The Nepali Mathematical Sciences Report
Regd No.: 53/033-034, ISSN: 2392–411X
http://www.journal.cdmathtu.edu.np
Year: 2024, Volume: 41, No. 1
Pages: 1–14
https://doi.org10.3126/nmsr.v41i1.67446
Research Article

Generalizing the Mittag-Leffler Function for Fractional Differentiation and Numerical Computation

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Received: 26 March, 2024 Accepted: 26 May 2024 , Published Online: 30 June, 2024

Abstract

This work aims to investigate fractional differential equations using the Magnus Gösta Mittag-Leffler (GML) function and compare the finding with convention calculus approaches. It examines the solutions with one, two, and three parameters using the GML function for different values of α, β, and γ. We also test the convergence of the GML function of two parameters and check the validity and the computational time complexity. Moreover, we extend the GML function into three dimensions within the domain of complex variables utilizing numerical computing software. Graphs of the single-parameter GML E_α(x), illustrates diverse disintegration rates across various α values, emphasizing dominant asymptotic trends over time periods.

Keywords: Convention Calculus, Fractional Calculus, Gösta Mittag-Leffler Function, Numerical Solution.

AMS(MOS) Subject Classification: 33E12, 26A33, 65R10, 65D20, 34A08.

1 Introduction

A fractional differential equation is an equation with fractional derivatives, while a fractional integral equation is an equation with fractional integrals. A system with different orders can be described by a set of these equations or by combining them [1, 2, 3]. Due to its applicability in a growing number of scientific and technical fields, where scientists have utilized it to model a range of physical, biological, and chemical processes, fractional calculus (FC) has gained significant research interest in recent times [7, 9, 10, 21]. FC is frequently

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utilized in physical systems, which include wave equations, harmonic oscillators, frictional forces, viscoelastic materials, chaotic systems, polymer material science, random walks, and anomalous diffusion processes [12]. In engineering, FC is essential for interpreting signals and images as well as for creating and assessing control systems [13]. In economics, FC mimics risk management techniques and stock price fluctuations. n biology, it simulates complex dynamics like population expansion and disease transmission [10, 14, 15, 21]. One of the most widely utilized of these concepts is the Riemann-Liouville (RL) fractional calculus [1, 5, 10]. However, the RL approaches imply that the derivatives of a constant term are not zero. This poses challenges when applying classical calculus to analyze fractional calculus. This poses challenges when applying classical calculus to analyze FC. Jumarie [1] replaced the prior RL-type fractional calculus approach to address this issue. Caputo and Grunwald-Letnikov (GL) formulations effectively handle non-zero derivatives of constant functions [8, 9]. Jumarie refined the Riemann-Liouville (RL) fractional derivatives to address similar issues. Although the Caputo and RL definitions are commonly used in analytical contexts, the Grünwald-Letnikov (GL) definition is particularly useful for numerical applications [3].

Modern research has placed significant emphasis on the GML function, well known for its transcendental properties and crucial role in solving fractional-order differential and integral equations[17, 18]. A solution to the problem of summing divergent series was created in the early 1900s by the Swedish mathematician GML. Since then, because of its crucial significance in solving fractional-order integrals and derivatives, researchers from a wide range of scientific and technical sectors have been enthralled by this singular transcendental function, also referred to as the GML function [4, 19]. In exploring the fractional extension of superdiffusive transport, random walks, kinetic equations, and complex systems, the GML function effectively addresses fractional-order derivatives and integrals [5, 20]. The exponential function e^z is important in an integer-order differential equation, and it was first introduced by the GML function and is now represented by $E_\alpha(z)$, which is its one-parameter generalization [2, 4, 19]. A second complex parameter was added to this formulation immediately after it was first introduced, Goreflo et al [7] and Agarwal [8] established the two-parameter GML function β, $R(\beta) > 0$, the function $E_{\alpha,\beta}$(z) is known as Wiman function and it is crucial to FC. Using the Laplace transform (LT) for this function, Humbert and Agrawal [1, 8] were able to establish several connections. Perhaps the Agarwal function would have been a more appropriate name for this function. Humbert and Agarwal prudently left the one-parameter GML function, that is why the two-parameter function is currently referred to as the GML. In 1971, Prabhakar [3] introduced the GML function, which includes three parameters α, β and γ in the form of $E^{\gamma}_{\alpha,\beta}$(z) which is a generalization of the Wiman function. The work demonstrates how FC expands the practical applications of classical calculus, explaining previously unknown phenomena. Additionally, it examines issues of fractional differentiation using the GML function, comparing them with conventional calculus techniques. This comparison helps clarify the unique advantages of FC in solving real-world problems [1, 11].

The paper is structured as follows: Section 2 introduces preliminary concepts. Section 3 explores the application of fractional derivatives using the generalized GML function. Section 4 focuses on the numerical solution of the GML function in fractional relaxation. Finally, Section 5 concluding remarks.

2 Principles and Notations of Fractional Calculus

2.1 Cauchy's formula for integrating n-times [7]

For $n \in \mathbb{N}$, $a, t \in \mathbb{R}$, $t > a$, Cauchy formula for n-fold integration is given by
$I^n f(t) = \int_a^t \int_a^\tau \cdots \int_a^{\tau(n-1)} f(\tau_n) d\tau \cdots d\tau_2 d\tau_1 = \frac{1}{(n-1)!} \int_a^t f(\tau)(t-\tau)^{(n-1)} d\tau.$

2.2 Riemann-Liouville (RL) Fractional Integral[1]

The RL integral formula of order α is the extension of Cauchy's formula, where the integer value $'n'$ is replaced by a positive real number represented by the symbol α,

$$I^\alpha g(m) = \frac{1}{\Gamma(\alpha)} \int_a^m g(\tau)(m-\tau)^{(\alpha-1)} d\tau.$$

Properties of RL Fractional Integral

- $I^0 g(t) = g(t)$.
- If $g(t)$ is continuous for $t \geq 0$, then $I^\alpha(I^\beta g(t)) = I^\beta(I^\alpha g(t))$, $\alpha, \beta \in \mathbb{R}_+$.
- $I_a^\alpha (t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)}(t-a)^{\alpha+\nu}$, $\nu > -1$, $\alpha \in \mathbb{R}_+$.

2.3 Riemann-Liouville Fractional Derivative [7]

Consider $\alpha > 0$, $t > a$, where α, a, and t are real numbers. The Riemann-Liouville fractional derivative, also known as the Riemann-Liouville fractional differential operator of order α, is defined as follows:

$$D_a^\alpha (f(t)) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \int_a^t \dfrac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & \text{if } \alpha \in \mathbb{R}^+, \ n-1 < \alpha < n, \\ \dfrac{d^n}{dt^n} f(t), & \text{if } \quad \alpha = n \in \mathbb{N}. \end{cases}$$

2.4 Caputo Fractional Derivative [7]

The Caputo differential operator is considered an alternative to the RL operator. Let $f \in C^n[a,b], \alpha > 0, t > a; \alpha, a, t \in \mathbb{R}$. Then the Caputo fractional derivative or Caputo fractional differential operator of order α is defined as

$$ {}^{c}D_a^{\alpha}\left(f^n(t)\right) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\int_a^t \dfrac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, & \text{if } \alpha \in \mathbb{R}^+, \ n-1<\alpha<n, \\ \dfrac{d^n}{dt^n}f(t), & \text{if } \alpha = n \in \mathbb{N}. \end{cases} $$

Note: ${}^{c}D_a^{\alpha}\left(f^n(t)\right) = 0$, where C is a constant.

2.5 Mittag-Leffler function with single parameter [3]

In 1903, the Swedish mathematician Magnus Gösta Mittag-Leffler introduced a unique special function with the following form:

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)} \quad \alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0. $$

where z is a complex variable and $\Gamma(.)$ is a gamma function.

2.6 Mittag-Leffler function with two parameters

In a subsequent development, Wiman [4] introduced the two-parameter Mittag-Leffler function as $E_{\alpha,\beta}(z)$ in the following form:

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)} \quad (\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0). $$

The Wiman function, a generalization of GML, marked a significant advancement. Early research focused on the theoretical aspects of GMLs as entire functions. However, practical applications for GMLs emerged nearly thirty years later.

2.7 Mittag-Leffler function with three parameters

A three-parameter multi-parameter GML function $E^{\gamma}_{\alpha,\beta}(z)$ was first suggested by Prabhakar [19] in 1971 as a further generalization of the GML function. Its definition is as follows:

$$ E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)^k z^k}{\Gamma(k\alpha+\beta)k!} \quad (\alpha, \beta, \gamma \in \mathbb{C}, \mathrm{Re}(\alpha) > 0). \tag{2.1} $$

2.8 Error Function [6]

The complementary error function (erf_c) is a special function in mathematics often denoted by erf_c, expressed as $\mathrm{erf_c}\, z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, \mathrm{d}t$.

3 Applications of FC using the Generalized GML Function

A logical expansion of the definition of one parameter, two parameters, and three parameters of the GML includes one of the three fundamental definitions of fractional derivatives, accompanied by mathematical expressions. The formula for the GML function of a single parameter

$$\begin{aligned} E_{0,1}(x) &= \frac{1}{1-x}, \\ E_{1,1}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x, \\ E_{1,2}(x) &= \frac{1}{x} \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{e^x - 1}{x}, \\ E_{1,3}(x) &= \frac{(-1 - x + e^x)}{x^2}, \\ \vdots \qquad & \quad \vdots \qquad \quad \vdots, \\ \vdots \qquad & \quad \vdots \qquad \quad \vdots. \\ \text{In general, }\ E_{1,m}(x) &= \frac{1}{x^{m-1}} \left(e^x - \sum_{k=0}^{m-2} \frac{x^k}{k!} \right). \end{aligned}$$

The particular cases of the *MLF* are hyperbolic sine and hyperbolic cosine functions.

$$\begin{aligned} E_{2,1}(x^2) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh(x), \\ E_{2,2}(x^2) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{\Gamma(2k+2)} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \frac{\sinh(x)}{x}. \end{aligned}$$

As an extension of the hyperbolic sine and cosine, the GML function can also be used to define the hyperbolic function of order n.

$$h_i(x,p) = \sum_{k=0}^{\infty} \frac{x^{pk+i-1}}{(pk+i-1)!} = x^{i-1} E_{p,i}(x^p),$$

as well as the n-order trigonometric function that yields the sine and cosine functions.

$$\begin{aligned} k_r(x,n) &= \sum_{j=0}^{\infty} \frac{(-1)^j x^{nj+r-1}}{(nj+r-1)!} = x^{r-1}E_{n,r}(-x^n), \\ E_{\frac{1}{2},1}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\frac{k}{2}+1)}, \\ &= e^{x^2}\mathrm{erf}_c(-x), \end{aligned} \tag{3.1}$$

where erf_c is complementary to the error function erf. Modeling fractional order viscoelastic materials involves utilizing the GML function. Experimental examination of these materials reveals an initial rapid decrease in stress, succeeded by a gradual decline over extended periods. This complex behavior necessitates the inclusion of numerous Maxwell components for accurate description, posing challenges in optimizing the identification of multiple material parameters.

3.1 Recurrence Relation

We compute the recurrence relation algebraically on the GML function using its series. A technique to create a sequence or function repeatedly in terms of its past values is to use a recurrence relation. The GML function, denoted by the symbol $E_{\alpha,\beta}$(z) is a unique function used in the study of FC. A particular recurrence relation is satisfied by the GML. According to the GML, the recurrence relation can be deduced as,

$$\begin{aligned} E_{\alpha,\beta}(x) &= xE_{\alpha,\alpha+\beta}(x) + \frac{1}{\Gamma(\beta)}, \\ &= \sum_{\nu=0}^{\infty} \frac{x^\nu}{\Gamma(\alpha\nu+\beta)}, \\ &= \sum_{k=-1}^{\infty} \frac{x^{k+1}}{\Gamma(\alpha(k+1)+\beta)}, \\ &= \frac{1}{\Gamma(\beta)} + \sum_{k=0}^{\infty} \frac{x^{k+1}}{\Gamma(\alpha k+\alpha+\beta)}, \\ &= \frac{1}{\Gamma(\beta)} + x\cdot E_{\alpha,\alpha+\beta}(x), \\ &= x\cdot E_{\alpha,\alpha+\beta}(x) + \frac{1}{\Gamma(\beta)}. \end{aligned}$$

It is a recurrence connection because it is expressed in the specified form.

3.2 Derivatives of Mittag-Leffler function

The derivatives of the GML function can be formed using multiple definitions of the fractional derivatives (FD) operator and have some distinctive and fascinating properties, such as non-locality and memory effects.

$$\begin{aligned}\frac{d}{dz}E_{\alpha,\beta}(z) &= \frac{d}{dz}\sum_{\nu=0}^{\infty}\frac{z^{\nu}}{\Gamma(\alpha\nu+\beta)},\\ &= \sum_{\nu=0}^{\infty}\frac{\nu z^{\nu-1}}{\Gamma(\alpha\nu+\beta-1+1)},\\ &= \frac{1}{\alpha}\sum_{\nu=0}^{\infty}\frac{[(\alpha\nu+\beta-1-(\beta-1)]z^{\nu-1}}{(\alpha\nu+\beta-1)\Gamma(\alpha\nu+\beta-1)},\\ &= \frac{1}{\alpha\cdot z}\sum_{\nu=0}^{\infty}\frac{z^{\nu}}{\Gamma(\alpha\nu+\beta-1)}-\frac{(\beta-1)}{\alpha\cdot z}\sum_{\nu=0}^{\infty}\frac{z^{\nu}}{\Gamma(\alpha\nu+\beta-1+1)},\\ &= \frac{1}{\alpha\cdot z}E_{\alpha,\beta-1}(z)-\frac{(\beta-1)}{\alpha\cdot z}E_{\alpha,\beta}(z).\end{aligned}$$

The GML derivatives enable the extension of differentiation to FC, which is crucial for modeling the nonlinear evolution of mean squared displacement in anomalous diffusion analysis. Within this context, particle behavior is described using the GML function and its derivatives.

3.3 Fractional Differential Equation Using GML Function

Consider the single parameter GML function, $y = E_\alpha(ax^\alpha) = \sum_{n=0}^{\infty}\frac{a^n x^{2n}}{\Gamma(\alpha n+1)}$. The fractional differential equation mentioned below can be solved using Caputo fractional derivatives,

$$6\times\frac{d^{3\alpha}y}{dx^{3\alpha}}+5\times\frac{d^{2\alpha}y}{dx^{2\alpha}}+\frac{d^{\alpha}y}{dx^{\alpha}}=0.$$

$$6\times{}_0^C D_x^{3\alpha}\left[\sum_{n=0}^{\infty}\frac{a^n x^{n\alpha}}{\Gamma(\alpha n+1)}\right]+5\times{}_0^C D_x^{2\alpha}\left[\sum_{n=0}^{\infty}\frac{a^n x^{n\alpha}}{\Gamma(\alpha n+1)}\right]+{}_0^C D_x^{\alpha}\left[\sum_{n=0}^{\infty}\frac{a^n x^{n\alpha}}{\Gamma(\alpha n+1)}\right]=0.$$

$$6\times\sum_{k=-3}^{\infty}\frac{a^{k+3}}{\Gamma(\alpha k+1)}x^{k\alpha}+5\times\sum_{k=-2}^{\infty}\frac{a^{k+2}}{\Gamma(\alpha k+1)}x^{k\alpha}+\sum_{k=-1}^{\infty}\frac{a^{k+1}}{\Gamma(\alpha k+1)}x^{k\alpha}=0.$$

$$\sum_{k=0}^{\infty}\left[6\times a^{(k+3)}+5\times a^{(k+2)}+a^{(k+1)}\right]\frac{x^{\alpha k}}{\Gamma(\alpha k+1)}=0.$$

$$\because a^k\neq 0 \implies 6a^3+5a^2+a=0 \implies a=0,-\frac{1}{2},-\frac{1}{3}.$$

$y = E_\alpha(ax^\alpha)$: $y_1 = E_\alpha(0.x^\alpha)$, $y_2 = E_\alpha(-\frac{1}{2}x^\alpha)$, $y_3 = E_\alpha(-\frac{1}{3}x^\alpha)$.

The general solution is $y = k_1 + k_2\ E_\alpha(-\frac{1}{2}x^\alpha) + k_3\ E_\alpha(-\frac{1}{3}x^\alpha)$.

Compared to the result of conventional calculus,

$6\times\frac{d^{3\alpha}y}{dx^{3\alpha}}+5\times\frac{d^{2\alpha}y}{dx^{2\alpha}}+\frac{d^{\alpha}y}{dx^{\alpha}}=0.$

At $\alpha = 1$, $6D^3+5D^2+D=0$. Its A.E. is $m = 0, -\frac{1}{2}, -\frac{1}{3}$.

The general solution is $y = k_1 + k_2 e^{-\frac{1}{2}.x} + k_3 e^{-\frac{1}{3}.x}$. $y = k_1 + k_2\ E_1(-\frac{1}{2}x) + k_3\ E_1(-\frac{1}{3}x)$.

At $\alpha = 1$, $y = k_1 + k_2 e^{-\frac{1}{2}.x} + k_3 e^{-\frac{1}{3}.x}$.

As a result, the fractional-order differential equation's solution matched with the classical differential equation, ensuring accurate results.

4 Numerical solution of GML function in fractional relaxation

This expansion illustrates how the gamma function, with variable parameters E_α, is involved in several terms that make up the GML, as seen in Equation 2.5. It is important to notice because it signifies the potential for infinite growth or divergence.

$$E_\alpha(x) = 1 + \frac{x}{\Gamma(\alpha+1)} + \frac{x^2}{\Gamma(2\alpha+1)} + \frac{x^3}{\Gamma(3\alpha+1)} + \ldots + \frac{x^n}{\Gamma(n\alpha+1)} + \ldots \tag{4.1}$$

Specifically, we are curious about the function $E_\alpha(t)$ for $t > 0$, $0 < \alpha \leq 1$

$$e_\alpha(t) := E_\alpha(-t_\alpha) = 1 - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{-3\alpha}}{\Gamma(1-3\alpha)} + \ldots \tag{4.2}$$

We display graphs of $E_\alpha(x)$ for various α values in Fig. 1, illustrating different rates of decay

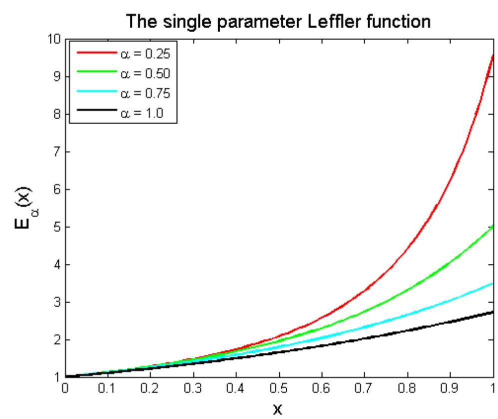

Figure 1: $E_\alpha(x)$ in (4.1) for $\alpha = 0.25, 0.50, 0.75, 1$

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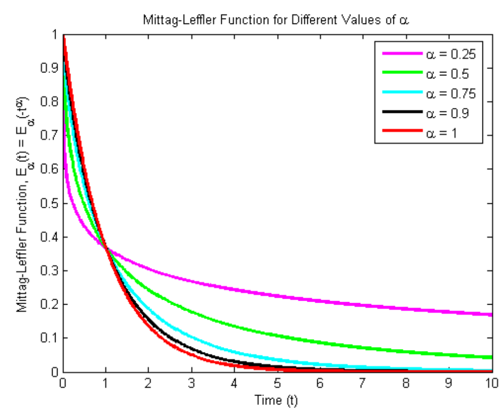

Figure 2: $E_\alpha(t)$ in (4.2) for $\alpha = 0.25, 0.50, 0.75, 0.90, 1$

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over short and long durations. The graph significantly slows as x tends to $+\infty$ and speeds up as x tends to $0+$. Plots of $E_\alpha(x)$ shows that fractional relaxation has different features from the exponential function for $a = 1$ (Fig. 1). We highlight dominant asymptotic trends on both small and large time periods. Moreover, when $\alpha = 1$, it simplifies to an exponential function, potentially involving a complex parameter α, provided that $Re(\alpha) > 0$. In fig., 2, we show multiple graphs for different values of the parameter $e_\alpha(\mathrm{t})$ for various value of α. Differentiate the decay rates of $e_\alpha(\mathrm{t})$ across time scales. Decay decreases greatly as time near positive infinity and increases significantly as time approaches zero from the right. The exponential extended is used instead of the fast decreasing expression $\frac{(1-t^\alpha)}{\Gamma(1+\alpha)}$ from (4.2).

4.1 Generalised GML Function of complex variable in three Dimension

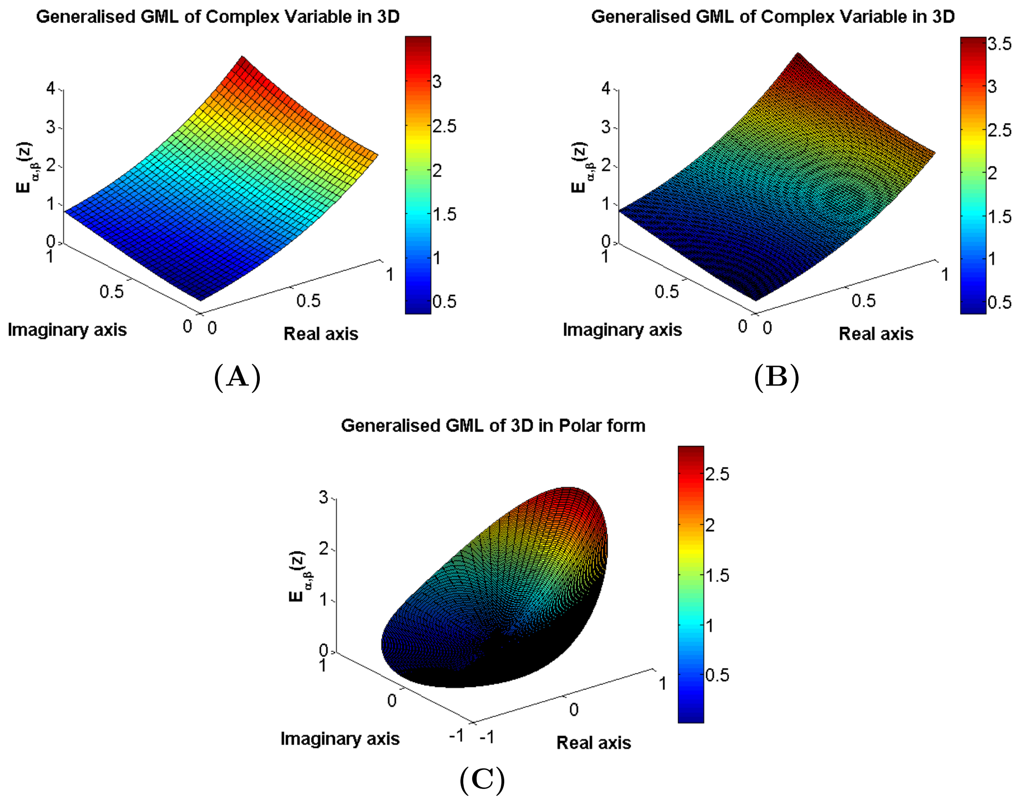

Figure 3: The GML function in 3 dimention **A:** at $\alpha = 0 : 0.03 : 1$ and **B:** $\alpha = 0 : 0.01 : 1$

When X is the real component, Y is the imaginary part, and Z is a complex variable in figures 3 **A** and 3 **B** at $\alpha = 0 : 0.03 : 1$ and $\alpha = 0 : 0.01 : 1$ respectively. The complex variable $z = x + iy$ will then pass through the GML function, the initial parameters $a = 1$ and $b = 0.3$ generates a grid of complex numbers (x,y) in the range [0,1] and evaluates the GML function for each point in the grid. There are various collections of points in the complex plane. The real component in figure 3 **C** is represented by X, imaginary component by Y, complex variable by Z, whereas in the base of the surface figure, the 3 dimension contour is clearly visible in the complex plane and the surface is generated in the polar coordinate at an angle and radius vector in $0 : 05 : 2\pi$.

4.2 Generalised the GML Function of three Independent Parameters

The GML function is determined using the most precise and efficient method, which is based on numerically inverting the Laplace Transform (LT). In 1971, Prabhakar introduced the function involving three parameters α, β, γ in the form of

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!\Gamma(\alpha n+\beta)} \cdot z^n, \quad R(\alpha) > 0, R(\beta) > 0 \text{ and } R(\gamma) > 0 \tag{4.3}$$

$$\text{where,}(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},$$

γ indicates the Pochhammer symbol. When $\gamma = 1$, then it reduces to two-parameter Mittage leffler function.

$$E^1_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{4.4}$$

Like wise, when β =1,then the function reduces to one parameter Mittag Liffler function.

$$E^1_{\alpha,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \tag{4.5}$$

when $\gamma = 1$, $\beta = 1$ and $\alpha = 1$,

$$E^1_{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z. \tag{4.6}$$

While exponential functions are often used in classical calculus and GML function is frequently used in fractional calculus, the former is the generalization of the latter.

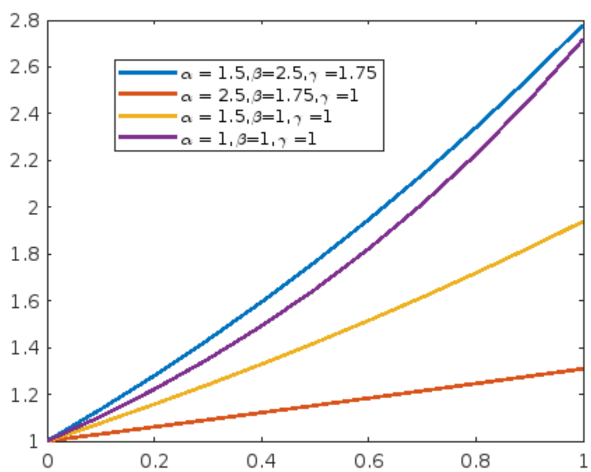

Figure 4: The GML function $E^{\gamma}_{\alpha,\beta}$ (z)= (α = 1.5, β= 2.5, γ =1.75) $E^1_{\alpha,\beta}$ (z)= (α = 2.5, β= 1.75, γ =1), $E^1_{\alpha,1}$ (z)= (α = 1.5, β= 1, γ =1), and $E^1_{1,1}$ (z)= (α = 1, β= 1, γ =1)

The equation (4.4) shows that when γ is replaced with one, the three-parameter GML, also known as the Prabhakar function, reduces to a two-parameter the GML. Likewise, when β is replaced with one then equation (4.5), reduces to one parameter. The equation (2.7) is the definition of the three-parameter GML. The exponential function is a classical function in calculus that is referred to as the GML if alpha is also equal to 1. The GML is a flexible and convertible extension of the popular exponential function 4.6. The GML is an important part of FC, just as the exponential function is in classical calculus.

The solutions are shown in Table 1 for the various values of three independent parameters α, β and γ in the time interval $0 \leq z \leq 1$. The adjacent table shows that the approximate values closely match the exponential function e^z when the values of α, β, and γ are equal to 1. For different values of α, the single-parameter the GML function acts as a generalization of the exponential function. On the other hand, the GML function with a second parameter is derived and defined in reference (4.5), while the GML function with three parameters is generated by the function specified in reference (4.4). Utilizing the GML technique provides a dependable and efficient solution for fractional order differential equations. As α, β, and γ approach 1, the error diminishes, often resulting in the analytical solution being a suitable approximation for nearby values of these parameters.

Table 1: GML function of three independent parameters α, β and γ.

Steps sizes	$E^{1.75}_{(1.5,2.5)}$(z)	$E^{1}_{(2.5,1.75)}$ (z)	$E^{1}_{1.5,1}$ (z)	$E^{1}_{1,1}$ (z)
0	1.0000	1.0000	1.0000	1.0000
0.1000	1.1357	1.0302	1.0769	1.1052
0.2000	1.2798	1.0605	1.1573	1.2214
0.3000	1.4326	1.0910	1.2412	1.3499
0.4000	1.5946	1.1217	1.3288	1.4918
0.5000	1.7660	1.1525	1.4203	1.6487
0.6000	1.9473	1.1836	1.5157	1.8221
0.7000	2.1390	1.2147	1.6151	2.0138
0.8000	2.3414	1.2461	1.7188	2.2255
0.9000	2.5550	1.2776	1.8269	2.4596
1.0000	2.7802	1.3093	1.9395	2.7183

4.3 Test of Convergence of the GML function for two Parameters.

Table 2 shows that if we increase n to 1 : 3, the sum of the four terms of the GML is equal to 1.1070 while the value has been rising. If we choose $n = 1 : 1$, it returns the sum of the two terms is equal to 1.1000 in the numerical value corresponding to the parameters $a = 1$, $b = 2$, and $z = 0.2$. Now, if we increase n by a factor of 1 : 5, $n = 1 : 100$, or $n = 200$, the value is equal to 1.1070. Similarly, if we add the series of 201 terms together, then it is also equal to 1.1070. Since we examined several the GML values, just five terms are required for convergence. It is concluded that convergence requires no more than five values. The time complexity for this function is more. For the convergence of a series, the first hundred terms are sufficient.

Table 2: Numerical values of GML for two parameters

Steps Sizes	Numerical Value
$n = 1 : 1$	1.1000
$n = 1 : 3$	1.1070
$n = 1 : 5$	1.1070
$n = 1 : 100$	1.1070
$n = 1 : 500$	1.1070

5 Conclusion:

In this article, we explored the convergence of the GML function with two parameters. Our findings showed that it typically converges within the first five terms, but achieving this convergence comes with higher time complexity. Because of its flexibility, the GML function simulated well-known functions like the exponential and Prabhakar functions, particularly as parameters approach 1, thereby facilitating accurate solutions in fractional-order differential equations. The single-parameter GML function exhibits different rates of disintegration of $E_\alpha(x)$ over time within a range of $0 < \alpha \leq 1$, particularly when α is equal to 1. Variable decay rates over time are displayed in $e_\alpha(t)$ graphs for various α values; the rates are slower toward infinity and quicker near zero. We found that the GML method offers a stable and effective solution. As the values of α, β, and γ approach 1, the error in the solution diminishes, often resulting in the approximate solution matching the analytical solution. The applicability of the GML function in many cases involving fractional calculus is demonstrated in this paper. Additionally, our study demonstrated that the GML function is quite similar to classical differential equations, ensuring accurate findings. The significance of representing the nonlinear growth of mean squared displacement in anomalous diffusion analysis is highlighted by this result.

Acknowledgement: Shankar Pariyar sincerely thanks the anonymous referees for their valuable suggestions, which significantly enhanced this article, and acknowledges the Nepal Mathematics Society (NMS) for providing financial support through the PhD fellowship-2024. Additionally, Jeevan Kafle expresses gratitude to the Nepal Academy of Science and Technology (NAST) for the individual research grant for young scientists in 2024.

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