The Nepali Mathematical Sciences Report
Regd No.: 53/033-034, ISSN: 2392–411X
http://www.journal.cdmathtu.edu.np

Year: 2024, Volume: 41, No. 2
Pages: 79-89
https://doi.org/10.3126/nmsr.v41i2.73227

Research Article

Double Integrals Involving Generalized Hypergeometric Function and Its Applications

Ganesh Bahadur Basnet[1], Narayan Prasad Pahari[*2], and Resham Prasad Paudel[3]

[1,3]Department of Mathematics, Tri-Chandra Multiple Campus, Tribhuvan University, Nepal
gbbmath@gmail.com, reshamprpaudel@gmail.com
[*2]Central Department of Mathematics, Tribhuvan University, Nepal
nppahari@gmail.com

Received: 12 August 2024, Accepted: 02 December 2024, Published Online: 31 December 2024

Abstract

This work aims to evaluate double integrals involving generalized hypergeometric functions. The integration of generalized hypergeometric functions, which extends the classical Gauss hypergeometric function ${}_pF_q$ plays a crucial role in various areas of mathematical analysis. We explore both definite and indefinite double integrals, highlighting special cases where the results can be expressed in simple form. These integrals offer valuable insights into the structure and properties of the hypergeometric function, with potential implications for various areas of mathematical analysis..

Keywords: Hypergeometric Function, Gamma function, Gauss's Theorem, Watson's Theorem, Dixon's Theorem, Whipple's Theorem, Dougal's Theorem, Double Integration.
AMS(MOS) Subject Classification: 33C20.

1 Introduction and Preliminaries:

Double integrals with generalized hypergeometric functions play an important role in mathematics, physics, and engineering. This is the extension of the concept of single integral to higher dimension, helping to analyze complex systems with multiple variables [7],[9]. When generalized hypergeometric functions, denoted as ${}_pF_q$, are involved, these integrals become powerful tools for solving advanced mathematical problems. Generalized hypergeometric functions are special functions with wide-ranging applications. Their unique properties,

*Corresponding author

such as convergence, transformation, and summation rules, make them useful in evaluating complex integral expressions [2], [5]. Incorporating them into double integrals allows for the discovery of new identities, closed-form solutions, and connections with other special functions [2].

This paper explores double integrals involving generalized hypergeometric functions. The main goals are to establish new double integral results, analyze convergence conditions, and highlight their special cases which are applicable in various areas of mathematical analysis.

Gamma function:

The gamma function of n is denote by $\Gamma(n)$ and is defined by [1]

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, Re(n) > 0$$

where $\Gamma(n+1) = n\Gamma(n)$, $\Gamma(n+1) = n!$, & $\Gamma(1/2) = \sqrt{\pi}$

$$\frac{\Gamma(z)}{\Gamma(z-n)} = (-1)^n \frac{\Gamma(-z+n+1)}{\Gamma(-z+1)} \tag{1.1}$$

Beta function: Beta function of m and n is denoted by $B(m,n)$ and is defined by [1]

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, Re(m) > 0, Re(n) > 0 \;\&\; B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{1.2}$$

Pochhammer Symbol: Pochhammer Symbol was introduced by the German mathematician Leo Pochhammer (1841-1920). Pochhammer symbol is defined by [1],[2]

$$(b)_n = \prod_{k=1}^{n} (b+k-1, \quad (b)_n = \frac{\Gamma(b+n)}{\Gamma(b)}, \quad (b)_0 = 1, \quad (1)_n = n! \tag{1.3}$$

where n is a non- negative integer.

Gauss's Hypergeometric Function: In 1812, Gauss systematically studied the series [2]

$$1 + \frac{\alpha.\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + ... \tag{1.4}$$

The ordinary hypergeometric series is another name for this series (1.4), which is frequently referred to as the Gauss's series or function. It can be regarded as an extension of the geometric series

$$1 + x + x^2 + x^3 + ... \tag{1.5}$$

The series (1.4) is denoted by ${}_2F_1[\alpha, \beta; \gamma; x]$ or ${}_2F_1\left[\begin{matrix} \alpha, & \beta, & \\ & & ;x \\ \gamma & & \end{matrix}\right]$ that is

$${}_2F_1\left[\begin{matrix} \alpha, & \beta, & \\ & & ;x \\ \gamma & & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!} \tag{1.6}$$

where α and β are numerator parameters, while γ is the denominator parameter. For $\gamma \neq 0, -1, -2, -3, \ldots$ and α or β is a negative integer, the series (1.6) will terminate. The Gauss's hypergeometric series (1.6) is [2]
i. convergent if $|x| < 1$, diverges is $|x| > 1$,
ii. convergent if $R(\gamma - \alpha - \beta) > 0$ when $x = 1$,
iii. divergent if $R(\gamma - \alpha - \beta) \leq 0$ when $x = 1$,
iv. convergent absolutely if $R(\gamma - \alpha - \beta) > 0$ when $x = -1$,
v. convergent but not absolutely if $-1 \leq R(\gamma - \alpha - \beta) < 0$ when $x = -1$,
vi. divergent if $R(\gamma - \alpha - \beta) \leq -1$ when $x = -1$.
The natural generalization of the Gauss's hypergeometric function ${}_2F_1$ is called the generalized hypergeometric function ${}_pF_q$ and is defined by [2], [3], [4]

$$ {}_pF_q \left[\begin{array}{ccc} \alpha_1, & ..., & \alpha_p, \\ & & ; x \\ \beta_1, & ..., & \beta_q \end{array} \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_p)_n}{(\beta_1)_n ... (\beta_q)_n} \frac{x^n}{n!} \tag{1.7} $$

The generalized hypergeometric function (1.7) is [2]
i. convergent for all finite x if $p \leq q$,
ii. convergent for $|x| < 1$ if $p = q + 1$,
iii. divergent for $|x| > 1$ if $p = q + 1$,
iv. divergent for $x \neq 0$ if $p > q + 1$,
v. absolutely convergent on $|x| > 1$ if $p = q + 1$ and $Re(\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i) > 0$.

2 Classical Summation Theorems and Some fundamental Double Integrals:

a. Classical Summation Theorems:

Classical summation theorems, like Gauss's for the hypergeometric series ${}_2F_1$, Watson's, Dixon's and Whipple's for the series ${}_3F_2$, are important role in the theory of hypergeometric series. These theorems are presented in this section to ensure that the paper remain self-contained. Here, we provide a brief of the classical summation theorems for the series ${}_2F_1$, ${}_3F_2$, as mentioned below [2], [4], [11];
i. Gauss's Summation Theorem: For $R(\gamma - \alpha - \beta) > 0$ and $\gamma \neq 0, -1, -2, -3, \ldots$

$$ {}_2F_1 \left[\begin{array}{cc} \alpha, & \beta \\ & ; 1 \\ \gamma & \end{array} \right] = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \tag{2.1} $$

ii. Watson's Summation Theorem: For negative integer α

$$ {}_3F_2\left[\begin{array}{ccc} \alpha, & \beta, & \gamma \\ & & ;1 \\ \frac{\alpha+\beta+1}{2}, & 2\gamma & \end{array}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\gamma+\frac{1}{2})\Gamma(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2})\Gamma(\gamma-\frac{\alpha}{2}-\frac{\beta}{2}+\frac{1}{2})}{\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{\beta}{2}+\frac{1}{2})\Gamma(\gamma-\frac{\alpha}{2}+\frac{1}{2})\Gamma(\gamma-\frac{\beta}{2}+\frac{1}{2})} \tag{2.2} $$

iii. Dixon's Summation Theorem: The well-poised ${}_3F_2$ with unit argument,

$$ {}_3F_2\left[\begin{array}{ccc} \alpha, & \beta, & \gamma \\ & & ;1 \\ 1+\alpha-\beta, & 1+\alpha-\gamma & \end{array}\right] $$
$$ = \frac{\Gamma(1+\frac{\alpha}{2})\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\frac{\alpha}{2}-\beta-\gamma)}{\Gamma(1+\alpha)\Gamma(1+\frac{\alpha}{2}-\beta)\Gamma(1+\frac{\alpha}{2}-\gamma)\Gamma(1+\alpha-\beta-\gamma)} \tag{2.3} $$

iv. Whipple's Summation Theorem: When $\alpha+\beta=1$ and $\delta_1+\delta_2=2\gamma+1$,

$$ {}_3F_2\left[\begin{array}{ccc} \alpha, & \beta, & \gamma \\ & & ;1 \\ \delta_1, & \delta_2 & \end{array}\right] = \frac{2^{2c-1}\Gamma(\frac{1}{2})\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\frac{\alpha}{2}+\frac{\delta_1}{2})\Gamma(\frac{\alpha}{2}+\frac{\delta_2}{2})\Gamma(\frac{\beta}{2}+\frac{\delta_1}{2})\Gamma(\frac{\beta}{2}+\frac{\delta_2}{2})} \tag{2.4} $$

b. Some fundamental and applicable important double integrals:[5], [6];

$$ \int_0^\infty \int_0^\infty x^{2p-1}y^{2q-1}e^{-(x^2+y^2)}dx\, dy = \frac{1}{4}\Gamma(p)\Gamma(q) \tag{2.5} $$

$$ (b)_n = \frac{\Gamma(b+n)}{\Gamma(b)} = \frac{1}{\Gamma(b)}\int_0^\infty e^{-s}s^{b+n-1}ds \tag{2.6} $$

$$ \int_0^1 \int_0^1 x^{c-1}y^{c+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{l-\alpha-\beta}dx\, dy $$
$$ = \frac{\Gamma(\alpha)\Gamma(\beta}{\Gamma(\alpha+\beta}\frac{\Gamma(c)\Gamma(l)}{\Gamma(c+l)} \tag{2.7} $$

3 Main Results

In this section, a notable double integral of the generalized hypergeometric function is computed in terms of the gamma function, along with some remarkable special cases.

Theorem 3.1:[9], [10]
For $0 \le p, q \le 1/4$,

$$ \int_0^\infty \int_0^\infty x^{-2p}y^{4p-1}e^{-(x^2+y^2)} \times {}_2F_1\left[\begin{array}{cc} p, & q \\ & ;-\frac{x^2}{y^2} \\ \frac{p+q+1}{2} & \end{array}\right] dx\, dy $$

$$= \frac{1}{4}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-p)\Gamma(2p)\Gamma(\frac{3}{2}-2p)\Gamma(\frac{p+q+1}{2})\Gamma(\frac{3}{2}-\frac{5p}{2}-\frac{q}{2})}{\Gamma(\frac{1}{2}+\frac{p}{2})\Gamma(\frac{1}{2}+\frac{q}{2})\Gamma(\frac{3}{2}-\frac{5p}{2})\Gamma(\frac{1}{2}-2p-\frac{q}{2})} \tag{3.1}$$

Proof:

Let left hand side of (3.1) be denoted by I.

$$Let\ I = \int_0^\infty \int_0^\infty x^{-2p}y^{4p-1}e^{-(x^2+y^2)} \times {}_2F_1\left[\begin{matrix} p, & q \\ & & ;-\frac{x^2}{y^2} \\ \frac{p+q+1}{2} \end{matrix}\right] dx\ dy$$

$$= \int_0^\infty \int_0^\infty x^{-2p}y^{4p-1}2e^{-(x^2+y^2)} \sum_{n=0}^{\infty} \frac{(p)_n(q)_n(-1)^n}{(\frac{p+q+1}{2})_n n!}\frac{x^{2n}}{y^{2n}} dx\ dy$$

Interchanging the order of integration and summation, we get

$$I = \sum_{n=0}^{\infty} \frac{(p)_n(q)_n(-1)^n}{(\frac{p+q+1}{2})_n n!} \int_0^\infty \int_0^\infty x^{2(-p+1/2+n)-1}y^{2(2p-n)-1}e^{-(x^2+y^2)} dx\ dy$$

Using (2.5) and then (1.1), we obtain

$$I = \frac{1}{4}\Gamma(\frac{1}{2}-p)\Gamma(2p) \times {}_3F_2\left[\begin{matrix} p, & q, & \frac{1}{2}-p \\ & & & ;1 \\ \frac{p+q+1}{2}, & 1-2p \end{matrix}\right]$$

Using Watson's theorem (2.2), we obtain the required result

Theorem 3.2 (Laplace's Integral):

The following double integral formula hold true [5]

i. For $Re(\alpha), Re(\beta) > 0$,

$$\int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-(s+t)}{}_0F_1\left[\begin{matrix} - \\ & ;xts \\ \gamma \end{matrix}\right] dt\ ds = \Gamma(\alpha)\Gamma(\beta)\ {}_2F_1\left[\begin{matrix} \alpha, & \beta \\ & & ;x \\ \gamma \end{matrix}\right] \tag{3.2}$$

ii. For $Re(\gamma-\alpha-\beta) > 0$

$$\int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-(s+t)}{}_0F_1\left[\begin{matrix} - \\ & ;ts \\ \gamma \end{matrix}\right] dt\ ds = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{3.3}$$

iii.

$$\int_0^\infty \int_0^\infty t^{\beta-1} s^{\gamma-1} e^{-(s+t)} {}_1F_2 \left[\begin{matrix} \alpha & & \\ & & ; ts \\ \frac{\alpha+\beta+1}{2} & , 2\gamma & \end{matrix} \right] dt\, ds$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(\beta)\Gamma(\gamma)\Gamma(\gamma)\Gamma(\gamma+\frac{1}{2})\Gamma(\frac{\alpha+\beta+1}{2})\Gamma(\gamma+\frac{1}{2}-\frac{\alpha}{2}-\frac{\beta}{2})}{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta+1}{2})\Gamma(\gamma+\frac{1}{2}-\frac{\alpha}{2})\Gamma(\gamma+\frac{1}{2}-\frac{\beta}{2})} \tag{3.4}$$

iv.

$$\int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-(s+t)} {}_1F_2 \left[\begin{matrix} \gamma & & \\ & & ; ts \\ 1+\alpha-\beta & , 1+\alpha-\gamma & \end{matrix} \right] dt\, ds$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{\alpha}{2}+1)\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\frac{\alpha}{2}-\beta-\gamma)}{\Gamma(1+\alpha)\Gamma(1+\frac{\alpha}{2}-\beta)\Gamma(1+\frac{\alpha}{2}-\gamma)\Gamma(1+\alpha-\beta-\gamma)} \tag{3.5}$$

v. If $\alpha+\beta=1, \delta_1+\delta_2=2\gamma+1$

$$\int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-(s+t)} {}_1F_2 \left[\begin{matrix} \gamma & & \\ & & ; ts \\ \delta_1 & , \delta_2 & \end{matrix} \right] dt\, ds$$

$$= \pi\, 2^{1-2\gamma}\Gamma(\alpha)\Gamma(\beta) \frac{\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\frac{\alpha+\delta_1}{2})\Gamma(\frac{\alpha+\delta_2}{2})\Gamma(\frac{\beta+\delta_1}{2})\Gamma(\frac{\beta+\delta_2}{2})} \tag{3.6}$$

Proof (i):

Let left-hand side of (3.2) be denoted by I.

$$Let \quad I = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-(s+t)} {}_0F_1 \left[\begin{matrix} - & \\ & ; xts \\ \gamma & \end{matrix} \right] dt\, ds$$

$$= \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-(s+t)} \sum_{n=0}^{\infty} \frac{(1}{(\gamma)_n} \frac{x^n t^n s^n}{n!} dt\, ds$$

Interchanging the sign of integration and the summation, we obtain

$$I = \Gamma(\beta) \int_0^\infty e^{-t} t^{\alpha-1} \sum_{n=0}^{\infty} \left[\frac{1}{\Gamma(\beta)} \int_0^\infty e^{-s} s^{\beta+n-1} ds \right] \frac{(1}{(\gamma)_n} \frac{x^n t^n}{n!} dt$$

Using (2.6), we obtain

$$\int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-(s+t)} {}_0F_1 \left[\begin{matrix} - & \\ & ; xts \\ \gamma & \end{matrix} \right] dt\, ds = \Gamma(\alpha)\Gamma(\beta)\, {}_2F_1 \left[\begin{matrix} \alpha, & \beta & \\ & & ; x \\ \gamma & & \end{matrix} \right]$$

Proof (ii):

Putting $x = 1$ in (3.3) and let L.H.S be I, we obtain

$$I = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-(s+t)} {}_0F_1 \left[\begin{matrix} - \\ \\ \gamma \end{matrix} ; ts \right] dt\, ds$$
$$= \Gamma(\alpha)\Gamma(\beta)\ {}_2F_1 \left[\begin{matrix} \alpha, \quad \beta \\ \\ \gamma \end{matrix} ; 1 \right]$$

Using Gauss's summation theorem (2.1), we obtain the required result.

Similarly, to prove(iii), putting x=1 and using Watson's theorem (2.2).
For (iv), putting x=1 and using Dixon's theorem (2.3).
For (v), putting x=1 and using Whipple's theorem (2.4).

Theorem 3.3:

For $\delta \in \mathbb{Z}, Re(\alpha), Re(\beta), Re(\gamma), Re(\gamma+\delta+1) > 0, n = 0, 1, 2, 3, ...$, the following double integral formulas hold true [7] [8], [10]

$$i. \int_0^1 \int_0^1 x^{\gamma-1} y^{\gamma+\alpha-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{\gamma+\delta-\alpha-\beta+1} \times$$
$${}_3F_2 \left[\begin{matrix} -2n, \quad a+2n, \quad 2\gamma+\delta+1 \\ \\ \frac{a+1}{2} \qquad , 2\gamma \end{matrix} ; xy \right] dx\, dy$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \frac{\Gamma(\gamma)\Gamma(\gamma+\delta+1)}{\Gamma(2\gamma+\delta+1)} \frac{(\frac{1}{2})_n (\frac{a}{2}-\gamma+\frac{1}{2})_n}{(\gamma+\frac{1}{2})_n (\frac{a}{2}+\frac{1}{2})_n} \tag{3.7}$$

$$ii. \int_0^1 \int_0^1 x^{\gamma-1} y^{\gamma+\alpha-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{\gamma-\alpha-\beta} \times$$
$${}_3F_2 \left[\begin{matrix} -2n-1, \quad a+2n+1, \quad 2\gamma+\delta+1 \\ \\ \frac{a+1}{2} \qquad , 2\gamma \end{matrix} ; 1-xy \right] dx\, dy = 0 \tag{3.8}$$

$$iii. \int_0^1 \int_0^1 x^{\gamma+\delta} y^{\gamma+\delta+\alpha} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{\gamma-\alpha-\beta} \times$$
$$ {}_3F_2 \left[\begin{matrix} -2n, & a+2n, & 2\gamma+\delta+\alpha \\ & & & ;1-xy \\ \frac{a+1}{2} & ,2\gamma & \end{matrix} \right] dx\,dy$$
$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \frac{\Gamma(\gamma)\Gamma(\gamma+\delta+1)}{\Gamma(2\gamma+\delta+1)} \frac{(\frac{1}{2})_n (\frac{a}{2}-\gamma+\frac{1}{2})_n}{(\gamma+\frac{1}{2})_n (\frac{a}{2}+\frac{1}{2})_n} \tag{3.9}$$

$$iv. \int_0^1 \int_0^1 x^{\gamma+\delta} y^{\gamma+\delta+\alpha} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{\gamma-\alpha-\beta} \times$$
$$ {}_3F_2 \left[\begin{matrix} -2n-1, & a+2n+1, & 2\gamma+\delta+1 \\ & & & ;xy \\ \frac{a+1}{2} & ,2\gamma & \end{matrix} \right] dx\,dy = 0 \tag{3.10}$$

Proof 3.3(i): Let left-hand side of (i) be denoted by I. Then

$$I = \int_0^1 \int_0^1 x^{\gamma-1} y^{\gamma+\alpha-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{\gamma+\delta-\alpha-\beta+1} \times$$
$$ {}_3F_2 \left[\begin{matrix} -2n, & a+2n, & 2\gamma+\delta+1 \\ & & & ;xy \\ \frac{a+1}{2} & ,2\gamma & \end{matrix} \right] dx\,dy$$

$$= \int_0^1 \int_0^1 x^{\gamma-1} y^{\gamma+\alpha-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{\gamma+\delta-\alpha-\beta+1} \times$$
$$\sum_0^\infty \frac{(-2n)_n \ (a+2n)_n \ (2\gamma+\delta+1)_n}{(\frac{a+1}{2})_n \ (2\gamma)_n} \frac{x^n y^n}{n!} dx\,dy$$

Interchanging the order of integration and summation signs, we obtain

$$I = \sum_0^\infty \frac{(-2n)_n \ (a+2n)_n \ (2\gamma+\delta+1)_n}{(\frac{a+1}{2})_n \ (2\gamma)_n \ n!} \times \int_0^1 \int_0^1 x^{\gamma+n-1} y^{\gamma+\alpha+n-1} (1-x)^{\alpha-1} (1-y)^{\beta-1}$$
$$(1-xy)^{\gamma+\delta-\alpha-\beta+1} dx\,dy$$

Using (2.7), we obtain

$$I = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \frac{\Gamma(\gamma)\Gamma(\gamma+\delta+1)}{\Gamma(2\gamma+\delta+1)} {}_3F_2 \left[\begin{matrix} -2n, & a+2n, & \gamma \\ & & & ;1 \\ \frac{a+1}{2} & ,2\gamma & \end{matrix} \right]$$

Using Watson's theorem (2.2) and gamma function (1.1), we obtain the required result. Similarly, we can prove remaining (ii), (iii) and (iv).

3.1 Special cases:

In this section, we introduce several significant special cases of our main integrals from above mentioned theorems, as described in the following corollaries;

Corollary (3.1.1): If we put $\alpha = \beta = \frac{1}{2}, \gamma = 2$ in (3.3), then we obtain the following result

$$\int_0^\infty \int_0^\infty \frac{e^{-(s+t)}}{\sqrt{ts}} {}_0F_1 \left[\begin{matrix} - \\ \\ 2 \end{matrix} ; ts \right] dt\, ds = 4$$

Corollary (3.1.2): If we put $\alpha = \beta = \gamma = \frac{1}{2}$ in (3.4), then we obtain the following result

$$\int_0^\infty \int_0^\infty \frac{e^{-(s+t)}}{\sqrt{ts}} {}_1F_2 \left[\begin{matrix} \frac{1}{2} & , \\ & , \\ 1, 1 & \end{matrix} ; ts \right] dt\, ds = \frac{\pi^{5/2}}{\Gamma^4(\frac{3}{4})}$$

Corollary (3.1.3): If we put $\alpha = \beta = \delta_1 = \delta_2 = \frac{1}{2}, \gamma = 1$ in (3.6), then we obtain the following result

$$\int_0^\infty \int_0^\infty \frac{e^{-(s+t)}}{\sqrt{ts}} {}_1F_2 \left[\begin{matrix} 1, & \\ & , \\ \frac{1}{2}, \frac{1}{2} & \end{matrix} ; ts \right] dt\, ds = \frac{\pi}{2}$$

Corollary (3.1.4): If we put $a = 2, \alpha = \beta = \gamma = \delta = 1$ in (3.7), then we obtain the following result

$$\int_0^1 \int_0^1 y(1-xy) \times {}_3F_2 \left[\begin{matrix} -2n, & 2+2n, & 4 \\ & & \\ \frac{3}{2} & , 2 & \end{matrix} ; xy \right] dx\, dy = \frac{1}{3(2n+1)^2} \qquad (3.11)$$

Corollary (3.1.5): If we put $a = 2, \alpha = \beta = \gamma = \delta = 1$ in (3.9), then we obtain the following result

$$\int_0^1 \int_0^1 \frac{x^2 y^3}{(1-xy)} \times {}_3F_2 \left[\begin{matrix} -2n, & 2+2n, & 4 \\ & & \\ \frac{3}{2} & , 2 & \end{matrix} ; 1-xy \right] dx\, dy = \frac{1}{3(2n+1)^2} \qquad (3.12)$$

Corollary (3.1.6): If we put $a = 1, \alpha = \beta = \gamma = \frac{1}{2}, \delta = 1$ in (3.7), then we obtain the following result

$$\int_0^1\int_0^1 \frac{(1-xy)^{3/2}}{\sqrt{x(1-x)(1-y)}} \times {}_3F_2\left[\begin{matrix} -2n, & 1+2n, & 3 \\ & & & ;xy \\ 1 & ,1 & \end{matrix}\right] dx\,dy = \frac{\pi^{3/2}}{(n!)^4}\left[\frac{(2n)!}{2^{2n}}\right]^2 \tag{3.13}$$

Corollary (3.1.7): If we put $a = 1, \alpha = \beta = \gamma = \frac{1}{2}, \delta = 1$ in (3.9), then we obtain the following result

$$\int_0^1\int_0^1 \frac{x^{3/2}y^2}{\sqrt{(1-x)(1-y)(1-xy)}} \times {}_3F_2\left[\begin{matrix} -2n, & 1+2n, & 5/2 \\ & & & ;1-xy \\ 1 & ,1 & \end{matrix}\right] dx\,dy = \frac{\pi^{3/2}}{(n!)^4}\left[\frac{(2n)!}{2^{2n}}\right]^2 \tag{3.14}$$

Corollary (3.1.8): If we put $a = 3, \alpha = \beta = \gamma = \frac{3}{2}, \delta = 1$ in (3.7), then we obtain the following result

$$\int_0^1\int_0^1 y^2\sqrt{x(1-x)(1-y)(1-xy)} \times {}_3F_2\left[\begin{matrix} -2n, & 3+2n, & 5 \\ & & & ;xy \\ 2 & ,3 & \end{matrix}\right] dx\,dy = \frac{5\pi^2}{1024(n!)^4}\left[\frac{(2n)!}{2^{2n}(n+1)}\right]^2 \tag{3.15}$$

Corollary (3.1.9): If we put $a = 3, \alpha = \beta = \gamma = \frac{3}{2}, \delta = 1$ in (3.9), then we obtain the following result

$$\int_0^1\int_0^1 \frac{x^{5/2}y^4\sqrt{(1-x)(1-y)}}{(1-xy)^{3/2}} \times {}_3F_2\left[\begin{matrix} -2n, & 3+2n, & 11/2 \\ & & & ;1-xy \\ 2 & ,3 & \end{matrix}\right] dx\,dy = \frac{5\pi^2}{1024(n!)^4}\left[\frac{(2n)!}{2^{2n}(n+1)}\right]^2 \tag{3.16}$$

4 Conclusion

This study introduces three new theorems that offer finite and infinite double integral representation of the generalized hypergeometric functions, broadening its theoretical framework and potential uses. Theorem complemented by corollaries, showcasing applications of these results. These contributions not only deepen the understanding of hypergeometric function but also create opportunities for their application in areas like mathematical Physics and engineering. This research provides both theoretical advancements and practical tools, paving the way for future research in this domain.

References

[1] J. P. Hannah. Identification of the gamma function: an overview from Euler to the present, Thesis *(University of the Witwatersrand)*, South Africa, (2013).

[2] E. D. Rainville. Special Functions, *The Macmillan Company*, New York, USA, (1960).

[3] Y. S. Kim, A. K. Rathie and R. B. Paris. A note on a hypergeometric transformation formula due to Slater with an application, *arXiv-1409.8527v1* [Math CA], (2014).

[4] L. J. Slater. Generalized Hypergeometric Functions, *Cambridge University Press*, (1966).

[5] H. Exton. Handbook of Hypergeometric Integrals, Theory, Applications, Tables, Computer Program, *John Wiley and Sons*, New York, (1978).

[6] N. M. Temme. Special Function an Introduction to the Classical Function of Mathematical Physics, *John Wiley and Sons*, (1996).

[7] I. Kim. A new class of double integral involving generalized hypergeometric function, *Tamkang Journal of Mathematics* , vol. 51, no. 1 (2018).

[8] W. Chammam, A. K. Rathie and M. Khlifi. A note on two new integral representations for the Gauss hypergeometric function with an application, *Bulletin of the Belgian Mathematical Society*, (2024).

[9] J. Choi and A.K. Rathie. A new class of double integrals involving generalized hypergeometric function, *Advanced Studies in Contemporary Mathematics* 27(2017), N0. 2,189-198.

[10] E. Diekema. A remarkable double integral of the product of two Gaussian hypergeometric functions, *arXiv.2302.13901v1 [Math CA]*, (2023)

[11] Y. S. Kim, M. A. Rakha and A. K. rathie. Extensions of certain classical summation theorems for the series ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$ with applications in Ramanujan's summations, *International Journal of Mathematics and Mathematical Sciences, Hindawi Publishing Corporation* (2010).