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Linear Programming Problems: Determination of Optimal Value of Real Life Practical Problems

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Abstract

Nowadays mathematical method are widely applied in planning of natural economy, organization of industry control, business decision, transportation, engineering, telecommunications, elaboration of military operations etc. From the general point of view, the problems of control and planning are usually reduced to a choice of a certain system of numerical parameters or a function ensuring the most effective achievement of the preplanned aim (optimum plan) with the limited possible resources taken into account. To estimate the effectiveness of a plan, introduce the plan quantity index expressed in term of the plan characteristics and attaining the extremism value for an optimal plan. For the large number of practically interesting problems the objective function is expressed linearly in term of plan characteristics, the permissible values of the parameters also obeying linear equalities or inequalities.

Key words: Mathematical method, linear programming, problems, determination, optimal value, graphical method and simplex method.

Introduction

Linear programming is a kind of mathematical procedure or tool. It is extremely useful in modeling real life problem as simple as household management of income and expenditure. In a real life we face with problems of this nature in many common diverse situations, ranging from the allocation of production facilities of products, to the allocation of natural resources, to domestic needs, from portfolio selection to the shipping patterns, from agriculture planning to design of radiation therapy and so on. In each of these situations, linear programming is one of the most common tools used by decision makers for planning or analysis in order to find best possible outcome. Most of the business and economic activities may have the problem of planning. The problem may be due to the limited resources. But we are concerned with the objective of obtaining the maximum production or to minimum the cost of production or to get maximum profit etc, with the use of limited resources. Such problems are referred to as the problems of optimization. Nowadays linear programming is used in a variety of applications such as maximizing profits, minimizing costs, find the most efficient transporting schedules, minimizing waste, securing the proper mix of ingredients, controlling inventories, and find the most efficient assignment of personnel.

Linear programming can be applied to various fields of study. It is utilized for some engineering problems, industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, routing, scheduling, assignment and design. Such type of problems of optimization are tackled by the mathematical technique known as mathematical programming. The simplest and mostly used type of technique to solve the optimization problem is the linear programming. Thus linear programming is a mathematical technique of finding the optimum solution to a linear desired objective using limited resources or satisfying certain conditions or restrictions. The term linear means the variable measured is of degree one. Mathematically, the problem of linear programming is formulated in the following way; it is required to find the absolute extremism (the least or the largest value depending on the sense of the problem) of the linear function.

$$F=c_1x_1+c_2x_2+c_3x_3+\dots\dots\dots+c_nx_n$$

is the objective function, provided the variables $x_1, x_2, \dots\dots, x_n$ are restricted with limitations in the form of linear equalities or inequalities;

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i=1, 2, \dots\dots m)$$

And non-negative conditions;

$x_1 \geq 0, x_2 \geq 0, \dots\dots\dots\dots, x_n \geq 0$ are called the constraints.

Historical Background

The problems of solving a system of linear inequalities dates back at least as far as Fourier, after whom the method is called Fourier-Motzkin elimination method. The linear programming was first developed by a Russian Mathematician L.V. Kantorovich in 1939. He developed the earliest linear programming problem in 1939 for use during World War II to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy. George B. Dantzig developed the linear programming problems in 1947, for the purpose of scheduling the complicated procurement activities of air force of United States. George B. Dantzig developed the theory of simplex method, it provides the systematic way of examining the vertices of the feasible reason to determine the optimal value of the objective function.

John Von Neumann developed the theory of duality as a linear optimization solution, and applied it in the field of game theory, postwar, many industries found its use in their daily planning. The linear programming problem was first shown to be solvable in polynomial time by Leonid Khanhiyan in 1979. But a large theoretical and practical in the field cam in 1984 when Narendra Karmarkar introduced a new interior point method for solving linear programming problems. Nowadays these above methods are popular mathematical methods in solving wide range of real-life practical business, agricultural, industrial and economics problems.

Uses of Linear Programming

Linear programming is considerable field of optimization for several seasons. Many practical problems in operations research can be expressed as linear programming problems. Certain special cases of linear programming, such as network flow problems and multicommodity flow problems are consider important enough to have generated much research on specialized algorithms for their solution. A number of algorithms for other types of optimization problems work by solving their programming problems as sub problems. Historically, ideas from linear programming have inspired many of the central concepts of optimization theory, such as duality, decomposition, and importance of convexity and its generalizations. Likewise, linear programming is heavily used in microeconomics, company management planning, production, transportation, technology, engineering, industries, telecommunications and other issues. Although the modern management issues are ever-changing, most companies would like to maximize profit or minimize costs with limited resources. Therefore many issues can be characterized as linear programming problems.

Notations and Terminologies Used in Linear Programming

Linearity: The relationship amongst the variables representing different phenomena are linearly related. The term linear means the variable measured is of degree one.

Decision Variable**:** Decision variables are those non-negative independent variables that are to be determined in the solution of the linear programming problem.

Objective function: The linear which is to be optimized is called the objective function. Thus the goal of the linear objective function is to be maximized or to minimize the function, depending upon the situations. Profits or output are generally maximized and losses or costs are generally minimized. Mathematically, the objective function is expressed as

$$f(x_1, x_2, \ldots\ldots, x_n) = (c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n)$$

$$f(x_1, x_2, \ldots\ldots x_n) = \sum_{i=1}^{n} c_i x_i$$

Where $c_1, c_2, \ldots\ldots c_n$ are constants and $x_1, x_2, \ldots\ldots x_n$ are decision variables.

Constraints

Decision variables have to satisfy certain limitations or conditions or restrictions in solving the linear programming problem. This is, the restrictions imposed on the decision variable are known as the constraints.

The constraints are expressed in linear equalities or inequalities.

$$a_1x_1 + a_2x_2 + \ldots. a_nx_n \geq or \leq d_1$$

$$b_1x_1 + b_2x_2 + \ldots\ldots b_nx_n \geq or \leq d_2$$

$$c_1x_1 + c_1x_2 + \ldots\ldots\ldots\ldots. + c_nx_n \geq or \leq d_n$$

where $a_1, a_2, \ldots.. a_n, b_1, b_2, \ldots\ldots, b_n, c_1, c_2, \ldots., c_n$ and $d_1, d_2, \ldots\ldots, d_n$ are constants. $x_1, x_2 \ldots\ldots, x_n$ are non-negative decision variables. $x_1 \geq 0, x_2 \geq 0, \ldots\ldots x_n \geq 0$. The non-negative conditions imposed on the decision variables is also a kind of constraints.

Feasible Solution

All these possible solutions, which can be worked upon under given constraints of a linear programming problem, is called feasible solution. Actually, those solutions which satisfy the all non-negative criteria of linear programming problem are the feasible solutions. Any feasible solution which optimizes (maximum or minimum) the objective function of a given linear programming problem is termed as optimal feasible solution. Feasible Region or Feasible Area: The region or area covered by all possible feasible solution is referred as feasible region or area.

Some Examples of Linear Programming Problems

Management Problem: In everyday life, a common employee often likes to purchase maximum number or quantity of goods or items or objects. But he cannot do so because of his limited incomes (sources). Then he starts to think of planning or programming in order to have a stable or maximum income. Suppose that a farmer has a piece of land L sq. km. to be planted with either maize or rice or some combination of the two. The farmer has a limited amount of fertilizer Fkg and insecticide Ikg. Every square Km. of maize requires F_1kg of fertilizer and I_1kg of insecticide. While every square Km. of rice required F_2kg of fertilizer and I_2 Kg. of insecticide. Let S_1 be the selling price of maize per sq. km. andS_2 be the price of rice. If we denote the area of land planted with maize and rice by x and y respectively, then profit (income) can be maximized by choosing optimal value of x and y. This problem can be expressed with the following linear programming problem in the standard form:

Maximize: $z=S_1x+S_2y$ (objective function)

Subject to: $x+y\leq L$ (limit on total area)

$F_1x+F_2y\leq F$ (limit on fertilizer)

$I_1x+I_2y\leq I$ (limit on insecticide)

$x\geq 0,\ y\geq 0$ (cannot plant a negative area)

which can be written as matrix form

Maximize: $\begin{bmatrix} S_1 & S_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Subject to: $\begin{bmatrix} 1 & 1 \\ F_1 & F_2 \\ I_1 & I_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} L \\ F \\ I \end{bmatrix}$

And $\begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The example above is converted into augmented form (before being solved by the Simplex algorithm)

Maximize: $z=s_1x+s_2x$ (objective function)

Subject to: $x+y+p=L$ (augmented constraint)

$F_1x+F_2y+q=F$ (augmented Constraint)

$I_1x+I_2y+r=I$ (augmented constraint)

$x_1y_1,p,q,r\geq 0$

where *p, q, r* are non-negative slack variables, representing in this example the unused area, the amount of unused fertilizer and the amount of unused insecticide.

In matrix form this becomes:

$$\begin{bmatrix} 1 & -S_1 & -S_2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & F_1 & F_2 & 0 & 1 & 0 \\ 0 & I_1 & I_2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x \\ y \\ z \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} O \\ L \\ F \\ I \end{bmatrix}, \begin{bmatrix} x \\ y \\ p \\ q \\ r \end{bmatrix} \geq 0$$

Transportation Problem: Transportation models play an important role in logistics and supply chain management for reducing cost and improving service. Therefore, the goal is to find the most cost effective way or programming to transport the goods.

Let the points $A_1, A_2, \ldots\ldots, A_n$ produce a certain homogenous product which is consumed by the centres situated at points $B_1, B_2, \ldots\ldots, B_n$.

Suppose the transport expenses due to the delivery of a unit of goods from point A_i to point B_j amount C_{ij} money units and the amount of the goods delivered from A_i to B_j is equal to x_{ij}. Where $i=1, 2, \ldots..m$ and $j=1, 2, 3, \ldots\ldots, n$. Then the total transport expenses will amount to

$$S = \sum_{i=1}^{m} \sum_{j=1}^{n} C_j\, x_j$$

The volume of production in point A_i is limited by a certain quantity $a_i (i=1, 2, \ldots\ldots m)$ which depends on the production capacity of the manufacturing enterprise. Assuming that the whole amount of the manufactured goods is delivered to the consumer, to consuming points. Then we have the following conditions.

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad (i=1, 2, \ldots.m)$$

The demand for the product in the point B_j is dictated by producer's interests and amounts to a certain quantity b_j $(j=1, 2, 3, \ldots\ldots, n)$. Therefore

$$\sum_{i=1}^{n} x_{ij} = b_j \qquad (j=1, 2, \ldots\ldots, n)$$

It is required to work out such a delivery plan which would ensure minimum transport expenses *S* on the first condition that the product manufactured at points A_i is entirely consumed and the need of the consuming points is completery met in the second condition. This is a typical problem of linear programming. It is possible to prove that for a transport problem to be solvable, it is necessary and sufficient to fulfill that following condition,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

That is the total volume of production must be equal to the total volume of consumption.

Theorems of Linear Programming

Theorem1: A linear objective function can reach its strict absolute extremum only at terminal points of the permissible domain.

Theorem2: The linear objective function:
$f(x, y)=ax+by+c$, in two variables, defined on a convex polygon regions R takes on its optimum value at vertex of R. If R is undounded there may or may not be an optimum value, but if there is then it must occur at a vertex of R.

Theorem3: Fundamental theorem of linear programming: "If there is a solution, it occurs on the boundary of the feasible region, not inside the region."

Solution of the Linear Programming Problems

Different types of real problems happen in our life. The problems which contain the number of relations not always equal to the number of variables. The most of the relations will be in the inequalities form. To solve a linear programming problem means to determine the values of the set of decision variables that satisfy the given conditions of constraints. Such types of solution is known as the feasible solution. A feasible solution to a linear programming problem is known as optimal solution of linear programming problem. Thus solving a linear programming problem implies the determination of the variables defining the objective conditions restricted by the system of inequalities.

Types of Solution of Linear Programming Problems

Unique Optimal Solution**:** When a optimal solution to a linear programming problem occurs only one vertex of a feasible region then it is known as a unique optimal solution. Alternative Optimal Solution**:** When a optimal solution to a linear programming problem occurs at least two vertices of a feasible region then it is known as alternative optimal solution. In such a case, there will be an infinite number of optimal solution. Unbounded Solution: When an optimal solution of a linear programming problem is infinite, then it is known as unbounded optimal solution. No solution: If there does not form any plane region satisfying all the given constraints, then it is known as no solution of the linear programming problem.

Graphical Methods of Solving a Linear Programming Problem

If the objective function is a function of two variables, we can solve linear programming problems by the graphical method. Procedure of Solving the Linear Programming Problems are listed below and also can be seen in figure 1:

- Identify the decision variables, objective function, constraints and other restrictions.
- Formulate the given linear programming problem into the mathematical form.
- Convert all the linear constraints given in the form of inequalities in term of equalities.
- Draw a graph of all equations includes constraints and restrictions in two dimensional plane.

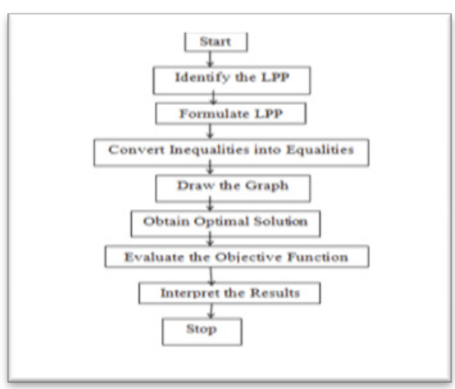

Figure 1. **Graphic Methods**

- Identify the feasible region and find the vertices of the resulting feasible region.
- Obtain the points on feasible region which optimizes the objective function. Thus, obtained solution will be the optimal solution.
- Evaluate the objective function and interpret the results.

Simplex Method of Solving a Linear Programming Problem

The simplex method is another algorithm for solving linear programming problems. For problems involving more than two variables or problems involving a large number of constraints, it is better to use methods that are adaptable to computers. One such method is the simplex method. The simplex method is a modification of a algebraic method, which avoids the vertices that are not feasible. It provides us with a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function.

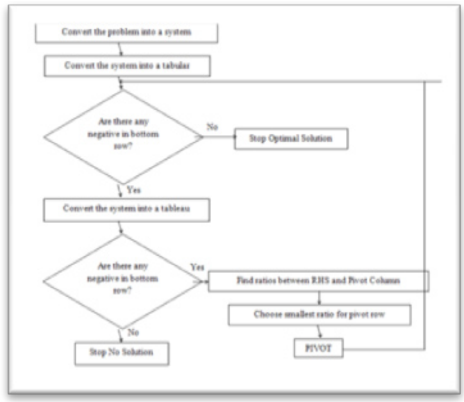

Figure 2. **Simplex Method**

The simplex method is also a tabular algorithm just like the algebraic method. In this method, each tableau corresponds to a movement from one basic variable set to another, making sure that the objective function improves at each stage of interaction until the optimal solution is reached. More specifically, those steps are presented below and also presented in figure 2.

- Identify decision variables, objective function, constraints and other restrictions.
- Formulate the linear programming problem into the mathematical form.
- Introduce slack variables to the linear programming problem.
- Putting the equations into a tableau.
- Select the pivot row.
- Use elementary row operation to change the pivot of the tableau into 1 and then to make all entries in the pivot column equal to 0 expect for the pivot that remains to be 1.
- In final tableau, make all entries in the bottom row are zero or positive.
- If we obtain a final tableau, write the improved solution and the optimal value of the objective function that is given by the entry in the lower-right corner of the tableau.

Conclusion

This paper introduce the liner programming problems to the determination of optimal value of year life problems. Such as management, economics, transportation etc. History, notations, terminologies, real like problem in terms of examples are introduces widely. Some theorems on liner programing are state with fundamental theorem of liner programming. Different methods of solving of liner programming problems is used, in this paper mostly graphics and simplex methods are presented with flow chart also. I hope this paper is very useful to new researcher.

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