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Separable Topological Space of Hereditary

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Abstract

A property of a topological space is termed hereditary ifand only if every subspace of a space with the property also has the property. The purpose of this article is to prove that the topological property of separable space is hereditary. In this paper we determine some topological properties which are hereditary and investigate necessary and sufficient condition functions for sub-spaces to possess properties of sub-spaces which are not in general hereditary.

Key words : hereditary, metric space, separable space, sub-spaces, topological space

Introduction

A property of topological space is termed hereditary if and only if every sub-space of a space with the property also has the property.

We determine some topological properties which are hereditary and investigate necessary and sufficient conditions for sub-spaces to possess properties of sub-spaces which are not in hereditary. This property of topological space is hereditary on open sub-sets or open-sub space closed i.e. any open sub-sets of a topological space having this property, also has this property.

By a property of topological spaces, we mean something that every topological space either satisfy or does not satisfy.

Definition 1. Let (X, Ω) be a topological space. Then (X, Ω) is a second axiom space if and only if there exists a countable base for Ω. This is known as the second axiom of countability.

Axiom 1. Let (X, Ω) be a second axiom space and $Z \subseteq X$. Then $(Z, Z \cap \Omega)$ is a second axiom space.

Definition 1. Let (X, Ω) be a topological space and $E \subseteq X$. Then E is dense in X if and only if $\kappa E = X$.

Definition 2. Let (X, Ω) be a topological space. Then (X, Ω) is separable if and only if there exists a countable dense subset of X.

We will show that the property of being separable is not hereditary by showing that every topological space is a subspace of a separable topological space. [2, p. 84]

Theorem 1. Let (X, Ω) be a topological space (in particular a non-separable space). Let∞be a point such that$\infty \subseteq X$. Then $X^* = X \subseteq (\infty)$ with the topology

$$\Omega^* = \left\{o^* : o^* = o \cup (\infty) \text{ for } o \in \Omega\right\} \cup \left\{\phi\right\}$$

is a separable topological space and (X, Ω) is a subspace.

Proof: First we will show that (X^*, Ω^*) is a topological space.

(i)Let $\lambda^* \subseteq \Omega^*$. If $\lambda^* = \varphi$ then

$$\bigcup_{H \in \lambda^*} H = \phi \in \Omega^*.$$

Assume $\lambda^* \neq \varphi$ and $\lambda = \{o : o \in \Omega$ such that $o \cup (\infty) = o^*$ for$o^* \in \lambda^*$. Then $\bigcup_{o \in \lambda^*} o^* = \bigcup_{o \in \lambda} o \cup (\infty)) = \bigcup_{o \in \lambda} o \cup (\infty) \in \Omega^*$, since $\bigcup_{o \in \lambda} o \in \Omega.$.

(ii) Now suppose $\lambda^* \subseteq \Omega^*$with λ^* finite and $\phi \notin \lambda^*$. If $\lambda^* \neq \varphi$then

$$\bigcap_{H \in \lambda^*} o = x^* \in \Omega^*.$$

Otherwise

$$\bigcap_{o \in \lambda^*} o^* = \bigcap_{o \in \lambda} (o \cup (\infty)) = \left(\bigcap_{o \in \lambda} o\right) \cup (\infty) \in \Omega^*,$$

since

$$\bigcap_{o \in \lambda} o \in \Omega$$

as λ^* finite implies$\lambda$is finite also. Therefore $(X^*, \Omega^*)$is a topological space.

Now we will show that (X^*, Ω^*) is separable. Le t$o^* \in \Omega^*$. Then $(\infty) \in o^*$, since $o^* = o \cup (\infty)$ where $o \in \Omega.$.

Let $x \in X^*$. Then for every o^* such that $x \in o^*$ we have

$$\phi \neq o^* \cup (\infty)$$

and $x \in \kappa$ whence $X^* \subseteq \kappa(\infty)$. Thus $\kappa(\infty) = X^*$, and $\{\infty\}$ is a countable dense subset of X. Therefore (X, Ω) is separable.

Clearly $(X, \Omega)$is a subspace of (X^*, Ω^*), since

$$X \subseteq x^* = X \cup (\infty)$$

and

$$\begin{aligned} X \cap \Omega^* &= \{X \cap o^* : o^* \in \Omega^*\} \\ &= \{X \cap (o(o \cup (\infty)) : o \in \Omega\} \\ &= \{X \cap o : o \in \Omega\} \\ &= (o : o \in \Omega\} = \Omega. \end{aligned}$$

Theorem 2. [3, p. 60) In every second axiom topological space separability is hereditary.

Proof. Since each sub space of a second axiom space is also a second axiom space, we need to show that every second axiom space is separable. Now (X, Ω)second axiom implies that there is a countable base β for Ω. Thus for every $o \in \Omega$ there exists a countable family -$\beta^* \subseteq \beta$ such that

$$o = \bigcup_{B \in \beta^*} B.$$

Let$\beta = \{B_n : n$ is a positive integer$\}$. Choose $x_n \in B_n$.

We may assume $n \geq 1$ implies that $B_n \neq \varphi$. Put

$$E = \bigcup_{n=1}^{\infty} (x_n).$$

Clearly E is a countable subset of X.

Now let $x \in X$ and $\beta_x \in \{B : B \in \beta\}$ and $x \in \beta\}$. Here β_x is a base at x and $B \in \beta_x$ implies that there exists an n such that $B = B_n$, whence

$$x_n \in B_n \cap E = B \cap E$$

so that $B \cap E \neq \phi$. Thus $x \in \kappa\, E,$ and $x \subseteq \kappa E.$ Therefore=X, and E is a countable dense subset of X, whence(X, Ω) is separable.

Theorem 3. Every separable metric space is hereditarily separable.

Proof. This follows immediately from theorem 3 and the fact that a metric space is separable if and only if it is second axiom. [1, p. 121]

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