cdots, x_n\,;\, x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)}) = U\,(x_1, x_2, \cdots, x_n) - U(x_1^{(1)}, x_2^{(1)}, \cdots x_n^{(1)}) \qquad \text{(ix)}$$

If the system's state is changed from $x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)}$ to $x_1, x_2, \cdots, x_n$ by applying an amount of work performed W but not adhering to the fact that the system is adiabatic enclosed. The change in internal energy U($x_1, x_2, \cdots, x_n$) – U($x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)}$), which can be measure by the amount in work but is not comparable to mechanical work W.

The differences between the two a is the amount is the amount of heat Q that the system absorbs during the non-adiabatic process [4]. The first law of thermodynamics can be expressed in writing

$$Q = U - U_0 \qquad \text{(x)}$$

Due to change in internal energy, the equation (iv) becomes,

ΔQ = dU$-\Delta W$ (xi)

Taking P and V as the thermo dynamical variables,

$\Delta W = -Pdv$ then

ΔQ = P dP + V dV (xii)

where P = $\frac{\partial U}{\partial P}$; V = $\frac{\partial U}{\partial V}$ +P then there exists $\mu(p,v)$ and ϕ (p,v) such that

$\mu\Delta Q$ = dΦ (xiii)

where ΔQ is not an accurate differential equation in and of itself. The value of thermo dynamical variable μ is set in such a way that $\mu\Delta Q$ becomes an accurate differential. If the process is characterized by the n thermo dynamical variables $x_1, x_2, \cdots, x_n$ then the equation (xi) takes the Pfaffian form of below type shown below.

$\Delta Q = \sum_{k=1}^{n} F_k \, dx_k = \sum_{I=1}^{n} X_i \, dx_i$ (xiv)

is in Pfaffian equation form.

To prove that all thermo dynamical systems occur in nature, a new physical axiom should be added. This new axiom will provide support for the second law of thermodynamics [4]. The second law of thermodynamics states that no change of state is physically possible. In other words, without external control, heat cannot flow from a cold body to a warm one.

The function in equation (xiv) is known as the thermo dynamical system's entropy. Based on the equation of state of a perfect gas, the gas thermometer scale defines a temperature that is directly proportional to T; equation (xiv) explains the second law of thermodynamics.

$\frac{\Delta Q}{T}$ = d Φ (xv)

Pfaff's Equation of Motion [7]

Let $\Delta Q = \sum_{k=1}^{n} F_k \, dx_k = \sum_{I=1}^{n} X_i \, dx_i = (X_1, X_2, \cdots \cdots X_n$ (xvi)

is Pfaff's differential equation and vector of the system.

X. d x = X_I dX^i is the Pfaff's expression (Pfaff, 1815) [1]. We can fit curl of vector X as

a_{ij} = curl X = $\frac{\partial X_i}{\partial x_j}$ - $\frac{\partial X_j}{\partial X_i}$ = A - A^T (xvii)

is the skew – symmetric tensor. The bilinear covariant is represented by the Pfaff vector X, which is a scalar number.

C = $\sum_{i,j}^{m} \left(\frac{\partial X_i}{\partial (dx^i)} - \frac{\partial X_j}{X_i} \right) d^{x^i} \delta x^j$ (xviii)

where dx = (dx^i) and $\delta x = (\delta x^i)$ indicate two different differentials.

$\frac{dx_i}{dt} = f_i\,(x_1, x_2, \ldots . x_n t)$ i = 1,2,3……n.

Let the three coordinates of the Pfaff phase space be X= (q, p, t). We can alternatively use the Pfaff vector defined by X= (p, 0, -E), where E = H is the system's energy [5].

X = (P, 0, -H) then Pfaffian Φ will be the

Φ= Pdq – Hdt. The association of Pfaff system is equivalent to the equation of motion [6],

$$A = \left(\frac{\partial x_i}{\partial x_j}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -\frac{\partial H}{\partial q} & -\frac{\partial H}{\partial j} & -\frac{\partial H}{\partial t} \end{bmatrix},$$

Now, Curl = A- $A^T = \begin{bmatrix} 0 & 1 & \frac{\partial H}{\partial q} \\ -1 & 0 & \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} & -\frac{\partial H}{\partial p} & 0 \end{bmatrix}$ (xix)

Discussion and methods

By multiplying the previous matrix on the right side of column vectors by (dq, dp, dr) the associated first Pfaff system is explained [8], [9], [10].

dp+$\frac{\partial H}{\partial q}$ dt = 0 is canonical equation of motion.

In Hamiltonian form, the equation of motion of a dynamical system of a degree are expressed as

$$\frac{dp_i}{dt} = -\frac{dH}{\partial q_i}, \quad \frac{dq_i}{dt} = -\frac{dH}{\partial p_i} \quad i = 1,2,3\cdots\cdots\cdots n \qquad \text{(xx)}$$

where H ($q_1, q_2 \cdots\cdots q_n p_1 \cdots\cdots p_n t$) Hamiltonian Equation of Motion.

The equation of motion reduced as

$$\frac{dp}{dt} = -\frac{dH}{\partial q}, \quad \frac{dq_i}{dt} = -\frac{dH}{\partial p} \qquad \text{(xxi)}$$

where H(p, q, t) is the Hamiltonian of the system.

It can be written

$$-\frac{dH}{\partial q} = \frac{P(p,q,t}{R(p,q,t)}, \quad \frac{dH}{\partial p} = \frac{Q(p,q,t}{R(p,q,t)}$$

Substituting in equation (xxi),

$$\frac{dp}{P(p,q,t)} = \frac{dq}{Q(p,q,t)} = \frac{dt}{R(p,q,t)} \qquad \text{(xxii)}$$

By trivial change of variable, it can be written equation (xxii) in to the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{(xxiii)}$$

where P, Q, and R are function of x, y, and z. Equation (xxi) represents the three variables Pf DE of the equation (v)

P dx +Q dy +R dz = 0

Conclusions

The study concludes with a brief review, ideas, and concepts about Pfaff's equation and its applications. The pfaff differential equation represents an equation of motion.

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