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Research Article

Hermite-Hadamart Integral Inequality for Differentiable m-Convex Functions

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Abstract

Convexity in connection with integral inequalities is an interesting research domain in recent years. The convexity theory plays a fundamental role in the development of various branches of applied sciences since it includes the theory of convex functions that possesses the two important attributes viz. a boundary point is where the maximum value is reached and any local minimum value is a global one. Convexities and inequalities are connected which has a basic character in many branches of pure and applied disciplines. The most important inequality related to convex function is the Hermite-Hadamard integral inequality. The extensions, enhancements and generalizations of this inequality has motivated the researchers in recent years. This paper is an extension of some inequalities connected with difference of the left-hand part as well as the right-hand part from the integral mean in Hermite- Hadamard's inequality for the case of m- convex functions.

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1 Introduction

A convex set is the set where the line segments joining any two points lie entirely on it. A convex function is one whose epigraph is a convex set. The theory of convex functions falls under the umbrella of convexity. It is incorporated in almost all branches of Mathematics. The study of classical inequalities is one of the greatest applications of the theory of convex

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functions since it offers a simple, beautiful, and unified solution to some of the most well-known mathematical inequalities. The topic of convexity is fairly broad and encompasses the convex function theory. It is seen as a characteristic inherent in function. Furthermore, it is distinctive, new, and advantageous due to its minimizing property. It holds a prominent position in the fields of probability theory, calculus of variations, and optimization theory. Convex function has a lengthy history. At the end of the eighteenth century is when it all started. Its first influences can be traced to the basic contributions made by Ch. Hermite (1881), O. Holder (1889), J. Hadamard (1893), and O. Stolz (1898). The first mathematician to recognize the significance of convex functions and begin a systematic study of them was J. L. W. V. Jensen (1905), and subsequent research on it has given rise to the theory of convex function as an independent field of mathematical analysis. For details see [1, 2, 4, 6, 8, 9]. The Hermite-Hadamard (H-H) inequality substantially impacted in the study of convex functions. In the mathematical literature, the following inequality, see[3]

$$\Phi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(x)\,dx \leq \left(\frac{\Phi(\alpha)+\Phi(\beta)}{2}\right) \tag{1.1}$$

is usually connected to Hadamard's name who proved it in 1893. In 1974, D.S. Mitrinivic found a modest note which was published in the journal Mathesis in 1883. It was an extract from a letter by Ch. Hermite in 1881 announcing the inequality 1.1 . Therefore, it seems that it was Ch. Hermite who obtained it for the first time. However, it is interesting that Hermite's note remained unknown for a so long time for experts in the history and theory of convex function. So, the aforementioned inequality is nowadays known as Hermite-Hadamard's integral inequality which is formally defined as follows: Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $\alpha, \beta \in I$ with $\alpha < \beta$. Then, the following double inequality

$$\Phi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)\,dx \leq \frac{\Phi(\alpha)+\Phi(\beta)}{2} \tag{1.2}$$

is known as Hermite-Hadamard's type inequality. For a function to be convex, this is both the necessary and sufficient condition. For a particular choice of the function Φ in above inequality yields some classical inequalities of means. If the function Φ is concave, then both inequalities hold in the opposite direction. The definition of classical convex function in literature is given as follows:

Definition 1.1. A function $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ is said to be a convex function on I or arithmetically-arithmetic convex function or simply the convex function if the following inequality

$$\Phi(\nu\alpha + (1-\nu)\beta) \leq \nu\Phi(\alpha) + (1-\nu)\Phi(\beta)$$

holds for every $\alpha, \beta \in I$with $\alpha < \beta$ and $0 \leq \nu \leq 1$. The concavity of the function Φ holds if the inequality is reversed.

Definition 1.2. [5] A function $\Phi : [0, \beta] \to \mathbb{R}$ is said to be an m-convex function on $[0, \beta]$, if

$$\Phi(\nu\alpha + m(1-\nu)\beta) \leq \nu\Phi(\alpha) + m(1-\nu)\Phi(\beta)$$

holds good for every $\alpha, \beta \in I$, $\beta > 0$ with $\alpha < \beta$ and $m \in [0, 1]$.

Remark 1.3. The m-convex function reduces to the classical convex function if $m = 1$.

In this paper, the first section includes a brief history of Hermite-Hadamard type integral inequality together with the concept of convex functions. The second section incorporates some preliminary results on integral mean function related to left-hand part as well as the right-hand part of Hermite-Hadamard type integral inequality. The third section, the main results, incorporates the extended results on inequalities connected with the left-hand part as well as right-hand part of inequality 1.2 for the case of m-convex functions with the help of the results given in preliminary section.

2 PRELIMINARY RESULTS

In this section, the definition of well established line segment and some previous results are stated which will be further extended to m-convex functions. The results connected with the left hand part of inequality 1.2 are given by U.S. Kirmaci [10] in the following lemma and theorems.

Definition 2.1. The line segment joining any two points α, $\beta \in \mathbb{R}^n$ is denoted by $L[\alpha, \beta]$ is defined as

$$L[\alpha, \beta] = \{\nu\alpha + (1-\nu)\beta : \quad \nu \in [0, 1]\}$$

Lemma 2.2. *Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0. And, $\alpha, \beta \in I$ (I^0 is the interior of I) with $\alpha < \beta$. If $\Phi' \in L[\alpha, \beta]$, then we have*

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi(x)\,dx - \Phi\left(\frac{\alpha+\beta}{2}\right) = (\beta-\alpha)\left[\int_0^{\frac{1}{2}} \nu\Phi'(\nu\alpha + (1-\nu)\beta)\,dt + \int_{\frac{1}{2}}^1 (\nu-1)\Phi'(\nu\alpha + (1-\nu)\beta)\,d\nu\right].$$

U.S. Kirmaci[10] also obtained the following inequalities using Lemma 2.2 .

Theorem 2.3. *Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0. Let $\alpha, \beta \in I$ with $\alpha < \beta$. If $|\Phi'|$ is convex on $L[\alpha, \beta]$, then we have*

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi(x)\,dx - \Phi\left(\frac{\alpha+\beta}{2}\right)\right| \leq \frac{\beta-\alpha}{8}(|\Phi'(\alpha)| + |\Phi'(\beta)|).$$

Theorem 2.4. *[10] Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0. And, $\alpha, \beta \in I$ with $\alpha < \beta$, and let $p > 1$. If the mapping $|\Phi'|^{\frac{p}{p-1}}$ is convex on $[\alpha, \beta]$, then we have*

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\,dx - \Phi\left(\frac{\alpha+\beta}{2}\right)\right| \leq \frac{\beta-\alpha}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}$$
$$\left(\left(|\Phi'(\alpha)|^{\frac{p}{p-1}} + 3|\Phi'(\beta)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(3|\Phi'(\alpha)|^{\frac{p}{p-1}} + |\Phi'(\beta)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right).$$

Theorem 2.5. *[10] Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0. And, $\alpha, \beta \in I$ $\alpha < \beta$, and let $p > 1$. If the mapping $|\Phi'|^{\frac{p}{p-1}}$ is convex on $[\alpha, \beta]$, then the following inequality holds:*

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi(x)\,dx - \Phi\left(\frac{\alpha+\beta}{2}\right)\right| \leq \frac{\beta-\alpha}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}} (|\Phi'(\alpha)| + |\Phi'(\beta)|.$$

Some results connected with the right hand part of inequality 1.1 are also established by Dragomir and Agrawal [7] in the following lemma and theorems.

Lemma 2.6. *Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, \alpha, \beta \in I^0$ with $\alpha < \beta$. If $\Phi' \in L[\alpha, \beta]$, then the following equality holds:*

$$\frac{\Phi(\alpha)+\Phi(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi(x)\,dx = \frac{\beta-\alpha}{2}\int_0^1 (1-2\nu)\Phi'(\nu\alpha + (1-\nu)\beta)\,d\nu.$$

Theorem 2.7. *[7] Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, \alpha, \beta \in I^0$ with $\alpha < \beta$. If $|\Phi'|$ is convex on $[\alpha, \beta]$, then the following inequality holds:*

$$\left|\frac{\Phi(\alpha)+\Phi(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi(x)\,dx\right| = \frac{(\beta-\alpha)(|\Phi'(\alpha)| + |\Phi'(\beta)|}{8}.$$

Theorem 2.8. *[7] Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, \alpha, \beta \in I^0$ with $\alpha < \beta$ and let $p > 1$. If the new mapping $|\Phi'|^{\frac{p}{p-1}}|$ is convex on $[\alpha, \beta]$, then the following inequality holds:*

$$\left|\frac{\Phi(\alpha)+\Phi(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi(x)\,dx\right| \leq \frac{\beta-\alpha}{2(p+1)^{\frac{1}{p}}}\left[\frac{|\Phi'(\alpha)|^{\frac{p}{p-1}} + |\Phi'(\beta)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}.$$

3 MAIN RESULTS

In this section, we obtain two lemmas of equality and then some inequalities connected with the left hand part as well as the right hand part of inequality 1.2 for the case of m-convex functions.

Lemma 3.1. *Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0. And, $\alpha, \beta \in I$ with $\alpha < \beta$. If $\Phi' \in L[\alpha, \beta]$, and m-convex function, then we have*

$$\frac{1}{m\beta-\alpha}\int_\alpha^{m\beta} \Phi(x)\,dx - \Phi\left(\frac{\alpha+m\beta}{2}\right) = (m\beta-\alpha)[\int_0^{\frac{1}{2}} \nu\Phi'(\nu\alpha + m(1-\nu)\beta)\,d\nu$$
$$+ \int_{\frac{1}{2}}^1 (\nu-1)\Phi'(\nu\alpha + m(1-\nu)\beta)\,d\nu].$$

Proof. By using integration by parts, we deduce

$$\int_0^{\frac{1}{2}} \nu\Phi'(\nu\alpha+m(1-\nu)\beta)\,d\nu + \int_{\frac{1}{2}}^1 (\nu-1)\Phi'(\nu\alpha+m(1-\nu)\beta)\,d\nu = \frac{\Phi(\nu\alpha+m(1-\nu)\beta)}{\alpha-m\beta}\nu|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{\Phi(\nu\alpha+m(1-\nu)\beta)}{\alpha-m\beta}\,d\nu + \frac{\Phi(\nu\alpha+m(1-\nu)\beta)}{\alpha-m\beta}(\nu-1)|_0^{\frac{1}{2}} - \int_{\frac{1}{2}}^1 \frac{\Phi(\nu\alpha+m(1-\nu)\beta)}{\alpha-m\beta}\,d\nu$$

$$= \frac{1}{2(\alpha-m\beta)}\Phi\left(\frac{\alpha+m\beta}{2}\right) - \frac{1}{\alpha-m\beta}\int_0^1 \Phi(\nu\alpha+m(1-\nu)\beta)\,d\nu + \frac{1}{2(\alpha-m\beta)}\Phi\left(\frac{\alpha+m\beta}{2}\right)$$

$$= \frac{1}{(\alpha-m\beta)}\Phi\left(\frac{\alpha+m\beta}{2}\right) - \frac{1}{\alpha-m\beta}\int_0^1 \Phi(\nu\alpha+m(1-\nu)\beta)\,d\nu$$

Put $x=\nu\alpha+m(1-\nu)\beta$. So, $dx=(\alpha-m\beta)\,d\nu$ when $\nu=0$, then $x=m\beta$, and when $\nu=1$, then $x=\alpha$. On substituting these values, we have

$$=\frac{1}{\alpha-m\beta}\Phi\left(\frac{\alpha+m\beta}{2}\right) - \frac{1}{\alpha-m\beta}\int_{m\beta}^{\alpha} \frac{\Phi(x)\,dx}{\alpha-mb\beta}$$
$$=-\frac{1}{m\beta-\alpha}\Phi\left(\frac{\alpha+m\beta}{2}\right) + \frac{1}{(\alpha-m\beta)^2}\int_{m\beta}^{\alpha}\Phi(x)dx$$
$$=\frac{1}{m\beta-\alpha}\left[\frac{1}{m\beta-\alpha}\int_{\alpha}^{m\beta}\Phi(x)\,dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right]$$

On simplifying, we obtain

$$\frac{1}{m\beta-\alpha}\int_{\alpha}^{m\beta}\Phi(x)\,dx - \Phi\left(\frac{a\alpha+m\beta}{2}\right) = (m\beta-\alpha)[\int_0^{\frac{1}{2}} \nu\Phi'(\nu\alpha+m(1-\nu)\beta)\,d\nu + \int_{\frac{1}{2}}^1 (\nu-1)\Phi'(\nu\alpha+m(1-\nu)\beta)\,d\nu].$$

This completes the proof. □

Theorem 3.2. *Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0 with $\alpha<\beta$. If $|\Phi'| \in L[\alpha,\beta]$ is an m-convex function on $[\alpha,\beta]$, then we have*

$$\left|\frac{1}{m\beta-\alpha}\int_{\alpha}^{m\beta}\Phi(x)\,dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right| \le \frac{m\beta-\alpha}{8}(|\Phi'(\alpha)|+m|\Phi'(\beta)|).$$

Proof. Using Lemma 2.6 and m-convexity of $|\Phi'|$, we have

$$\left|\frac{1}{m\beta-\alpha}\Phi(x)\,dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right| = |(m\beta-\alpha)[\int_0^{\frac{1}{2}} \nu\Phi'(\nu\alpha+m(1-\nu)\beta)\,d\nu + \int_{\frac{1}{2}}^1 (\nu-1)\Phi'(\nu\alpha+m(1-\nu)\beta)\,d\nu]|$$

$$\le (m\beta-\alpha)[\int_0^{\frac{1}{2}} (\nu^2|\Phi'(\alpha)|+m\nu(1-\nu)|\Phi'(\beta)|)\,d\nu + \int_{\frac{1}{2}}^1 \nu(\nu-1)(|\Phi'(\alpha)|+m(1-\nu)^2|\Phi'(\beta)|)\,d\nu]$$

Here, we have

$$\int_0^{\frac{1}{2}} \nu^2 d\nu = \int_{\frac{1}{2}}^1 (1-\nu)^2\, d\nu = \frac{1}{24}$$

And,

$$\int_0^{\frac{1}{2}} \nu(1-\nu)\, d\nu = \int_{\frac{1}{2}}^1 \nu(1-\nu)\, d\nu = \frac{1}{12}$$

On substituting these values, we have

$$\begin{aligned}
\left|\frac{1}{m\beta-\alpha}\int_\alpha^{m\beta}\Phi(x)\,dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right| &\leq (m\beta-\alpha)\left(\frac{1}{24}|\Phi'(\alpha)| + m\frac{1}{12}|\Phi'(\beta)| + \frac{1}{12}|\Phi'(\alpha)| + m\frac{1}{24}|\Phi'(\beta)|\right)\\
&= (m\beta-\alpha)\left(\frac{1}{8}|\Phi'(\alpha)| + m\frac{1}{8}|\Phi'(\beta)|\right)\\
&= \frac{m\beta-\alpha}{8}\left(|\Phi'(\alpha)| + m|\Phi'(\beta)|\right).
\end{aligned}$$

This completes the proof. □

Remark 3.3. If $m = 1$, then it reduces to the Theorem 2.5.

Theorem 3.4. *Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, \alpha, \beta \in I$ with $\alpha < \beta$ and, $p > 1$. If the mapping $|\Phi'|^{\frac{p}{p-1}}$ is an m- convex function on $[\alpha, \beta]$, then we have*

$$\left|\frac{1}{m\beta-\alpha}\int_\alpha^{m\beta}\Phi(x)\,dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right| \leq \frac{m\beta-\alpha}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}$$
$$\left(\left(|\Phi'(\alpha)|^{\frac{p}{p-1}} + 3m|\Phi'(\beta)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(3|\Phi'(\alpha)|^{\frac{p}{p-1}} + m|\Phi'(\beta)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right).$$

Proof. Using Lemma 2.6 and Holder's integral inequality, we deduce

$$\begin{aligned}
\left|\frac{1}{m\beta-\alpha}\int_\alpha^{m\beta}\Phi(x)\,dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right| &\leq (m\beta-\alpha)[\int_0^{\frac{1}{2}}|\nu||\Phi'(\nu\alpha+m(1-\nu)\beta)|\,d\nu\\
&+\int_{\frac{1}{2}}^1|\nu-1||\Phi'(\nu\alpha+m(1-\nu)\beta)|\,d\nu]\\
&\leq (m\beta-\alpha)[(\int_0^{\frac{1}{2}}\nu^p d\nu)^{\frac{1}{p}}(\int_0^{\frac{1}{2}}|\Phi'(\nu\alpha+m(1-\nu)\beta)^q d\nu)^{\frac{1}{q}}
\end{aligned}$$
$$+(\int_{\frac{1}{2}}^1(\nu-1)^p d\nu)^{\frac{1}{p}}\int_{\frac{1}{2}}^1(|\Phi'(\nu\alpha+m(1-\nu)\beta)|^q d\nu)^{\frac{1}{q}}]$$

where, $\frac{1}{p}+\frac{1}{q}=1$ Using the property of m-convexity of $|\Phi'|^q$, we obtain

$$\int_0^{\frac{1}{2}}|\Phi'(\nu\alpha+m(1-\nu)\beta|^q d\nu \leq \int_0^{\frac{1}{2}}[\nu|\Phi'(\alpha)|^q + m(1-\nu)|\Phi'(\beta)|^q]\,d\nu$$

Here,

$$\int_0^{\frac{1}{2}}\nu\,d\nu = \frac{1}{8}$$

And,

$$\int_0^{\frac{1}{2}} (1-\nu)\, d\nu = \frac{3}{8}$$

$$\begin{aligned} &\leq \frac{|\Phi'(\alpha)|^q}{8} + \frac{3m}{8}|\Phi'(\beta)|^q \\ &= \frac{|\Phi'(\alpha)|^q + 3m|\Phi'(\beta)|^q}{8} \end{aligned}$$

And,

$$\begin{aligned} \int_{\frac{1}{2}}^1 |\Phi'(\nu\alpha + m(1-\nu)\beta)|^q\, d\nu \leq & \int_{\frac{1}{2}}^1 [\nu|\Phi'(\alpha)|^q + m(1-\nu)|\Phi'(\beta)|^q]\, d\nu \\ =& \frac{3}{8}|\Phi'(\alpha)|^q + m\frac{1}{8}|\Phi'(\beta)|^q \\ =& \frac{3|\Phi'(\alpha)|^q + m|\Phi'(\beta)|^q}{8}. \end{aligned}$$

Furthermore, we have

$$\int_0^{\frac{1}{2}} \nu^p\, d\nu = \frac{1}{(p+1)2^{p+1}}$$

And,

$$\int_{\frac{1}{2}}^1 (\nu-1)^p\, d\nu = \frac{1}{(p+1)2^{p+1}}$$

Thus, on combining the above results, we have

$$\left|\frac{1}{m\beta-\alpha}\int_\alpha^{m\beta} \Phi(x)\, dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right| \leq \frac{m\beta-\alpha}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}$$
$$\left((|\Phi'(\alpha)|^{\frac{p}{p-1}} + 3m|\Phi'(\beta)|^{\frac{p}{p-1}})^{\frac{p-1}{p}} + (3|\Phi'(\alpha)|^{\frac{p}{p-1}} + m|\Phi'(\beta)|^{\frac{p}{p-1}})^{\frac{p-1}{p}}\right).$$

This completes the proof. □

Remark 3.5. If $m = 1$, then it reduces to the Theorem 2.7.

Theorem 3.6. *Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, \alpha, \beta \in I$ with $\alpha < \beta$ and, $p > 1$. If the mapping $|f'|^{\frac{p}{p-1}}$ is an m- convex function on $[\alpha, \beta]$, then we have*

$$\left|\frac{1}{m\beta-\alpha}\int_\alpha^{m\beta} \Phi(x)\, dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right| \leq \frac{m\beta-\alpha}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}(|\Phi'(\alpha)| + m|\Phi'(\beta)|)$$

Proof. We consider the inequality of Theorem 3.4

$$\left|\frac{1}{m\beta-\alpha}\int_\alpha^{m\beta} \Phi(x)\, dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right| \leq \frac{m\beta-\alpha}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}$$
$$\left((|\Phi'(\alpha)|^{\frac{p}{p-1}} + 3m|\Phi'(\beta)|^{\frac{p}{p-1}})^{\frac{p-1}{p}} + (3|\Phi'(\alpha)|^{\frac{p}{p-1}} + m|\Phi'(\beta)|^{\frac{p}{p-1}})^{\frac{p-1}{p}}\right)$$

Let,$\alpha_1 = |\Phi'(\alpha)|^q$; $\beta_1 = 3|\Phi'(\beta)|^q$; $\alpha_2 = 3|\Phi'(\alpha)|^q$; $\beta_2 = |\Phi'(\beta)|^q$
Here, $0 \leq \frac{p-1}{p} < 1$ for $p > 1$ Using the fact that,

$$\Sigma_{k=1}^{n}(\alpha_k + \beta_k)^s \leq \Sigma_{k=1}^{n}\alpha_k^s + \Sigma_{k=1}^{n}\beta_k^{s}"$$

for $0 \leq s \leq 1, \alpha_1, \alpha_2, ..., \alpha_n \geq 0, \beta_1, \beta_2, ...\beta_n \geq 0$, we obtain

$$\begin{aligned}\left|\frac{1}{m\beta-\alpha}\int_{\alpha}^{m\beta}\Phi(x)\,dx - \Phi\left(\frac{\alpha+m\beta}{2}\right)\right| &\leq \frac{m\beta-\alpha}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}} 4(|\Phi'(\alpha)| + m|\Phi'(\beta)|)\\ &= \frac{m\beta-\alpha}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}} (|\Phi'(\alpha)| + m|\Phi'(\beta)|).\end{aligned}$$

This completes the proof. □

Now, we obtain some results connected with the right-hand part of inequality 1.2 in case of m-convex function.

Lemma 3.7. *Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, \alpha, \beta \in I^0$ with $\alpha < \beta$. If $\Phi' \in L[\alpha, \beta]$, then the following equality holds:*

$$\frac{\Phi(\alpha) + \Phi(m\beta)}{2} - \frac{1}{m\beta-\alpha}\int_{\alpha}^{m\beta}\Phi(x)\,dx = \frac{m\beta-\alpha}{2}\int_0^1 (1-2\nu)\Phi'(\nu\alpha + m(1-\nu)\beta)\,d\nu.$$

Proof. Let,

$$I = \int_0^1 (1-2\nu)\Phi'(\nu\alpha + m(1-\nu)\beta)\,d\nu$$

Integrating by parts,

$$\begin{aligned}&= \frac{\Phi(\nu\alpha + m(1-\nu)\beta}{(\alpha-m\beta)}(1-2\nu)|_0^1 - \int_0^1 (-2)\frac{\Phi(\nu\alpha + m(1-\nu)\beta)}{(\alpha-m\beta)}\,d\nu\\ &= \frac{\Phi(\nu\alpha + m(1-\nu)\beta)}{\alpha-m\beta}(-1) - \frac{\Phi(m\beta)}{\alpha-m\beta} - \frac{2}{\alpha-m\beta}\int_0^1 \Phi(\nu\alpha + m(1-\nu)\beta)\,d\nu\\ &= -\frac{\Phi(\alpha)+\Phi(m\beta)}{\alpha-m\beta} - \frac{2}{\alpha-m\beta}\int_0^1 \Phi(\nu\alpha + m(1-\nu)\beta)\,d\nu\end{aligned}$$

Put

$$x = \nu\alpha + m(1-\nu)\beta$$

when $\nu = 0$, then $x = m\beta$, and, $\nu = 1$, then $x = \alpha$

$$d\nu = \frac{dx}{\alpha - m\beta}$$

On substituting these values, we obtain

$$\begin{aligned}&= -\frac{\Phi(\alpha)+\Phi(m\beta)}{\alpha-m\beta} - \frac{2}{\alpha-m\beta}\int_{m\beta}^{\alpha}\Phi(x)\frac{dx}{\alpha-m\beta}\\ &= -\frac{\Phi(\alpha)+\Phi(m\beta)}{\alpha-m\beta} - \frac{2}{(\alpha-m\beta)^2}\int_{m\beta}^{\alpha}\Phi(x)\,dx\\ &= \frac{2}{m\beta-\alpha}\left[\frac{\Phi(\alpha)+\Phi(m\beta)}{2} - \frac{1}{m\beta-\alpha}\int_{\alpha}^{m\beta}\Phi(x)\,dx\right].\end{aligned}$$

And, thus, we have

$$\frac{\Phi(\alpha)+\Phi(m\beta)}{2}-\frac{1}{m\beta-\alpha}\int_{\alpha}^{m\beta}\Phi(x)\,dx=\frac{m\beta-\alpha}{2}\int_{0}^{1}(1-2\nu)\Phi'(\nu\alpha+m(1-\nu)\beta)\,d\nu.$$

This completes the proof. □

Remark 3.8. If $m=1$, then it reduces to the equality as given by 2.6.

Theorem 3.9. *Let $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, \alpha, \beta \in I^0$ with $\alpha < \beta$. If $|\Phi'|$ is an m-convex function on $[\alpha, \beta]$, then the following inequality holds:*

$$\left|\frac{\Phi(\alpha)+\Phi(m\beta)}{2}-\frac{1}{m\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)\,dx\right|\leq(m\beta-\alpha)\frac{|\Phi'(\alpha)+m|\Phi'(\beta)}{8}.$$

Proof. Using Lemma 3.7, it follows that

$$\begin{aligned}\left|\frac{\Phi(\alpha)+\Phi(m\beta)}{2}-\frac{1}{m\beta-\alpha}\int_{\alpha}^{m\beta}\Phi(x)\,dx\right|&=\left|\frac{m\beta-\alpha}{2}\int_{0}^{1}(1-2\nu)\Phi'(\nu\alpha+m(1-\nu)\beta)\,d\nu\right|\\&\leq\frac{m\beta-\alpha}{2}\int_{0}^{1}|1-2\nu||\Phi'(\nu\alpha+m(1-\nu)\beta)|d\nu\\&\leq\frac{m\beta-\alpha}{2}\left[|\Phi'(\alpha)|\int_{0}^{1}\nu|1-2\nu|d\nu+m|\Phi'(\beta)|\int_{0}^{1}(1-\nu)|1-2\nu|d\nu\right]\end{aligned}$$

Here,

$$\int_{0}^{1}\nu|1-2\nu|\,d\nu=\int_{0}^{\frac{1}{2}}\nu(1-2\nu)\,d\nu+\int_{\frac{1}{2}}^{1}\nu(2\nu-1)\,d\nu=\frac{1}{4}$$

And,

$$\int_{0}^{1}(1-\nu)|1-2\nu|\,d\nu=\frac{1}{4}$$

On substituting these values, we obtain

$$\begin{aligned}&=\frac{m\beta-\alpha}{2}\left[|\Phi'(\alpha)|\frac{1}{4}+m|\Phi'(\beta)|\frac{1}{4}\right]\\&=\frac{m\beta-\alpha}{8}\left[|\Phi'(\alpha)|+m|\Phi'(\beta)|\right].\end{aligned}$$

This concludes the proof. □

Remark 3.10. If $m=1$, then it reduces to Theorem 2.7 .

Another result is embodied in the following theorem.

Theorem 3.11. *Let, $\Phi : I^0 \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, \alpha, \beta \in I^0$ with $a < b$, and let $p > 1$. If the new mapping $|\Phi'|^{\frac{p}{p-1}}|$ is an m-convex on $[\alpha, m\beta]$, then the following inequality holds:*

$$\left|\frac{\Phi(\alpha)+\Phi(m\beta)}{2}-\frac{1}{m\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)\,dx\right|\leq\frac{m\beta-\alpha}{2(p+1)^{\frac{1}{p}}}\left[\frac{|\Phi'(\alpha)|^{\frac{p}{p-1}}+m|\Phi'\beta|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}.$$

Proof. Using Lemma 3.7 and Holder's inequality, we obtain

$$\begin{aligned}\left|\frac{\Phi(\alpha)+\Phi(m\beta)}{2}-\frac{1}{m\beta-\alpha}\int_{\alpha}^{m\beta}\Phi(x)\,dx\right| &= \left|\frac{m\beta-\alpha}{2}\int_0^1(1-2\nu)\Phi'(\nu\alpha+m(1-\nu)\beta)\,d\nu\right| \\ &\le \frac{m\beta-\alpha}{2}\int_0^1|1-2\nu||\Phi'(\nu\alpha+m(1-\nu)\beta)|\,d\nu \\ &\le \frac{m\beta-\alpha}{2}\left(\int_0^1|1-2\nu|^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(ta+m(1-t)\beta|^q dt\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\frac{1}{p}+\frac{1}{q}=1.$$

Using m-convexity of $|\Phi'|^q$, we have

$$\begin{aligned}\int_0^1|\Phi'(\nu\alpha+m(1-\nu)\beta)|^q\,d\nu &\le \int_0^1[\nu|\Phi'(\alpha)|^q+m(1-\nu)|\Phi'(\beta)|^q]\,d\nu \\ &\le |\Phi'(\alpha)|^q\int_0^1\nu d\nu+m|\Phi'(\beta)|^q\int_0^1(1-\nu)\,d\nu \\ &= \frac{|\Phi'(\alpha)|^q+m|\Phi'(\beta)|^q}{2}\end{aligned}$$

Also,

$$\int_0^1|1-2\nu|^p\,d\nu=\frac{1}{p}$$

On substituting these values, we obtain

$$\left|\frac{\Phi(\alpha)+\Phi(m\beta)}{2}-\frac{1}{m\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)\,dx\right|\le\frac{m\beta-\alpha}{2(p+1)^{\frac{1}{p}}}\left[\frac{|\Phi'(\alpha)|^{\frac{p}{p-1}}+m|\Phi'(\beta)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}.$$

Now the proof is complete. □

Remark 3.12. If $m=1$, then it reduces to the Theorem 2.8 .

4 CONCLUSION

An estimate of a continuous convex function's integral mean value is provided by the Hermite-Hadamard integral inequality. In this study, we have expanded several results on H-H type integral inequalities for differentiable convex functions into differentiable m-convex functions, particularly on the results on the left hand portion and right hand portion of Hermite-Hadamard integral inequality. The interested readers can carry out this technique to enhance some more new results on H-H type inequalities for other kinds of convex functions.

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