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Research Article

Generalized Cesàro Summable Vector Valued Sequence Space of Bounded Type

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Abstract

This work aims to introduce and study new classes of generalized Cesàro summable vector-valued sequence spaces of bounded type . Besides exploring the completeness of the classes $\text{Ces}(X, p)$ and $\text{Ces}(X, p)_\infty$ when topologized with suitable natural p-normed, our primary interest is to study the β- dual of $\text{Ces}(X, p)$.

Keywords: Sequence space, p-normed space, Cesàro sequence space.
AMS(MOS) Subject Classification: $40A05$, $40C05$, $46A45$, $46B45$.

1 Introduction

Before proceeding with the main work, we recall some of the basic notations and definitions that are used in this paper.
A sequence space is defined to be a linear space of scalar (real or complex) valued functions on a countable set. The study of sequence spaces is a special case of the more general study of function spaces. The classical sequence spaces or scalar-valued sequence spaces (real or complex) have proven their worth as a big contribution in introducing the spaces of Cesàro almost convergent sequences (Kuddusi, and Şengönül [5]). Thus the vector-valued sequence spaces, are the natural generalizations of the scalar-valued sequence spaces where the sequence spaces are those of vectors from some vector spaces.

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Several workers like Ghimire and Pahari [2] , Kadak [3], Kolk [4], Maddox [6], Nath & Tripathy [7], Pahari [8],[9], Pokharel & Pahari [10], [11], Pokharel, Pahari & Paudel [12] , Ruckle [14], Tripathy [16], etc. have made their contributions and enriched the theory on sequence spaces in many directions.
The research in summability techniques had grown very fast by many mathematicians and played pioneer roles in the development of different summability methods for divergent sequences and series in various directions.

Let (x_n) be a sequence, and let s_k be its k^{th} partial sum. Then $\sum x_n$ is called Cesàro summmable to A if the sequence of arithmetic mean of its n^{th} partial sum tends to A as n tends to infinity. Here the limit A is called Cesàro sum of the series $\sum x_n$ and we write $\sum x_n = A(C, 1)$.
In particular, the Grandi's series $G = 1 - 1 + 1 - ...$ has partial sum (s_i) given by $\frac{1-(-1)^i}{2}, i = 1, 2, 3, ...$
Using Cesàro summable method we can observe that,

$$c_n = \frac{1}{n}\sum_{i=1}^{n} s_i = \frac{1}{n}\sum_{i=1}^{n} \frac{1-(-1)^i}{2} = \frac{1}{2} - \frac{1}{2n}\sum_{i=1}^{n}(-1)^i$$

and $\lim_{n\to\infty} c_n = \frac{1}{2}$.
This shows that G is Cesàro summable i.e. $(C, 1)-$ summable to $\frac{1}{2}$ and we have $G = \frac{1}{2}(C, 1)$.

2 Classical Sequence Spaces

Let $\overline{x} = (x_k)$ and $\overline{y} = (y_k)$ be two sequences with complex terms $x_i^{,}s$ and $y_i^{,}s$. Let us write

$$\overline{x} + \overline{y} = (x_k + y_k) \text{ and } \alpha\overline{x} = (\alpha x_k), \text{ where } \alpha \in \mathbb{C} \text{ is a complex number.}$$

Let ω be the collection of all sequences of complex numbers and $\overline{x}, \overline{y} \in \omega$ and $\alpha, \beta \in \mathbb{C}$ then $\alpha\overline{x} + \beta\overline{y} \in \omega$.

Definition 2.1:
If X denotes the normed space and $x_k^{,}s$ are the elements of X, then we have the following well-known sequence spaces.

$$c_0(X) = \{\overline{x} = (x_k) : x_k \in X, \|x_k\| \to 0 \; as \; k \to \infty\};$$

$$c(X) = \{\overline{x} = (x_k) : x_k \in X,\ and\ there\ exists\ l \in X\ such\ that\ ||x_k - l|| \to 0\ as\ k \to \infty\}; and$$

$$l_\infty(X) = \{\overline{x} = (x_k) : x_k \in X, sup_{k\geq 1}||x_k|| < \infty\}.$$

The spaces $c_0(X), c(X)$ and $l_\infty(X)$ are used by Maddox [6] and others.

Cesàro sequence spaces of absolute type were defined and studied in the work of Roopaei and Başar [13] and Saejung [15]. Ahmad and Mursaleen [1] defined Cesàro sequence spaces of a bounded type and almost convergent type.

Definition 2.2:
A $p-$ normed space $(X, ||.||)$ is a linear space X together with the mapping $||.|| : X \to \mathbb{R}_+$ (called p-norm on X) with $0 < p < 1$ such that for all $x, y \in X$ and $\alpha \in \mathbb{C}$, we have

$$||x|| \geq 0\ and\ ||x|| = 0\ if\ and\ only\ if\ x = \theta; ||\alpha x|| = |\alpha|^p ||x||,\ and\ ||x + y|| \leq ||x|| + ||y||.$$

A p-normed space is a pair (X, G), where X is a vector space and G is a p-norm on X.

3 The Class Ces (X, p) and Ces $(X, p)_\infty$

Let X be a Banach space over the field $\mathbb{C}$ of complex-numbers. Let $p, p, p, ...$ be a constant sequence of real numbers. For a fixed positive number n and $0 < p < 1$, we define the vector-valued Cesàro sequence spaces of bounded type as follows:

$$Ces(X, p) = \{\overline{x} = (x_k)\, and\, x_k \in X : sup_{k\geq 1, n\geq 0}||\frac{1}{k}\sum_{i=1}^{k} x_{n+i}||^p < \infty\} \tag{3.1}$$

$$Ces(X, p)_\infty = \{\overline{x} = (x_k)\, and\, x_k \in X : sup_{k\geq 1}(\frac{1}{k}\sum_{i=1}^{k} ||x_i||)^p < \infty\} \quad (3.2).$$

4 Main Results

In this section, we shall investigate the result that characterize the completeness of the classes Ces(X, p) and Ces $(X, p)_\infty$ when topologized with the suitable natural p-norm.
Let $\overline{x} \in Ces(X, p)$. Define

$$||\overline{x}|| = sup_{k\geq 1, n\geq 0}||\frac{1}{k}\sum_{i=1}^{k} x_{n+i}||^p \tag{4.1}$$

Theorem 4.1: Ces (X, p) is a complete p-normed space.

Proof:

It is easy to see that $\|x\|$ defined in (4.1) is a well-defined p-norm in Ces (X, p).

To prove the completeness of Ces(X, p), let $(\overline{x}^r)$ be a Cauchy sequence in Ces (X, p),

where $\overline{x}^r = (x_i^r)_{r=1}^{\infty}, r = 1, 2, 3, \ldots$ Given $0 < \epsilon < 1$, there exists a positive integer N_0 such that

$$||\overline{x}^r - \overline{x}^s|| < \epsilon, \forall r, s \geq N_0$$

$$\Rightarrow sup_{k\geq 1, n\geq 0} || \frac{1}{k} \sum_{i=1}^{k} x_{n+i}^r - x_{n+i}^s ||^p < \epsilon, \forall r, s \geq N_0$$

$$\Rightarrow ||x_{n+i}^r - x_{n+i}^s|| < \epsilon^{\frac{1}{p}} < \epsilon, \forall n \geq 0 \;\&\; r, s \geq N_0 \tag{4.2}$$

This shows that for a fixed $i(1 \leq i < \infty)$, the sequence $(x_i^r)_{i=1}^{\infty}$ is a vector-valued Cauchy sequence. Since the space X is complete, therefore $(x_i^r)_{i=1}^{\infty}$ converge in it.

Let $\overline{x_i}^r \to x_i$ as $r \to \infty$. Define $\overline{x} = (x_i)_{i=1}^{\infty}$ and taking limit $s \to \infty$ in (4.2), we get

$$||x_{n+i}^r - x_{n+i}|| \leq \epsilon^{\frac{1}{p}} < \epsilon, \forall n \geq 0 \;\&\; r \geq N_0.$$

Therefore, we have

$$sup_{k\geq 1, n\geq 0} || \frac{1}{k} \sum_{i=1}^{k} x_{n+i}^r - x_{n+i} ||^p \leq \epsilon, \forall r \geq N_0 \tag{4.3}$$

$$or, ||\overline{x}^r - \overline{x}|| \leq \epsilon, \forall\; r \geq N_0$$

Now,

$$||\overline{x}|| = sup_{k\geq 1, n\geq 0} || \frac{1}{k} \sum_{i=1}^{k} x_{n+i} ||^p$$

$$\leq sup_{k\geq 1, n\geq 0} || \frac{1}{k} \sum_{i=1}^{k} x_{n+i} - x_{n+i}^r ||^p + sup_{k\geq 1, n\geq 0} || \frac{1}{k} \sum_{i=1}^{k} x_{n+i}^r ||^p$$

$$< \infty \;\; \text{(using (4.3)}$$

$$\Rightarrow \overline{x} \in Ces(X, p).$$

Therefore, Ces (X,p) is a complete p-normed space. This completes the proof.

In the following, we have to determine the β-dual Ces^{β} (X,p) is $S(p)\cap S_0$, where;
$\text{Ces}^{\beta}(X,p) = \{\overline{a} = (a_k) \in \omega : \sum_{k=1}^{\infty} a_k x_k \text{ converges for every } \overline{x} = (x_k) \in Ces(X,p)\}$ (4.4)

$S(p) = \{\overline{a} = (a_k) \in \omega : \sum_{k=1}^{\infty} k\Delta a_k N^{\frac{1}{p}}$ converges for all $N > 1\}$
where $\Delta a_k = a_k - a_{k+1}$ (4.5)

$S_0 = \{\overline{a} = (a_k) \in \omega : \lim_{k\to\infty} ka_k = 0\}$ (4.6)

Theorem (4.2): $\text{Ces}^{\beta}(X,p) = S(p)\cap S_0$

Proof:
Let $\overline{a} \in S(p)\cap S_0$ and $\overline{x} \in$ Ces (X,p). Choose a natural number $N > 1$, such that

$$N > max\{1, sup_{k\geq 1, n\geq 0}\|\frac{1}{k}\sum_{i=1}^{k} x_{n+i}\|^p\}$$

Applying Abel's partial summation formula for a positive integer m, we have

$$\begin{aligned}\|\sum_{k=1}^{m} a_k x_k\| &= \|\sum_{k=1}^{m-1} k\Delta a_k(\frac{1}{k}\sum_{i=1}^{k} x_i) + ma_m(\frac{1}{m}\sum_{i=1}^{m} x_i)\| \\ &\leq sup_{k\geq 1}\|\frac{1}{k}\sum_{i=1}^{k} x_i\| \; |\sum_{k=1}^{m-1} k\Delta a_k| + |ma_m| \; \|\frac{1}{m}\sum_{i=1}^{m} x_i\| \\ &\leq sup_{k\geq 1, n\geq 0}\|\frac{1}{k}\sum_{i=1}^{k} x_{n+i}\| \; |\sum_{k=1}^{m-1} k\Delta a_k| + |ma_m| \; \|\frac{1}{m}\sum_{i=1}^{m} x_{n+i}\|\end{aligned}$$

Taking $m \to \infty$ and second term of the above relation vanishes because $\overline{a} \in S_0$ and $\frac{1}{m}\sum_{i=1}^{m} x_{n+i}$ is finite, therefore
$\|\sum_{k=1}^{\infty} a_k x_k\| \leq |\sum_{k=1}^{\infty} k\Delta a_k| N^{\frac{1}{p}} < \infty.(using (4.5))$

Thus, $\overline{a} \in \text{Ces}^{\beta}(X,p)$.

Conversely, suppose that $\overline{a} \in \text{Ces}^{\beta}(X,p)/S(p)\cap S_0$, then either $\overline{a} \notin S(p)$ or $\overline{a} \notin S_0$.

Let $\overline{a} \notin S(p)$ and $\overline{a} \in S_0$ i.e. $\sum_{k=1}^{\infty} k\Delta a_k N^{\frac{1}{p}} = \infty$, for some $N > 1$.

Choose a sequence $\overline{x} = (x_k)$ such that $x_k = N^{1/p}$ for some $N > 1$, then $\overline{x} \in \text{Ces}(X,p)$.

Again using Abel's partial summation

$$\|\sum_{k=1}^{m} a_k x_k\| = \|\sum_{k=1}^{m}(k\Delta a_k)(\frac{1}{k}\sum_{i=1}^{k} x_i)\| + \|m a_m(\frac{1}{m}\sum_{i=1}^{m} x_i)\|$$

$$\leq sup_{k\geq 1, n\geq 0}\|\frac{1}{k}\sum_{i=1}^{k} x_{n+i}\| \; |\sum_{k=1}^{m} k\Delta a_k| + |m a_m| \; \|\frac{1}{m}\sum_{i=1}^{m} x_{n+i}\|$$

Taking $m \to \infty$ and the second term in the above relation vanishes because $\overline{a} \in S_0$ and $\frac{1}{m}\sum_{i=1}^{m} x_{n+i}$ remain finite, therefore $\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} k\Delta a_k N^{\frac{1}{p}} = \infty$,

which contradicts that $\overline{a} \in \mathrm{Ces}^{\beta}(X,p)$.

Hence $\overline{a} \in \mathrm{Ces}(X,p)$.

Next, we suppose $\overline{a} \notin S_0$ but $\overline{a} \in S(p)$ then $l = \lim_{k\to 0} k a_k \neq 0$.

Define $\overline{x} = (x_k)$ by $x_k = (-1)^k k$.

Then $\overline{x} \in \mathrm{Ces}(X,p)$, but $\sum_{k=1}^{\infty} a_k x_k = l\sum_{k=1}^{\infty}(-1)^k$,

i.e. $\sum_{k=1}^{\infty} a_k x_k$ does not converge which contradicts that $\overline{x} \in \mathrm{Ces}^{\beta}(X,p)$. Hence $\overline{a} \in S_0$. This completes the proof.

Theorem 4.3: For $\overline{x} \in \mathrm{Ces}(X,p)_\infty$, the class

$\mathrm{Ces}(X,p)_\infty = \{\overline{x} = (x_k) \; and \; x_k \in X : sup_{k\geq 1}(\frac{1}{k}\sum_{i=1}^{k}\|x_i\|)^p < \infty\}$

is a complete p-normed space with p-norm G defined by

$$G(\overline{x}) = sup_{k\geq 1}\frac{1}{k}(\sum_{i=1}^{k}\|x_i\|)^p \qquad (4.7)$$

Proof:

It is easy to see that G defined in (4.7) is a well-defined p-norm in Ces(X,p) as it satisfies

$G(\overline{x}) \geq 0$ and $G(\overline{x}) = 0$ if and only if $\overline{x} = \overline{\theta}$, where $\overline{\theta}$ is the zero element ;

$G(\alpha\overline{x}) = |\alpha|^p G(\overline{x})$ and $G(\overline{x}+\overline{y}) \leq G(\overline{x}) + G(\overline{y})$.

Let $(\overline{x}^n)$ be a Cauchy sequence in Ces $(X,p)_\infty$, and let $0 < \epsilon < 1$ be given. Then there exists an $N_0 \in \mathbb{Z}^+$ such that

$$d(\overline{x}^n, \overline{x}^m) \leq \epsilon, \forall n, m \geq N_0$$

$$\Rightarrow G(\overline{x}^n - \overline{x}^m) \leq \epsilon, \forall n, m \geq N_0$$

$$\Rightarrow sup_{k\geq 1}(\frac{1}{k}\sum_{i=1}^{k} ||x_i^n - x_i^m||)^p < \epsilon, \forall n, m \geq N_0 \tag{4.8}$$

$or, (\frac{1}{k}\sum_{i=1}^{k} ||x_i^n - x_i^m||)^p < \epsilon, \forall n, m \geq N_0 \;\; \& \;\; \forall k \in \mathbb{Z}^+$

$or, ||x_i^n - x_i^m|| < \epsilon, \forall n, m \geq N_0 \;\; \& \;\; \forall i \in \mathbb{Z}^+$

This shows that (x_i^n) is a Cauchy sequence in X. But X is complete, so there exists $x_i \in X$ for each $i \in \mathbb{Z}^+$ such that $x_i^n \to x_i$ as $n \to \infty$.

Define $\overline{x} = (x_i)_{i=1}^{\infty}$ and taking limit $m \to \infty$ in (4.8), we get

$$sup_{k\geq 1}(\frac{1}{k}\sum_{i=1}^{k} ||x_i^n - x_i||)^p < \epsilon, \forall n \geq N_0 \tag{4.9}$$

$$\Rightarrow G(\overline{x}^n - \overline{x}) < \epsilon, \forall n \geq N_0$$

$$or, d(\overline{x}^n, \overline{x}) < \epsilon, \forall n \geq N_0$$

$$or, \overline{x}^n \to \overline{x} \;\; as \;\; n \to \infty.$$

We shall show that $\overline{x}$ actually lies in Ces $(X,p)_\infty$. By triangle inequality, and in view of (4.9), we have

$$d(\overline{x}, \overline{\theta}) \leq d(\overline{x}, \overline{x}^n) + d(\overline{x}^n, \overline{\theta})$$

$$or, G(\overline{x} - \overline{\theta}) \leq G(\overline{x} - \overline{x}^n) + G(\overline{x}^n - \overline{\theta}).$$

$$or, sup_{k\geq 1}(\frac{1}{k}\sum_{i=1}^{k}||x_i-\theta||)^p \leq sup_{k\geq 1}(\frac{1}{k}\sum_{i=1}^{k}||x_i-x_i^n||)^p + sup_{k\geq 1}(\frac{1}{k}\sum_{i=1}^{k}||x_i^n-\theta||)^p$$

$$or, sup_{k\geq 1}(\frac{1}{k}\sum_{i=1}^{k}||x_i-\theta||)^p \leq \epsilon + \ finite\ number\ < \infty$$

Thus,

$$sup_{k\geq 1}(\frac{1}{k}\sum_{i=1}^{k}||x_i||)^p < \infty.$$

This shows that $\overline{x} = (x_i) \in \mathrm{Ces}(X,p)_\infty$ and hence $\mathrm{Ces}(X,p)_\infty$ is a complete $p-$ normed space. This completes the proof.

5 Conclusion

In this work, we have introduced and studied a new class of generalized Cesàro summable vector-valued sequence space of bounded type with suitable natural $p-$norm. These results can be used in the fields of Functional Analysis, Fourier series, and Engineering to investigate other properties of many sequences spaces.

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