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Numerical Solution of Multiple Internal Rate of Return with Non-conventional Cash Flows

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Abstract

This paper focuses on solving methods of multiple internal rates of return (MIRR) of the series of non-conventional cash flow when the net present value is equal to zero. The internal rate of return is a popular rule for project acceptance/rejection. When a projects cash flow has three or more sign variations, the internal rate of return (IRR) rule is not easy to obtain multiple internal rates of returns. In his paper, we introduce Descartes' rule for the maximum positive roots, Bolzano Theorem for the maximum roots in the interval, Budan's Theorem (sequence), and Sturm's Theorem (sequence) for the exact numbers of positive real roots and their locations (intervals) where they lie and finally we solve a real problem using Newton-Raphson method for the approximate solution of the polynomial equation.

Keywords: Budan's theorem, cash flow, Descartes's rule, *IRR*, *MIRR*, net present value, Newton-Raphson method, polynomial, Sturm's theorem

Introduction

Internal rate of return (*IRR*) is the discount rate that makes the net present value (*NPV*) of all the cash flows (both positive and negative) equal to zero for a specific investment, that is the internal rate of return for an investment is the interest rate that is equal to the annual compound rate earned on a saving account with the same cash flow. The term internal refers to the fact that the calculation excludes external factors, such as the risk-free rate, inflation, or various financial risks. The *IRR* is designed to account for the time preference of money and investments. A given return on investment received at a given time is worth more than the same return received at a later time, so the later would yield a lower *IRR* than the former. The internal rate of return is based only on a project's cash flow. For *IRR* to exists, both costs and benefits must be defined if only a project's benefits are defined then the present value is positive for all interest rates. If there are only costs, then the present value is always negative. Only if there costs and benefits can the *NPV* may equal to zero for some interest rate. In general, *IRR* is an average discount rate at which *NPV* is equal to zero, *IRR* tells how much the rate of return is getting from the project.

There are some approaches to remove multiple internal rates, however they are not easy to understand or interpret appropriately (Blank & Tarquin, 1989) & (White, Case, Pratt, & Agee, 1998), inaccurate (Park, 1997) ambiguous and contradictory (Steiner, 1979). When multiple rates of return are found, there is no logical way to determine which one is best for measuring economic desirability (Thuesen & Fabrygky, 1989). Multiple (or even complex-valued) internal rates, each has meaningful interpretation as a rate of return on its own underlying investment, but problems of using multiple internal rates of return (Hazen, 2003). Studying complex solution of the *IRR* equation and its applications should not be forbidden or neglected, but it should also be understood that the introduction of such advanced concepts in the business or industry, which by necessity must be somewhat conservative, need for meticulous and thorough research as well as several expert talks (Yuri & Alexander, 2013). However, if there aren't any real-valued internal rates of return, the concept is useless because complex-valued IRR can't be used in practical applications.

The use of the *IRR* as a profit measure is very popular in the industrial and financial world. We shall define the *IRR* method as the method of economic evaluation which primarily uses *IRR* as the decision criteria that will include all possible conditions whether they occur often or rarely. Companies wishing to compare capital projects might utilize IRR as one of their tools. It can be valuable aid in project acceptance and selection. The *IRR* method refers not only to the use of the *IRR* as merit measure, but also the adoption of a set of decision-making guidelines that incorporate incremental analysis of mutually exclusive solutions and are based on the idea of IRR. *IRR* cannot be used to rank cash flows for mutually exclusive projects, excepts on an incremental basis.

Future and present values of cash flow

Financial institutions are willing to pay interest on deposits because they can lend the money to the investors. The future value (*FV*) of a present amount of money will be larger than the existing amount because of the accumulated interest over time. Conversely, the amount of money at the beginning of each period is computed backward by subtracting the interest for the period from the amount at the end of that period, the net effect is to discount or reduce the future sum to its present value (*PV*) (Tung and Thomas 1992).

If a series of cash flow $C_0, C_1, \dots\dots, C_n$ at time $t = 0, 1, \dots\dots, n$, where $C_k\ (k = 0, 1, 2, \dots, n)$ are maybe positive, negative, or zero which is designated as positive for receipts and negative for disbursement. Then

Net future value (*NFV*) = $\sum_{t=0}^{n} C_t(1+i)^{n-t}$ (1)

Net Present value (*NPV*) = $\sum_{t=0}^{n} C_t(1+i)^{-t}$ (2)

Using (1) and (2), we obtain the relation between *NPV* and *NFV* that is $NFV = (1+i)^n NPV$.

The internal rate of return (*IRR*) is defined by using a *NPV* equal to zero.

Let the internal rate of interest (*IRR*) be r, where *NPV* =0. Then, we obtain the series of cash flow is $C_0(1+r)^n + C_1(1+r)^{n-1} + C_2(1+r)^{n-2} + \cdots + C_n = 0$ (3)

This is a polynomial equation of degree *n* in (1 + *r*) and therefore it has exactly *n* solutions, including complex ones.

Solution methods of Multiple Internal Rate of Return of non-conventional cash flow

In the polynomial equation (3), the solution of *r* is generally quite complicated when *n* is large. There are many different methods for the solution of *MIRR* (Multiple Internal rates of return), like the 'Trial and Error' method, the 'Spreadsheet' method using the computer, 'Numerical Method' for approximate roots. In practice, a project does not always have a conventional cash flow stream. For instance, an iron company plans to develop a strip mine and requires a large amount of clean-up costs at the end (Ross, Randolph, & Bradford, 2008). Hence, in this paper, we use the numerical method to find the approximate *MIRR*, using 'Descartes' rule of sign' (Bajracharya, Basnet, & Phulara, 2014), 'Budan's Theorem', 'Sturm Theorem' (Panton, 1999) and finally we use Bolzano Theorem and, 'Newton-Raphson' method (Bajracharya, Basnet, & Phulara, 2014) for the non-conventional cash flow to obtain the proper multiple *IRRs*. Without properly identifying multiple *IRRs* for a project with non-conventional cash flows, it is possible to make the wrong decision based on the misleading result of a single *IRR*.

The Descartes' rule of sign provides an upper bound on the number of positive (or negative) real roots (counting multiplicities) and Budan's theorem gives an upper bound on the number of real roots (counting multiplicities) of a polynomial in a given interval, whereas by Sturm's theorem, we can find the exact number of real roots in a square-free polynomial.

Descartes' rule of sign

Descartes' rule for determining the maximum number of positive real roots of a polynomial $p(x)$ in one variable based on the frequency with which the real number coefficients' signs alter when the terms are ordered from highest power to lowest power. It is noted that zero is a root when $p(x)$ has no constant term. It is a technique for getting information on the number of positive real roots of a polynomial. Descartes' rule of signs, first described by the French philosopher and mathematician Rene Descartes in 1637.

Theorem: (Bolzano)

Let $p(x)$ be a real continuous function on $[a, b]$. If $p(a)p(b) < 0$, then an equation $p(x) = 0$ has at least one root between a and b (Bajracharya, Basnet, & Phulara, 2014).

Budan's Theorem

Budan's theorem is a theorem for bounding the number real roots of a polynomial $p(x)$ in an interval and computing the parity (integer of whether it is even or odd) of this number. It states that a given univariate polynomial $p(x)$ with real coefficient, let $v_h(p)$ be the number of sign changes in the sequence of the coefficients of the polynomial $p_h(x) = p(x + h)$. By Budan's theorem, the number of real roots in (a, b) is equal to $v_a(p) - v_b(p)$ is a non-negative. It was published in 1807 by Francois Budan de Boislaurent. In other words, if we set $x = 0$, the sign changes in $p(0), p'(0), p''(0), \ldots$ is the same as the sign changes in the coefficients of $p(x)$, where $p'(x)$ is the first derivative of $p(x)$ and $p(x),\ p'(x), p''(x), \ldots \ldots, p^n(x)$ is Budan's sequence . If $x = \infty$, then all of the signs of $p(\infty), p'(\infty), p''(\infty), \ldots$ will be the same. Thus, Descartes' rule of sign counts the same things as Budan for $a = 0$ and $b = \infty$, the positive real roots in $(0, \infty)$.

Sturm Theorem

Sturm sequence: Sturm sequence is a finite sequence of polynomials $p_0(x), p_1(x), p_2(x), \ldots, p_n(x)$ of decreasing degree with the following properties

$p_0 = p$, where $p(x)$ is the polynomial of degree n.

$p_1 = p'$ where p' is the derivative of p.

$p_{i+1} = -rem(p_{i-1}, p_i)$ for $i \geq 1$, where $rem(p_{i-1}, p_i)$ denote the remainder of Euclidean division of p_{i-1} by p_i.

The length of the Sturm sequence is at most the degree of p.

$p(x)$ is square-free polynomial i.e. $p(x)$ has no factor of the form $[q(x)]^2$, for any polynomial $q(x)$.

If a is a root of $p_i(x)$ for some i with $0 < i < n$, the sign of $p_{i-1}(a)$ is the opposite of the sign of $p_{i+1}(a)$.

$p_n(x)$'s sign is constant and non-zero for all x.

Statement of Sturm Theorem

Let $p(x)$ is a square-free univariate polynomial and any interval (a, b) such that $p_i(a)$, $p_i(b) \neq 0$, for any i. Let $p_0(x), p_1(x), p_2(x), \ldots, p_n(x)$ denote the Sturm sequence corresponding to $p(x)$. Then the number of distinct real roots of $p(x)$ in (a, b) is $v(a) - v(b)$, $a < b$, where $v(a)$ and $v(b)$ denote the no. of sign variation at $x = a$ and $x = b$ respectively in the Sturm sequence.

The theorem extends to unbound intervals by defining the sign at ∞ or at $-\infty$ of a polynomial. Sturm Theorem counts the number of distinct real roots and locates them in the intervals whereas the fundamental theorem of algebra readily yields the overall number of complex roots, counted with multiplicity. The Sturm sequence and Sturm theorem are named after *Jacques Charles Francois Sturm,* who discovered the theorem in 1829.

Newton-Raphson Method

The Newton-Raphson method is used to find the approximate real root of a non-linear equation. It is based on the principle that, if the initial guess of the root of $f(x) = 0$, then the point where the tangent to the curve at $(x_i, f(x_i))$ intersect the x-axis is an improved estimate of the root and iterate this process until we approach the approximate rate. Then using the definition of the slope of a function, we obtain the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}$$

Solution methods of multiple internal rates of return (MIRR) of a project cash flow (cash inflow and cash outflow) are not an easy task. There are many more solution methods of *MIRR* among them, we use a numerical method to find the approximate *MIRR*, that is 'Descartes' sign rule', 'Budan's Theorem', 'Sturm Theorem', 'Bolzano Theorem', and finally 'Newton-Raphson' method. For example

Consider the following unconventional cash flow profile (cash inflows have a positive sign and cash outflows have a negative sign);

Time (years)	0	1	2	3	4
Cashflow (,000)	−400	1050	−659	−210	216

We have NPV = $\sum_{t=0}^{n} \frac{C_t}{(1+r)^t}$

$$\Rightarrow \quad NPV = -\frac{400}{(1+r)^0} + \frac{1050}{(1+r)^1} - \frac{659}{(1+r)^2} - \frac{210}{(1+r)^3} + \frac{216}{(1+r)^4}$$

The rate r is the internal rate of return (*IRR*) if *NPV* = 0. Then

$$-400\,(1+r)^4 + 1050(1+r)^3 - 659(1+r)^2 - 210\,(1+r)^1 + 216 = 0$$

Setting $1 + r = x$, we can obtain the value of r corresponding to every positive real root of $p(x) = 0$ i.e.

$$r = \begin{cases} \text{positive} & if\ x > 1 \\ \text{zero} & if\ x = 1 \\ \text{negative} & if\ x < 1 \end{cases}$$

Then $-400\,x^4 + 1050x^3 - 659\,x^2 - 210\,x + 216 = 0$ (1)

Let $p(x) = -400\,x^4 + 1050\,x^3 - 659\,x^2 - 210\,x + 216$ (2)

is the polynomial function of degree 4.

In (2), the number of sign changes between consecutive non-zero coefficients is 3. By Descartes' rule, there exists a maximum of 3 real and positive roots.

Applying Budan's theorem, we first determine Budan's sequence $\{p(x), p'(x), p''(x), p'''(x), p^{iv}(x)\}$, where

$$p(x) = -400\,x^4 + 1050\,x^3 - 659\,x^2 - 210\,x + 216$$

$$p'(x) = -1600x^3 + 3150\,x^2 - 1318\,x - 210$$

$$p''(x) = -4800\,x^2 + 6300\,x - 1318$$

$$p'''(x) = -9600\,x + 6300$$
$$p^{iv}(x) = -9600$$

Since $p(0) = 216 > 0$, $p(0.5) = 52.5 > 0$, $p(1) = -3 < 0$, $p(1.15) = 0.29 > 0$, $p(2) = -840 < 0$ and $p(3) = -10395 < 0$. Then $p(0)p(0.5) > 0$, $p(0.5)p(1) < 0$, $p(1)p(1.15) < 0$, $p(1)p(2) < 0$ and $p(2)p(3) > 0$. By Bolzano theorem, there exists at least one real and positive root in each of the intervals $(0.5, 1)$, $(1, 1.15)$ and $(1.15, 2)$, no roots are present in $(0, 0.5)$ and $(2, 3)$.

Now a variation of signs of $p(x)$, $p'(x)$, $p''(x)$, $p'''(x)$, $p^{iv}(x)$ at points $0, \infty, 0.5, 1, 1.5$ *and* 2, shown in the following table

x	p	p'	p''	p'''	p^{iv}	variation of signs (v)
0	+	−	−	+	−	3
∞	−	−	−	−	−	0
0.5	+	−	+	+	−	3
1	−	+	+	−	−	2
1.5	+	+	−	−	−	1
2	−	−	−	−	−	0
3	−	−	−	−	−	0

By Budan's theorem, 3 real and positive roots exist between 0 and ∞ because $v(\infty) - v(0) = 3$. Similarly, one positive real root in $(0.5, 1)$, one positive real root $(1, 1.5)$, and one positive real root in $(1.5, 2)$. If $x = \infty$, then all of the signs of $p(\infty), p'(\infty), p''(\infty), \dots$ will be the same.

Again, we use Sturm's theory, Sturm's sequence is $\{p_0, p_1, p_2, p_3, p_4\}$ where

$$p_0(x) = p(x) = -400\,x^4 + 1050\,x^3 - 659\,x^2 - 210\,x + 216$$
$$p_1(x) = p'(x) = -1600x^3 + 3150\,x^2 - 1318\,x - 210$$
$$p_2(x) = -187.1\,x^2 + 373.65\,x - 181.56 \qquad [-\mathrm{rem}(p_0, p_1)]$$ [1]

and similarly, $p_3(x)$ and $p_4(x)$.

$$p_3(x) = -145x + 166.6 \qquad [-\mathrm{rem}(p_1, p_2)]$$
$$p_4(x) = -0.87. \qquad [-\mathrm{rem}(p_2, p_3)]$$

Now a variation of signs of $p_0(x)$, $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x)$ at points $0, \infty, 0.5, 1, 1.5$ *and* 2, shown in the following table

1 Note that $\frac{-400\,x^4 + 1050\,x^3 - 659\,x^2 - 210\,x + 216}{-1600\,x^3 + 3150\,x^2 - 1318\,x - 210} = \left(\frac{x}{4} - 0.169\right) + \frac{187.1\,x^2 - 373.65\,x + 181.56}{-1600\,x^3 + 3150\,x^2 - 1318\,x - 210}$
$p_2(x) = -187.1\,x^2 + 373.65\,x - 181.56 \qquad [-\mathrm{rem}(p_0, p_1)]$

Now,

x	p_0	p_1	p_2	p_3	p_4	variation of signs (v)
0	+	−	−	+	−	3
0.5	+	−	−	+	−	3
1	−	+	+	+	−	2
1.5	+	+	+	−	−	1
2	−	−	−	−	−	0
∞	−	−	−	−	−	0

By Sturm's theorem, 3 real and positive roots exist between 0 and ∞ because $v(\infty) - v(0) = 3$. Similarly, one positive real root in $(0.5, 1)$, one positive real root $(1, 1.5)$, and one positive real root in $(1.5, 2)$.

Hence, from both Budan's theorem and Sturm's theorem, we obtain the same result. For approximate real and positive roots, we use the Newton-Raphson method.

Newton-Raphson method

From Descartes' sign rule and the Bolzano theorem, we determine the maximum no. positive real roots of the polynomial, and from Budan's sequence and Sturm sequence, we find the location of the real and positive roots that is three positive and real roots in $(0, \infty)$, one positive real root in $(0.5, 1)$, one positive real root in $(5, 1.5)$ and one positive real root in $(1.5, 2)$.

By using the Newton-Raphson method, we can easily find the positive real root in $(0.5, 1)$.

We have $p(x) = -400x^4 + 1050x^3 - 659x^2 - 210x + 216$ and

$$p'(x) = -1600x^3 + 3150x^2 - 1318x - 210$$

Let the initial approximate root be $x_0 = 0.5$, then successive approximations

$$x_1 = x_0 - \frac{p(x_0)}{p'(x_0)} = 0.5 - \frac{p(0.5)}{p'(0.5)} = \frac{52.5}{281.5} = 0.6865$$

$$x_2 = 0.6865 - \frac{p(0.6865)}{p'(0.6865)} = 0.7685$$

$$x_3 = 0.7685 - \frac{p(0.7685)}{p'(0.7685)} = 0.7962$$

$$x_4 = 0.7962 - \frac{p(0.7962)}{p'(0.7962)} = 0.7999$$

$$x_5 = 0.7999 - \frac{p(0.7999)}{p'(0.7999)} = 0.80$$

$$x_6 = 0.80 - \frac{p(0.80)}{p'(0.80)} = 0.80$$

Hence, positive real root $x = 0.80$ in $(0.5, 1)$.

Similarly, using the Newton-Raphson method, positive real roots $x = 1.125$ in $(1, 1.5)$ and $x = 1.20$ in $(1.5, 2)$.

Since, $1 + r = x$ i.e. $r = x - 1$, we have $x = 0.80$, $x = 1.125$, $x = 1.2$,

Hence, multiple *IRR* are $r_1 = -0.20 = -20\,\%$, $r_2 = 0.125 = 12.5\,\%$ and $r_3 = 0.20 = 20\,\%$.

Analysis

For an unconventional cash flow profile that leads to multiple values of IRR, the decision criteria based on these multiple values are much more complex. The IRR can be either positive or zero because a negative value of r is not admissible as an internal rate of return. The positive IRR is an absolute measure that is independent of the minimum attractive rate of return (*MARR*) and must be compared to the *MARR* to evaluate the acceptability of an independent project, where *MARR* is specified by the top management in a private firm (or by a government agency in a public project) reflects the opportunity cost of capital of the firm, the market interest for lending and borrowing and risks associated with investment opportunities. The decision rule of the IRR method for a project with conventional cash flows is quite simple – acceptance (rejection) the project if the *IRR* is greater (less) than the project's *MARR*. The decision rules for pure investment and pure borrowing opportunities are as (Tung & Thomas, 1992)

For a pure investment opportunity, accept if $IRR \geq MARR$; reject otherwise

For a pure borrowing opportunity, accept if $IRR \leq MARR$; reject otherwise.

But in our above, example multiple IRR occurs for an unconventional investment or borrowing opportunity, none of these above values can be used as a merit measure because IRR represents the internal rate of return of investments for some periods and the internal rate of return of borrowing for other periods of the cash flow profile. In this situation, we ought to be able to assess the opportunity using the net present value (*NPV*) criterion (Tung & Thomas, 1992).

Conclusions

The multiple internal rates of return as an evaluation criterion of mutually exclusive and independent projects. It is a widely used criterion for evaluating the independent projects, however, it presents disadvantages like reinvesting the intermediate revenue, late cost, the existence of many roots during solving out the respective mathematical equation i.e. polynomial equation, so that financial analysts suggest to use it with more carefully. If non-conventional cash flow has three or more sign variations then it is not easy to find the positive real roots of the polynomial. Descartes' rule determines the upper limit of positive real roots and excludes the complex and negative roots of the polynomial because complex-valued and negative *IRR* cannot apply in the real situation and Bolzano Theorem determine the upper limit of real roots in the given interval. Budan's theorem and Sturm's theorem facilitate the location where the real positive roots lie then we can easily find roots numerically using the Newton-Raphson method. The main disadvantage of Sturm's theorem is tedious and time-consuming to find the Sturm sequence if the polynomial equation has higher order.

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