The Nepali Mathematical Sciences Report
Regd No.: 53/033-034, ISSN: 2392–411X
http://www.journal.cdmathtu.edu.np
Year: 2024, Volume: 41, No. 2
Pages: 134–156
https://doi.org/10.3126/nmsr.v41i2.73232
Research Article

On Metric Spaces with Its Major Variants

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Received: 15 July 2024, Accepted: 5 December 2024, Published Online: 31 December 2024

Abstract

A fundamental characteristics of the relative position of points of a space is the distance between them that leads to the idea of a metric space first introduced by M. Frechet in 1906. Metric spaces form a natural environment for exploring fixed points of single and multi-valued mappings. There have been several attempts to extend or to generalize the definition of a metric space in order to obtain possibilities for more fixed points results in Nonlinear analysis.

In this paper, we discuss a short review of original generalized forms of metric space with suitable examples and also obtain the interrelation between these forms.

Keywords: Distance function, Fixed point, Metric space.
AMS(MOS) Subject Classification: 49K35; 47H10; 20M12 .

The notion of metric space is a fundamental concept in mathematics particularly in the field of topology, functional analysis and non linear analysis. In the modern mathematical set-up, French mathematician Maurice Rene Frechet [2] is the first who axiomatically formulated the notion of metric space, under the name of L-space. Also, German mathematician Felix Hausdorff used the term "metric space" on the role of point-sets within abstract set theory. The idea stated with the measurement that it is non negative, symmetric in nature and follows indistancy property if the points are identical. Again, the measurement between two points is less than or equal to the total distance taken when if traveled via some other

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point. It provides a framework for studying the notion of distance between points in a set. In this way, Frechet got the structure as metric spaces and now it has attracted a considerable attention from several mathematicians because of its extension and generalization of metric spaces for the existence of fixed points.

Understanding the metric spaces is important as they provide a versatile framework for studying the concept of distance and convergence which are central to many areas of mathematics and its applications. Thus, a metric is a concept of measuring distance originated from the prototype of the ordinary distance between any points in the Euclidean plane. The purpose of this paper is to discuss some first and original generalized forms of metric space as major variants with examples and the interrelationship between them. We start with the following definition of metric space as follows:

Definition 0.1. [2]: Let X be a non empty set. Let d be a function on X $\times$ X into R. Then d is called a metric on X if the following axioms are satisfied :

M1. d(x, y) $\geq$ 0 for all x,y $\in$ X (non-negativity);

M2. d (x, y) = 0 if and only if x = y (Indiscrenibles);

M3. d(x,y) = d(y,x) for all x, y $\in$ X(Symmetricity); and

M4. d(x,z) $\leq$ d(x,y) + d (y, z) for all x, y, z $\in$ X (Triangle Inequality)
and the pair (X, d) is called a **metric space**.

Clearly, a metric space is continuous but is non-linear. In 1986, S.G. Matthews[3] discussed the concept of **dislocated metric space** with symetricity, triangle inequality and with the condition

$$d(x, y) = d(y, x) = 0 \text{ implies } x = y,$$

under the name of metric domain in domain theory.
We have the following example of Box metric space different from Euclidean space:

Example 0.2. Let us consider a function $d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_1((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

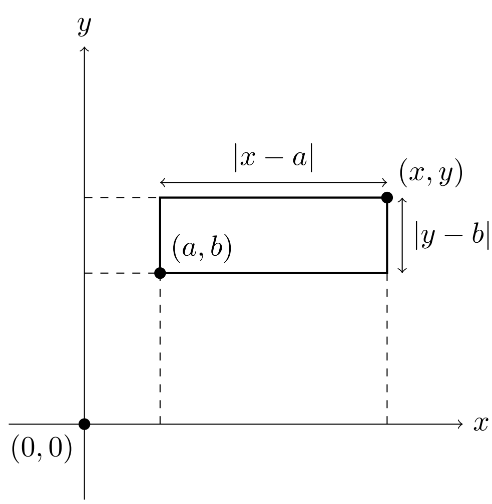
Then d_1 is a metric on $\mathbb{R}^2$ called the **box metric**.
For any points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $|x_1 - x_2| \geq 0$ and $|y_1 - y_2| \geq 0$,
so the max $\{d_1((x_1, y_1), (x_2, y_2))\} \geq 0$.
Also, $d_1((x_1, y_1), (x_2, y_2)) = 0 \iff |x_1 - x_2| = 0$ and $|y_1 - y_2| = 0$.
It implies $x_1 = x_2$ and $y_1 = y_2$.

i.e. $d_1((x_1, y_1), (x_2, y_2)) = 0 \iff (x_1, y_1) = (x_2, y_2)$,
so d_1 satisfies the first requirement in the definition of a metric on $\mathbb{R}^2$.
Also, for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, since $|x_1 - x_2| = |x_2 - x_1|$ and $|y_1 - y_2| = |y_2 - y_1|$, the max of $|x_1 - x_2|$ and $|y_1 - y_2|$ is equal to the max of $|x_2 - x_1|$ and $|y_2 - y_1|$, so $d_1((x_1, y_1), (x_2, y_2)) = d_1((x_2, y_2), (x_1, y_1))$, and so the symmetricity holds.
Finally, for $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$, using the triangle inequality, we get
$|x_1 - x_2| \leq |x_1 - x_3| + |x_3 - x_2|$ and $|y_1 - y_2| \leq |y_1 - y_3| + |y_3 - y_2|$.
Also, $|x_1 - x_3| + |x_3 - x_2| \leq \max\{|x_1 - x_3|, |y_1 - y_3|\} + \max\{|x_3 - x_2|, |y_3 - y_2|\}$,
and similarly for $|y_1 - y_3| + |y_3 - y_2|$.
i.e. both $|x_1 - x_2|$ and $|y_1 - y_2| \leq d_1((x_1, y_1), (x_3, y_3)) + d_1((x_3, y_3), (x_2, y_2))$, so their max $d_1((x_1, y_1), (x_2, y_2))$ is as also. This proves the triangle inequality for d_1, Hence d_1 is a metric on $\mathbb{R}^2$.
In particular, if we take $(1, 2), (0, 0) \in \mathbb{R}^2$. The usual distance on $\mathbb{R}^2 = \sqrt{1^2 + 2^2} = \sqrt{5}$.

But the box metric d_1 is

$$d_1((1, 2), (0, 0)) = \max\{|1 - 0|, |2 - 0|\} = 2.$$

Specifically, the distance from a point (x, y) to the origin $(0, 0)$ is given by the larger of $|x|$ and $|y|$. More generally, the distance between two points in this metric is defined as the length of the longest side of the rectangle formed with one vertex at the first point and the opposite vertex at the second point. This characteristic gives rise to the term "box metric."



1 Some Generalized Forms of Metric spaces

We discuss some initial and original generalized forms of metric space in crononical order with suitable standard notations and suitable examples.

1.1 Quasi-Metric Space

An American mathematician W. A. Wilson [28] introduced the Quasi-Metric space in 1931 as follows:

Definition 1.1. [28]: Let X be a non-empty set. A mapping d: X × X → [0, ∞) is called a quasi-metric if and only if it satisfy the following conditions:

QM1 d (x, y) ≥ 0 for all x, y ∈ X;

QM2 $d(x, y) = 0$ if and only if $x = y$ for all x, y ∈ X; and

QM3 d(x,z) ≤ d(x,y) + d (y, z) for all x, y, z ∈ X .

In this case, the pair (X, d) is called a **quasi-metric space**.

This concept is fundamental in topology and analysis allowing the relaxation of the symmetry condition typically required in metric spaces

We have the following example of quasi-metric space:

Example 1.2. Let $X = \mathbb{N} \cup \{0\}, \mathbb{N}$ being the set of natural numbers and we define the function d on X as follows:

$$d(0, n) = \frac{1}{n^2} \quad \text{for all } n \in \mathbb{N}, \quad d(n, x) = n^2 \quad \text{for} n \neq x,\ n \in \mathbb{N},$$

$d(x, x) = 0$ for all $x \in X$. Then (X, d) is not a metric space because $d(0, 2) = \frac{1}{4} \neq d(2, 0) = 4$, so the symmetricity is not satisfied but it is a quasi-metric space.

For this, if $x = z$ or $y = z$, then the inequality holds.
Also, if $x \neq y \neq z$ and $x = 0$, then $y \neq 0$, and thus we have

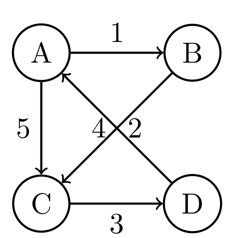
$$d(0, y) = \frac{1}{y^2} \leq d(0, z) + d(z, y) = \frac{1}{z^2} + z^2 \geq \frac{1}{y^2} = d(0, y).$$

Moreover, if $x \neq 0$, then

$$d(x, z) + d(z, y) = x^2 + d(z, y) \geq x^2 = d(x, y).$$

Example 1.3. [14]

A directed graph where the distance from node x to node y is defined as the shortest path length in the directed edges form a quasi-metric space. Here, the graph is not symmetric as $d(x, y)$ may not equal to $d(y, x)$ as shown in the graph .



1.2 Pseudo-metric space

Duro Kurepa a Serbian mathematician introduced the Pseudo metric space in 1934 as follows [24]:

Definition 1.4. [24]: A let X be a set. A mapping d: X × X → [0, ∞) is called a pseudo metric if the following properties are satisfied:

pSM 1. d (x, x) = 0;

pSM 2. d(x,y) = d(y,x) for all x, y ∈ X; and

pSM 3. d(x,z) ≤ d(x,y) + d (y, z) for all x, y, z ∈ X .
and the pair (X, d) is called a **pseudo metric space**.

Clearly, every metric space is pseudo metric space, but the converse may not be true. We have the following example of Pseudo metric space:

Example 1.5. [20]

Consider a function d:ℝ ×ℝ → [0,∞) defined by
$d(x, y) = |x^2 - y^2|$, then d is Pseudo metric but not metric.
Since d(-1,1) = 0 for -1 ≠ 1, in this example, we get x = y $\implies$ d(x,y) = 0 but d(x,y) = 0 $\not\Rightarrow$ x = y.

1.3 Probabilistic metric space

We have the following definition of distribution function useful to define the probabilistic metric space.[17, 21, 22]

Definition 1.6. [17]:

For the set $\mathbb{R}$ of real numbers, a function $F : \mathbb{R} \to [0, 1]$ is called a **distribution function** if it satisfies the followings:

1. F is non-decreasing;
2. F is left-continuous; and
3. $\inf_{x\in\mathbb{R}} F(x) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$.

If X is a non-empty set, $F : X \times X \to \Delta$ is called probabilistic distance on X, and $F(x, y)$ is usually denoted by F_{xy}. We denote by Δ the family of all **distribution functions** on $(-\infty, \infty)$ and Δ^+ on $[-0, \infty)$.

Example 1.7. Let $\mathcal{H}$ be a maximal element for Δ^+ then, the distribution function $\mathcal{H}$ is defined by

$$\mathcal{H}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

In 1942, Austrian Mathematician Karl Menger [17]introduced the concept of the probabilistic metric as follows:

Definition 1.8. A probabilistic metric space (briefly, PM-space)[21] is an order pair(X,F) where X is non-empty set and F is a function defined by F : X × X → Δ^+ (the set of all distribution functions) that associates a distribution function $F_(p, q)$ with every pair (p, q) of points in X. The distribution function is denoted by $F_(p, q)$ and the symbol $F_{p,q}(x)$ represents the value of $F_{p,q}$ at x ∈ℝ which satisfy following conditions:

PM(i) Fp,q(0) = 0;

PM (ii) $F_{p,q} = F_{q,p}$;

PM (iiI) $F_{p,q}(x) = 1$, for every $x > 0 \Leftrightarrow$ p = q; and

PM (iV) For every p,q,r ∈ X and for every $x, y > 0$,
$F_{p,q}(x) = 1$, $F_{q,r}(x) = 1 \Rightarrow F_{p,r}$ (x + y) = 1.

We have the following example of PM space:

Example 1.9. [21]: Let $X = \mathbb{R}$, $a * b = \min(a, b)$ for all a, b ∈ (0, 1), and $F_{u,w}(x) = \mathcal{H}(x)$ for $u \neq v$ and 1 for $u = v$, where

$$\mathcal{H}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } 0 < x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Then, $(X, F, *)$ is a PM space or a Menger Space.

1.4 Semi-metric space

Karl Menger[29],[30] introduced the notion of Semi-metric space in 1928 as follows:

Definition 1.10. [29]: Let X be a non empty set. A mapping d : X × X → [0, ∞) is called a semi-distance if and only if for any x, y ∈ X, we have

SM 1. d(x, y) = 0 if and only if x = y; and

SM 2. d(x, y) = d(y, x)
In this case, the pair (X, d) is called a **semi-metric space**.

We have the following example for semi-metric space:

Example 1.11. [29]:

Consider $X = [0, 1]$. Let a function d on X be defined as

$$d(x, y) = (x - y)^2, \qquad \forall x, y \in X.$$

Then (X, d) is a semi-metric space but not a metric space since d does not satisfy triangle inequality as

$$d(x, y) = (x - z + z - y)^2$$

$$\leq 2[(x - z)^2 + (z - y)^2]$$

$$= 2[d(x, z) + d(z, x)]$$

$$i.e. d(x, y) \leq 2[d(x, z) + d(z, x).$$

1.5 2-metric space

The concept of 2-metric space was introduced in the early 1963s by a German mathematician S. Gähler[26] as follows:

Definition 1.12. [26]: Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be a **2-metric** on X if for all x, y, z in X, we have

$(2 - M1.)$ given distinct elements x,y of X, there exists an element z of X such that d(x, y, z) $\neq$ 0;

$(2 - M2.)$ d(x, y, z) = 0 when at least two of x, y, z are equal;

$(2 - M3.)$ d(x, y, z) = d(x, z, y) = d(y, z, x) for all x, y, z $\in$ X; and

$(2 - M4.)$ d(x, y, z) $\leq$ d(x, y, w) + d(x, w, z)+d(w, y, z), x, y, z, w $\in$ X.

We have the following example of 2 metric space:

Example 1.13. Let a mapping d : $\mathbb{R} \to [0, \infty)$ be defined by
d(x, y, z) = min $\{|x - y|, |y - z|, |z - x|\}$
Then d is a 2-metric on $\mathbb{R}$, i.e., the following inequality holds:
d(x, y, z) $\leq$ d(x, y, t) + d(y, z, t) + d(z, x, t), for arbitrary real numbers x, y, z, t.

1.6 Fuzzy metric space

We have the following definition of continuous t-norms:

Definition 1.14. [16]: A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a **continuous t-norm** if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

M. Kramosil and J. Michalek [27] from Czech Republic introduced the concept of fuzzy metric space in 1975 as follows:

Definition 1.15. [27]:

A 3-tuple (X, M, ∗) is said to be a **fuzzy metric space** if X is an arbitrary set, ∗ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions: for all x; y; z ∈ X and $s, t > 0$,

(FM 1) M(x, y, t) > 0;

(FM 2) M(x, y, t) = 1 if and only if x = y;

(FM 3) M(x,y,t) = M(y, x, t);

(FM 4) M(x, y, t) ∗ M(y, z, s) ≤ M(x, z, t + s); and

(FM 5) M(x, y, .) :$(0, \infty) \to (0, 1]$ is continuous .
Then M is called a fuzzy metric on X. The function M(x, y, t) denote the degree of nearness between x and y with respect to t. Also, we consider the following condition in the fuzzy metric spaces (X, M, ∗) .

(FM 6) $\lim_{t\to\infty}$M(x, y, t) = 1 ∀ x; y ∈ X.

We have the following example of fuzzy metric space.

Example 1.16. [16], [27]:

Let (X, d) be a metric space. We define a ∗ b = ab for all a,b ∈ [0, 1] and let M be fuzzy set on $X^2 \times (0,\infty)$ defined as follows: M(x,y,t) = $\frac{t}{t+d(x,y)}$. Then M(x, y, ∗) is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard fuzzy metric.

For this, we verify the conditions (FM1) to (FM6).

We need only to verify (FM4) others are obvious.

For (FM4), we have

$$M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$$

$$M(x,y,t) = \frac{t}{t+d(x,y)}, \quad M(y,z,s) = \frac{s}{s+d(y,z)}$$

$$M(x,y,t) * M(y,z,s) = \frac{t}{t+d(x,y)} \cdot \frac{s}{s+d(y,z)}$$

We need to show that:

$$\frac{t}{t+d(x,y)} \cdot \frac{s}{s+d(y,z)} \leq \frac{t+s}{t+s+d(x,z)}$$

Using the inequality property of fractions and the triangle inequality of metric d, we get

$$d(x,z) \leq d(x,y) + d(y,z)$$

$$\frac{t \cdot s}{(t+d(x,y))(s+d(y,z))} \leq \frac{t+s}{t+s+d(x,y)+d(y,z)}$$

Since both sides have the same denominators' summation, this inequality holds.

Thus, the condition (FM4) is satisfied.

1.7 Ultra Metric Space

The concept of ultra-metric spaces was introduced by the Dutch mathematician J. V. Roovij in 1978 as follows [25]:

Definition 1.17. [25]: An ultra metric d on a set M is a real-valued function $d\colon M \times M \to \mathbb{R}$ such that for all $x, y, z \in M$ we have

UM 1. $d(x,y) \geq 0$;

UM 2. $d(x,y) = d(y,x)$;

UM 3. $d(x,x) = 0$;

UM 4. if $d(x,y) = 0$ then $x = y$; and

UM 5. $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ (strong triangle inequality or ultra metric inequality).

The pair (M,d) consisting of a set M together with an ultra metric d on M is called **ultra-metric space**. It is also known as super- metric space.

We have the following example of Ultra metric space.

Example 1.18. [25]: Let X $\neq \phi$, be a non-empty set and metric d is defined on X by

$$d(x,y) = \begin{cases} 0 & if \text{ x = y}, \\ 1 & if \text{ x} \neq y. \end{cases} \tag{1.1}$$

To verify that (X, d) is an ultrametric space, we need to show that d satisfies the ultrametric inequality for all $x, y, z \in X$: $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ Consider the possible cases of $d(x, y)$, $d(y, z)$, and $d(x, z)$:

1. If $x = y$ and $y = z$, then $x = z$, and we have
   $d(x, y) = d(y, z) = d(x, z) = 0$. Clearly, $0 \leq \max\{0, 0\}$.
2. If $x = y$ and $y \neq z$, then $d(x, y) = 0$, $d(y, z) = 1$, and $d(x, z) = 1$. Thus, we have$1 \leq \max\{0, 1\} = 1$.
3. If $x \neq y$ and $y = z$, then $d(x, y) = 1$, $d(y, z) = 0$, and $d(x, z) = 1$.
   Thus, we have $1 \leq \max\{1, 0\} = 1$.
4. If $x \neq y$, $y \neq z$, and $x \neq z$, then: $d(x, y) = 1$, $d(y, z) = 1$, and
   $d(x, z) = 1$.

Thus, we have $1 \leq \max\{1, 1\} = 1$. In all cases, the ultrametric inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ is satisfied. Therefore, the discrete metric is an ultra metric on X.

1.8 D-metric space

The concept of D-metric space was introduced by an Indian mathematician B.C. Dhage [32]in 1992 as follows [1]:

Definition 1.19. [31]: Let X be a nonempty set, and let $\mathbb{R}$ denotes the real numbers. A function D : $X^3 \to \mathbb{R}$ satisfying the following axioms

($DM1$) D(x, y, z) $\geq$ 0 for all x, y, z $\in$ X;

($DM2$) D(x, y, z) = 0 if and only if x = y = z;

($DM3$) D(x, y, z) = D(x,z,y) = D(y, x, z)= ...(symmetry in all three variables),

($DM4$) D(x, y, z) $\leq$ D(x, y, a) + D(x, a, z) + D(a, y, z) for all x, y, z, a $\in$ X. (rectangle inequality),

is called a generalized metric, or a D-metric on X.
The pair (X,D) is called the **D-metric space**.

We have the following example of D-metric space.

Example 1.20. [23, 31]: Let (X, d) be a metric space.Then, with metric D defined on X by;

(d1) D(x, y, z) = 1/3 (d(x, y) + d(y, z) + d(x, z)),

is a D- metric spaces.

1.9 b - Metric Space (or Metric Type Space)

In 1993, S. Czerwik[11] introduced the concept of b-metric spaces by relaxing the triangle inequality as follows:

Definition 1.21. [11]: Let X be a nonempty set and s $\geq$ 1 be a given real number. Then, a function d :X $\times$ X $\rightarrow [0, \infty)$ is a b-metric on X if, for all x, y, z $\in$ X, the following conditions hold:

($bM1$) d(x,y) = 0 if and only if $x = y$;

($bM2$) d(x,y) = d(y, x); and

($bM3$) d(x,z) $\leq s[d(x, y) + d(y, z)]$,for every s $\geq$ 1 (b-triangular inequality).

In this case, the pair (X, d) is called a **b - metric space** (or metric type space). If $s = 1$, then the triangle inequality in a metric space is satisfied.

We have the following examples of b-metric space:

Example 1.22. Let X = $\{-1, 0, 1\}$. We consider a function $d : X \times \ X \rightarrow \mathbb{R}^+$ given by d(x,y) = $(x - y)^2$ for all x, y $\in$ X,
Then

$$d(x, x) = 0, \forall x \in X, d(-1, 0) = 1, d(-1, 1) = 4, d(0, 1) = 1.$$

Then (X,d) is a b metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have

$$d(-1, 0) + d(0, 1) = 1 + 1 = 2 < 4 = d(-1, 1).$$

i.e. $d(-1, 1) \geq (d(-1, 0) + d(1, 0))$, so the triangle inequality is not satisfied.

Example 1.23. Let $M = \mathbb{R}$, the set of real numbers and the function $d : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ be defined as

$$d(x, y) = |x - y|^2.$$

Then d is a b-metric on $\mathbb{R}$. Clearly, the first two conditions are satisfied. For the third condition, we have

$$\begin{aligned} |x-y|^2 &= |x-z+z-y|^2 = |x-z|^2 + 2|x-z||z-y| + |z-y|^2 \\ &\leq 2\left[|x-z|^2 + |z-y|^2\right] \end{aligned}$$

since
$2|x-z||z-y| \leq |x-z|^2 + |z-y|^2$.

1.10 Partial Metric space

The notion of this partial metric space was introduced by English mathematician S.G. Matthews [3] in 1994.

Definition 1.24. [3]: Let X be a nonempty set and let p : X × X → [0,∞) satisfy

($pM1$) p(x, y) = p(y, x) for all x, y ∈ X;

($pM2$) if 0 ≤ p(x,x) = p(y, y) = p(x,y) then x = y;

($pM3$) p(x, x) ≤ p(x,y) ; and

($pM4$) p(x,z) ≤ p(x,y) + p(y,z) - p(y,y) ∀ x,y,z ∈X.
Then (X, p) is called a **partial metric space** and p is called a partial metric on X

We have the following example of partial metric space.

Example 1.25. [13]:

Consider the set $X = \{a, b\}$ and define the partial metric p as follows:

$$p(x,y) = \begin{cases} 0, & \text{if } x = y = a \\ 1, & \text{if } x = y = b \\ 1, & \text{if } x \neq y \end{cases}$$

Now, let's verify that this is indeed a partial metric space:

Reflexivity: For $x = a$: $p(a,a) = 0 \leq p(a,b) = 1$ and $0 \leq p(a,a) = 0$.
Also, for $x = b$: $p(b,b) = 1 \leq p(b,a) = 1$ and $1 \leq p(b,b) = 1$.

Symmetry: Clearly, the definition is symmetric:
$p(a,b) = 1 = p(b,a)$ and $p(a,a) = 0 = p(a,a)$ and $p(b,b) = 1 = p(b,b)$.

Self-Distance Property: $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$.
If $x = y$: $p(a,a) = 0, \quad p(b,b) = 1$.
If $x \neq y$:$p(a,b) = 1 \quad$ and $\quad p(b,a) = 1$.

Triangle Inequality: $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.
For $x = y = z = a$, we get $p(a,a) = 0 \leq p(a,a) + p(a,a) - p(a,a) = 0 + 0 - 0 = 0$.
For $x = y = z = b$, we have $p(b,b) = 1 \leq p(b,b) + p(b,b) - p(b,b) = 1 + 1 - 1 = 1$.
For $x = a, y = b, z = a$, we have $p(a,a) = 0 \leq p(a,b) + p(b,a) - p(b,b) = 1 + 1 - 1 = 1$.
For $x = b, y = a, z = b$, we get $p(b,b) = 1 \leq p(b,a) + p(a,b) - p(a,a) = 1 + 1 - 0 = 2$.

Thus, the function p defined above satisfies all the properties of a partial metric space.

1.11 Rectangular metric space

A. Branciari [10] in 2000 introduced the notion of a generalized (rectangular) metric space where the triangle inequality of a metric space was replaced by another inequality.

Definition 1.26. [15]: Let X be a nonempty set, and let $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$ (each of them different from x and y), satisfying the following conditions:

(RM 1) $d(x,y) = 0$ if and only if $x = y$;

(RM 2) $d(x,y) = d(y,x)$; and

(RM 3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ (the rectangular inequality).

Then (X,d) is called a Branciari type **generalized or rectangular metric space**. Every metric space is a Branciari type generalized metric space, but the converse is not necessarily true.

We have the following example of rectangular metric space:

Example 1.27. [15]

Let $X = A \cup B$, where $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ and $B = [1,2]$.Let us define the generalized metric d on X as follows:

$$d(x,y) = \begin{cases} d(y,x), & \text{if } x, y \in X, \\ 0, & \text{if } x, y \in X \text{ with } x = y, \\ 0.3, & \text{if } x = \frac{1}{2}, y = \frac{1}{3} \text{ or } x = \frac{1}{4}, y = \frac{1}{5}, \\ 0.2, & \text{if } x = \frac{1}{2}, y = \frac{1}{5} \text{ or } x = \frac{1}{3}, y = \frac{1}{4}, \\ 0.6, & \text{if } x = \frac{1}{2}, y = \frac{1}{4} \text{ or } x = \frac{1}{5}, y = \frac{1}{3}, \\ |x - y|, & \text{if } x, y \in B \text{ or } x \in A, y \in B. \end{cases}$$

Then (X, d) is a Branciari type generalized metric space, but it is not a metric space. In fact,

$$0.6 = d\left(\frac{1}{2}, \frac{1}{4}\right) > d\left(\frac{1}{2}, \frac{1}{3}\right) + d\left(\frac{1}{3}, \frac{1}{4}\right) = 0.5.$$

1.12 G-metric space

In 2003, Z. Mustafa and B. Sims [1], attempted to remove the weakness of the D-metric by introducing G-metrics.

Definition 1.28. [1]: Let X be a nonempty set. Suppose that G: $X \times X \times X \to [0,+\infty)$ is a function satisfying the following conditions:

GM 1. $G(x, y, z) = 0$ if and only if $x = y = z$;

GM 2. $0 < G(x, x, y)$ for all x,y $\in$ X with x $\neq$ y;

GM 3. $G(x, x, y) \leq G(x, y, z)$ for all x, y, z $\in$ X with x $\neq$ y;

GM 4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables); and

GM 5. G(x, y, z) $\leq$ G(x, a, a) + G(a, y,z) for all x, y, z, a $\in$ X.

Then G is called a G-metric on X and (X, G) is called a **G-metric space**.

We have the following example of G-metric space.

Example 1.29. [9] Let $X = \mathbb{R}$ be a set of real numbers, and d be a Euclidean metric on X. Consider the function $G : X \times X \times X \to \mathbb{R}$ defined by

$$G(x, y, z) = \frac{1}{3}[d(x, y) + d(y, z) + d(z, x)], \text{for all} \quad x, y, z \in X.$$

Then the function G(x,y,z) is a metric on X.
Solution: For this, we need to verify the following properties by G on X.

Property 1: $G(x, x, y) = 0$ if and only if $x = y$
For,

$$G(x, x, y) = \frac{1}{3}[d(x, x) + d(x, y) + d(y, x)] = \frac{1}{3}[0 + d(x, y) + d(x, y)] = \frac{2}{3}d(x, y)$$

Thus, $G(x, x, y) = 0$ if and only if $d(x, y) = 0$, which implies $x = y$ as d is a metric.
Property 2. $G(x, x, y) \geq 0$. Since d is a metric, $d(x, y) \geq 0$ for all $x, y \in X$. Therefore,

$$G(x, x, y) = \frac{2}{3}d(x, y) \geq 0$$

Property 3: Symmetry

$$G(x,y,z) = \frac{1}{3}[d(x,y) + d(y,z) + d(z,x)]$$

This function G is symmetric in x, y, z . we know the addition is commutative. So, $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(y,z,x) = G(z,x,y) = G(z,y,x)$

Property 4:Triangle Inequality

We need to show that

$$G(x,y,z) \leq G(x,y,w) + G(x,w,z) + G(w,y,z).$$

Since

$$G(x,y,z) = \frac{1}{3}[d(x,y) + d(y,z) + d(z,x)]$$

and $G(x,y,w) + G(x,w,z) + G(w,y,z) =$

$$\frac{1}{3}[d(x,y) + d(y,w) + d(w,x)] + \frac{1}{3}[d(x,w) + d(w,z) + d(z,x)] + \frac{1}{3}[d(w,y) + d(y,z) + d(z,w)]$$

Adding these terms, we get

$$G(x,y,w) + G(x,w,z) + G(w,y,z) = \frac{1}{3}[d(x,y) + d(y,w) + d(w,x)] +$$

$$[d(x,w) + d(w,z) + d(z,x)] + [d(w,y) + d(y,z) + d(z,w)]$$

By the triangle inequality property of the metric d, we get

$$d(x,z) \leq d(x,w) + d(w,z),$$

$$d(y,z) \leq d(y,w) + d(w,z),$$

and

$$d(x,y) \leq d(x,w) + d(w,y).$$

Using these inequalities, we get

$$d(x,y)+d(y,z)+d(z,x) \leq d(x,y)+d(y,w)+d(w,x)+d(x,w)+d(w,z)+d(z,x)+d(w,y)+d(y,z)+d(z,w)$$

Therefore,

$$G(x,y,z) \leq G(x,y,w) + G(x,w,z) + G(w,y,z)$$

Thus, $G(x,y,z) = \frac{1}{3}[d(x,y) + d(y,z) + d(z,x)]$ satisfies all the properties of a G-metric on X.

1.13 Modular metric space

In 2010, V. Chistyakov [7] introduced the concept of a metric modular on a set as follows:

Definition 1.30. [7]: Let X be a nonempty set. A function

$$w : (0, \infty) \times X \times X \to [0, \infty],$$

written as $(\lambda, x, y) \mapsto w_\lambda(x, y)$ is said to be metric modular if it satisfies the following three axioms:

MM 1. $w_\lambda(x, y) = 0$ if and only if $x = y$,for all $\lambda > 0$ and $x, y \in X$;

MM 2. $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$; and

MM 3. $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(y, z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

The ordered pair (X, w) is called a **metric modular space** where X is a set and w is a metric modular on X.

We have the following example of modular metric space:

Example 1.31. Let X be a nonempty set. A function w on X such that

$$w : (0, \infty) \times X \times X \to [0, \infty] \text{defined by}$$

$$w_\lambda(x, y) = \frac{d(x, y)}{\lambda}$$

is a metric modular. Here we think $w_\lambda(x, y)$ as the average velocity of traveling from x to y in time λ.

1.14 D^*-metric spaces

Iranian mathematician S. Sedghi [12] introduced the D^*-metric spaces in 2007 by weakening the triangle inequality.

Definition 1.32. [12]: Let X be a nonempty set. A **generalized metric (or D^*-metric)** on X is a function, $D^* : X^3 \to [0, \infty)$, that satisfies the following conditions for each x, y, z, a ∈ X:

(D*1) D^*(x,y,z) ≥ 0;

(D*2) D^*(x,y,z) $= 0$ if and only if $x = y = z$;

(D*3) D^*(x,y,z) $= D^*$p((x,y,z)),(symmetry) where p is permutation function; and

(D*4) D^*(x,y,z) $\leq$ D^*(x,y,a) + D^*(a,z,z).

We have the following example of D^*-metric space.

Example 1.33. [12]: Let $X = \mathbb{R}$ be a set, and consider the function $d^* : X \times X \times X \to \mathbb{R}$ defined by

$$d^*(x, y, z) = |x - y| + |y - z| + |z - x|$$

for all

$$x, y, z \in X$$

. Then d^* is a D^*metric space.

1.15 S-metric Space

S. Sedghi, N. Shobe and A. Aliouche [12] introduced the concept of S- metric space in 2011 as follows:

Definition 1.34. [12]: Let X be a nonempty set. An S-metric on X is a function S : X^3 $\to$ [0,+∞) that satisfies the following conditions for each x, y, z, a $\in$ X :

(S_1) S (x, y, z) = 0 if and only if x = y = z.

(S_2) S (x, y, z) $\leq$ S(x, x, a) + S (y, y, a) + S (z, z, a).

The pair (X,S) is called **S-metric space**.

We have the following example of S-metric space.

Example 1.35. [12]: Let $X = \mathbb{R}^n$ and let $\| \cdot \|$ be a norm on X. Then, the function $d : X \times X \times X \to \mathbb{R}$ defined by

$$d(x, y, z) = \| \, y + z - 2x \| \; + \| \, y - z \|$$

is an S-metric on X.

1.16 Complex valued metric space

A. Azam, B. Fisher and M. Khan [6]introduced the notion of a complex metric space in 2011 as follows:

Definition 1.36. [6]:

Let X be a nonempty set, where $\mathbb{C}$ is the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:

$(CM1)$: $0 \preccurlyeq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

$(CM2)$: $d(x,y) = d(y,x)$ for all $x,y \in X$; and

$(CM3)$: $d(x,y) \preccurlyeq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a complex-valued metric on X, and (X,d) is called a **complex-valued metric space** .

We have the following example of complex valued metric space.

Example 1.37. [6]:

Let $X = [0,1]$. Also, we define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1, z_2) = e^{i\frac{\pi}{6}}|z_1 - z_2|$. Then (X,d) is a complex-valued metric space.

The conditions to check are as follows:

(d1): For any $z_1, z_2 \in X$, $0 < e^{i\frac{\pi}{6}}|z_1 - z_2|$, and $e^{i\frac{\pi}{6}}|z_1 - z_2| = 0$ if and only if z_1=z_2.

(d2): $d(z_1, z_2) = e^{i\frac{\pi}{6}}|z_1 - z_2| = e^{i\frac{\pi}{6}}|z_2 - z_1| = d(z_2, z_1)$.

(d3): $d(z_1, z_2) = e^{i\frac{\pi}{6}}|z_1 - z_2|$ and $d(z_1, z) + d(z, z_2) = e^{i\frac{\pi}{6}}|z_1 - z| + e^{i\frac{\pi}{6}}|z - z_2|$. We need to show that $e^{i\frac{\pi}{6}}|z_1 - z_2| \leq e^{i\frac{\pi}{6}}|z_1 - z_2| + e^{i\frac{\pi}{6}}|z_1 - z_2|$, which follows from the triangle inequality for real numbers.

Since all three conditions are satisfied, (X,d) is a complex-valued metric space.

1.17 Circular metric space

In 2014, P. Chaipunya, Y.J. Cho and P. Kumam [8] introduced the concept of circular metric space as follows:

Definition 1.38. [8]: Let X be a non-empty set. A function $\delta : \mathbb{R}^+ \times X \times X \to \mathbb{R}^+$ is said to be circular metric if the following conditions are satisfied:

$(C1.)$ $\delta_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

$(C2.)$ $\delta_\lambda(x,y) = \delta_\lambda(y,x)$ for all $(\lambda, x, y) \in \mathbb{R}^+ \times X \times X$;

$(C3.)$ For any $\lambda > 0$ and $x,y,z \in X$, we can find $\mu in(0,\lambda)$ such that

$$\delta_\lambda(x,y) \leq \delta_\mu(x,z) + \delta_{\lambda\mu}(z,y).$$

In this case, the pair (X,δ) is called circular metric space. Additionally, if $\delta_{.}(x,y)$ is non-increasing at each $(x,y) \in X \times X$, we call (X,δ) a natural circular metric space.

Clearly, a metric space and a modular metric space are natural circular metric space but a natural circular metric space is not a modular metric space [7].

1.18 $\mathcal{F}$-metric space

In 2018, Saudi Arabian mathematicians M. Jleli Mohamd and Bessem Samet [19] introduced the concept of F-metric space which is a generalization of the Banach Contraction principle.

Definition 1.39. [19]: Let X be a nonempty set, and let $D : X \times X \to [0, +\infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that

(D 1) $D(x, y) = 0 \Leftrightarrow x = y \ \forall \ x, y \in X$

(D 2) $D(x, y) = D(y, x), \ \forall \ x, y \in X$

(D3) For every $(x, y) \in X \times X$, for every $N \in \mathbb{N}$, $N \geq 2$, and for every $(u_i)_{i=1}^{N} \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$D(x, y) > 0 \Rightarrow f(D(x, y)) \leq \sum_{i=1}^{N-1} D(u_i, u_{i+1}) + \alpha.$$

Then D is said to be an $\mathcal{F}$-metric on X, and the pair (X, D) is said to be an $\mathcal{F}$-**metric space**.

We have the following examples:

Example 1.40. [19]: Let $X = \mathbb{N}$, and let $D : X \times X \to [0, +\infty)$ be the mapping defined by

$$D(x, y) = \begin{cases} (x-y)^2, & \text{if } (x, y) \in [0, 3] \times [0, 3], \\ |x-y|, & \text{if } (x, y) \in [0, 3] \times [0, 3], \end{cases}$$

for all $(x, y) \in X \times X$.
Then D is an $\mathcal{F}$-metric on X

1.19 Cone metric space

We have the following definition of a cone and cone metric space [4]

Definition 1.41. [4]: Let E be a real Banach space. A subset P of E is called a cone if and only if the following hold:

(a1) P is closed, nonempty, and $P \neq \{0\}$,

(a2) For all $a, b \in \mathbb{R}$, $ab \geq 0$, and $x, y \in P$ imply that $ax + by \in P$,

(a3) If $x \in P$ and $-x \in P$, then $x = 0$.

Given a cone $P \subset E$, we define a partial ordering with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}(P)$, where $\text{int}(P)$ denotes the interior of P.

The cone P is called normal if there exists a number $K > 1$ such that $0 \leq x \leq y$ implies $x \leq Ky$ for all $x, y \in E$. The least positive number satisfying this condition is called the normal constant E_K.

The generalisation of metric spaces ware re- introduced in 2007 by Huang and Zhang under the name of cone metric spaces

Definition 1.42. [4]:

Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a cone metric on X if it satisfies the following conditions:

$C_1 :$ $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$,

$C_2 :$ $d(x, y) = d(y, x)$ for all $x, y \in X$,

$C_3 :$ $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then, (X, d) is called a cone metric space.

We have the following example of cone metric space.

Example 1.43. [4] Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E | x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space

2 Conclusion

On the basis of the above definitions and examples, we have obtained the following inter-relations among these generalized forms of metric space. The Psedu-metric space, partial metric space, probabilistic metric space, 2-metric space, rectangular metric space, D-metric space, G-metric space, D*-metric space, Fuzzy metric space, and S-metric spaces are all generalized forms of metric space. However, cone metric spaces are not the real generalization of metric spaces, by re-norming by Banach space. Consequently, every cone metric space is really a metric space and every theorem in metric spaces is valid for cone metric spaces automatically[33]. Also, Ultra metric space, b-metric space, Modular metric space, circular metric space are all the extension of metric space. However, the dislocated metric space, quasi metric space, semi metric space are the particular cases of the metric space.

Finally, the generalization of metric space has always played a crucial role as it provides a more flexible framework for measuring similarity of distance between objects in various mathematical contexts and even in the development of functional analysis. The generalizations of metric space are important to explore & study mathematical structures beyond the conventional Euclidean space $\mathbb{R}^n$ and to address the needs of complex analysis as well as differential equations relative to applications. There have been many attempts to generalize the definition of metric space in order to obtain possibility for more general fixed point results. The metric fixed point theory is a well-established topic of research activities & the famous Banach contraction principle has a huge impact on the establishment of this theory. The paper of Jha [5] deals with several examples of metric spaces with some applications. With a metric chosen good enough, that is, it satisfies the required properties, we can show the existence and uniqueness of fixed points of a given mapping in a metric space. The proof of this principle is constructive, so that it also demonstrates the technique of determining fixed points via iteration. Our paper has interestingly & significantly included the updated classical generalized forms of metric space with examples & also their interrelations. Also, this paper can definitely be a source for the researchers to study and to establish new fixed point results in these forms with applications.

3 Acknowledgment

The authors are thankful to the referees for their valuable suggestions to improve this paper.

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