The Nepali Math. Sc. Report, Vol. 40, No.1 and 2, 2023, 55-70
DOI: https://doi.org/10.3126/nmsr.v40i1-2.61498

GLOBAL REGULARITY CRITERIA FOR THE 2D MAGNETO-MICROPOLAR EQUATIONS WITH PARTIAL DISSIPATION

DURGA JANG K.C.[1], DIPENDRA REGMI[2]

1. Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal, email:durgajkc@gmail.com

2. Department of Mathematics, University of North Georgia, 3820 Mundy Mill Rd, Oakwood, GA 30566, email: dipendra.regmi@ung.edu

Abstract. The magneto-micropolar equations model the motion of electrically conducting micropolar fluids in the presence of a magnetic field. These equations have been the focus of numerous analytical, experimental, and numerical investigations. One fundamental problem concerning these equations is whether their classical solutions are globally regular for all time or if they develop finite time singularities. The global regularity problem can be particularly challenging when there is only partial dissipation. In this paper, we study the 2D incompressible magneto-micropolar equations with partial dissipation and prove two new regularity results. The first result addresses a weak solution, and the second result establishes global regularity criteria. As a consequence, we are able to single out one special partial dissipation case and establish the global regularity if $(\partial_y u_1, \partial_y u_2) \in L^\infty\left([0,T], \mathbb{R}^2\right)$. The proofs of our main results rely on anisotropic Sobolev-type inequalities and the appropriate combination and cancellation of terms.

Key Words: Global regularity, Magneto-micropolar equations, Partial dissipation
AMS (MOS) Subject Classification. 35, 35A, 35B.

1. Introduction

The standard 3D incompressible magneto-micropolar equations can be written as

$$
\begin{cases}
\partial_t u + (u\cdot\nabla)u + \nabla(p + \frac{1}{2}|b|^2) = (\mu+\chi)\Delta u + (b\cdot\nabla)b + 2\chi\nabla\times\omega,\\
\partial_t b + (u\cdot\nabla)b = \nu\Delta b + (b\cdot\nabla)u,\\
\partial_t \omega + (u\cdot\nabla)\omega + 2\chi\omega = \kappa\Delta\omega + (\alpha+\beta)\nabla\nabla\cdot\omega + 2\chi\nabla\times u,\\
\nabla\cdot u = 0, \quad \nabla\cdot b = 0,
\end{cases} \tag{1.1}
$$

where, for $\mathbf{x}\in\mathbb{R}^3$ and $t\geq 0$, $u = u(\mathbf{x},t), b = b(\mathbf{x},t), \omega = \omega(\mathbf{x},t)$ and $p = p(\mathbf{x},t)$ denote the velocity field, the magnetic field, the micro-rotation field and the pressure respectively, and μ denotes the kinematic viscosity, ν the magnetic diffusivity, χ the vortex viscosity, and α, β and κ the angular viscosities.

Received: 31 May, 2016 Accepted: 25 October, 2023 Published Online: 29 December, 2023 .

The 3D magneto-micropolar equations reduce to the 2D magneto-micropolar equations when

$$
\begin{aligned}
&u = (u_1(x,y,t), u_2(x,y,t), 0), \quad b = (b_1(x,y,t), b_2(x,y,t), 0),\\
&\omega = (0,0,\omega(x,y,t)), \quad \pi = \pi(x,y,t),
\end{aligned}
$$

where $(x,y) \in \mathbb{R}^2$ and we have written $\pi = p + \frac{1}{2}|b|^2$. The 2D magneto-micropolar equations can be written as

$$
\begin{cases}
\partial_t u + (u\cdot\nabla)u + \nabla\pi = (\mu+\chi)\Delta u + (b\cdot\nabla)b + 2\chi\nabla\times\omega,\\
\partial_t b + (u\cdot\nabla)b = \nu\Delta b + (b\cdot\nabla)u,\\
\partial_t\omega + (u\cdot\nabla)\omega + 2\chi\omega = \kappa\Delta\omega + 2\chi\nabla\times u,\\
\nabla\cdot u = 0, \quad \nabla\cdot b = 0,
\end{cases}
\tag{1.2}
$$

where $u = (u_1, u_2)$, $b = (b_1, b_2)$, $\nabla\times\omega = (-\partial_y\omega, \partial_x\omega)$ and $\nabla\times u = \partial_x u_2 - \partial_y u_1$.

A generalization of the 2D magneto-micropolar equations can be written as

$$
\begin{cases}
\partial_t u_1 + (u\cdot\nabla)u_1 + \partial_x\pi = \mu_{11}\partial_{xx}u_1 + \mu_{12}\partial_{yy}u_1 + (b\cdot\nabla)b_1 - 2\chi\partial_y\omega,\\
\partial_t u_2 + (u\cdot\nabla)u_2 + \partial_y\pi = \mu_{21}\partial_{xx}u_2 + \mu_{22}\partial_{yy}u_2 + (b\cdot\nabla)b_2 + 2\chi\partial_x\omega,\\
\partial_t b_1 + (u\cdot\nabla)b_1 = \nu_{11}\partial_{xx}b_1 + \nu_{12}\partial_{yy}b_1 + (b\cdot\nabla)u_1,\\
\partial_t b_2 + (u\cdot\nabla)b_2 = \nu_{21}\partial_{xx}b_2 + \nu_{22}\partial_{yy}b_2 + (b\cdot\nabla)u_2,\\
\partial_t\omega + (u\cdot\nabla)\omega + 2\chi\omega = \kappa_1\partial_{xx}\omega + \kappa_2\partial_{yy}\omega + 2\chi\nabla\times u,\\
\nabla\cdot u = 0, \quad \nabla\cdot b = 0,\\
u(x,y,0) = u_0(x,y),\ b(x,y,0) = b_0(x,y),\ \omega(x,y,0) = \omega_0(x,y).
\end{cases}
\tag{1.3}
$$

If $\mu_{11} = \mu_{12} = \mu_{21} = \mu_{21} = \mu$, $\nu_{11} = \nu_{12} = \nu_{21} = \nu_{21} = \nu$, and $\kappa_1 = \kappa_2 = \kappa$, then (1.3) reduces to the standard magneto-micropolar equations (1.2). For notational convenience, we set $\chi = \frac{1}{2}$ for the rest of the paper.

The equations for $\Omega = \nabla\times u$, the current density $j = \nabla\times b$, and $\nabla\omega$ can be expressed as

$$
\begin{cases}
\Omega_t + u\cdot\nabla\Omega = -\mu_{11}\partial_{xxy}u_1 - \mu_{12}\partial_{yyy}u_1 + \mu_{21}\partial_{xxx}u_2 + \mu_{22}\partial_{xyy}u_2 + (b\cdot\nabla)j - \Delta\omega,\\
j_t + u\cdot\nabla j = -\nu_{11}\partial_{xxy}b_1 - \nu_{12}\partial_{yyy}b_1 + \nu_{21}\partial_{xxx}b_2 + \nu_{22}\partial_{xyy}b_2 + b\cdot\nabla\Omega\\
\qquad +2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1),\\
\partial_t\nabla\omega + \nabla(u\cdot\nabla\omega) + \nabla\omega = \kappa_1\nabla\omega_{xx} + \kappa_2\nabla\omega_{yy} + \nabla\Omega,\\
\nabla\cdot u = 0, \quad \nabla\cdot b = 0.
\end{cases}
\tag{1.4}
$$

The above generalization can model the motion of anisotropic fluids for which the diffusion properties in different directions are different. The system (1.3) allows us to explore the smoothing effects of various partial dissipations.

The mathematical study of the magneto-micropolar equations started in the seventies and has been continued by many authors (see, e.g., [1, 2, 3, 6, 7, 9, 10, 11, 12, 15, 16, 17, 18] and references therein). Recent efforts have focused on addressing the well-posedness problem and studying various asymptotic behaviors. In the work of Yamazaki [17], the author obtained the global regularity of the 2D magneto-micropolar equation with zero angular viscosity, namely (1.2) with $\kappa = 0$ and other coefficients being positive. Another partial dissipation case for the 2D magneto-micropolar equation was studied in [5].

Global regularity has been established for the following three cases by D. Regmi and J. Wu in their work [14].

- $\mu_{11} = \mu_{22} = 0, \quad \nu_{21} = \nu_{22} = 0, \quad \kappa_2 = 0, \quad \mu_{12} = \mu_{21} = 1, \quad \kappa_1 = \nu_{11} = \nu_{12} = 1,$

- $\mu_{11} = \mu_{12} = 1, \quad \nu_{21} = \nu_{22} = 0, \quad \kappa_2 = 0, \quad \mu_{22} = \mu_{21} = 0, \quad \kappa_1 = \nu_{11} = \nu_{12} = 1,$

- $\mu_{12} = \mu_{22} = 1, \quad \nu_{21} = \nu_{22} = 0, \quad \kappa_2 = 0, \quad \mu_{11} = \mu_{21} = 0, \quad \kappa_1 = \nu_{11} = \nu_{12} = 1$

In addition, the global regularity of the following two cases has been settled recently by Y. Guo and H. Shang in [8].

- $\mu_{11} = \mu_{22} = 0, \quad \nu_{21} = \nu_{22} = 0, \quad \kappa_1 = 0, \quad \mu_{12} = \mu_{21} = 1, \quad \kappa_2 = \nu_{11} = \nu_{12} = 1,$

- $\mu_{11} = \mu_{21} = 1, \quad \nu_{12} = \nu_{21} = 1, \quad \kappa_2 = 1, \quad \mu_{12} = \mu_{22} = 0, \quad \kappa_1 = \nu_{11} = \nu_{22} = 0$

Furthermore, the global regularity for the following case has been stablished in [13] by D. Regmi.

- $\mu_{11} = \mu_{12} = 0, \quad \nu_{11} = \nu_{21} = 0, \quad \kappa_1 = 0, \quad \mu_{21} = \mu_{22} = 1, \quad \kappa_2 = \nu_{12} = \nu_{22} = 1.$

In this paper, we consider the following magneto-micropolar equations with partial dissipations.

$$(1.5) \qquad \begin{cases} \partial_t u_1 + (u \cdot \nabla) u_1 + \partial_x \pi = (b \cdot \nabla) b_1 - \partial_y \omega, \\ \partial_t u_2 + (u \cdot \nabla) u_2 + \partial_y \pi = \partial_{xx} u_2 + (b \cdot \nabla) b_2 + \partial_x \omega, \\ \partial_t b_1 + (u \cdot \nabla) b_1 = \partial_{yy} b_1 + (b \cdot \nabla) u_1, \\ \partial_t b_2 + (u \cdot \nabla) b_2 = \partial_{yy} b_2 + (b \cdot \nabla) u_2, \\ \partial_t \omega + (u \cdot \nabla)\omega + \omega = \partial_{yy}\omega + \nabla \times u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y),\ b(x,y,0) = b_0(x,y),\ \omega(x,y,0) = \omega_0(x,y). \end{cases}$$

Whether the classical solutions of (1.5) is globally regular for all time or they develop finite time singularities is an unsolved problem in fluid dynamics. In fact, this problem is extremely hard because the dissipation is not enough to control the non-linear terms.

In this paper, we prove the following new results related to regularity of the solution; the first result is about a weak solution and the second result is global regularity criteria.

Theorem 1.1. *Assume $(u_0, b_0, \omega_0) \in H^1(\mathbb{R}^2)$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then (1.5) has a global weak solution (u, b, ω) satisfying, for any $T > 0$,*

$$(u, b, \omega) \in L^\infty([0,T]; H^1(\mathbb{R}^2))$$

provided $\int_0^T \|u_1\|_\infty^2 \, dt < \infty$.

Theorem 1.2. *Assume $(u_0, b_0, \omega_0) \in H^2(\mathbb{R}^2)$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then (1.5) has a global solution (u, b, ω) satisfying, for any $T > 0$,*

$$(u, b, \omega) \in L^\infty([0,T]; H^2(\mathbb{R}^2))$$

provided $\int_0^T \|\partial_y u_1, \partial_y u_2\|_\infty \, dt < \infty$.

As a consequence, the above regularity conditions helps in establishing the well-posedness of the problem and it opens the future research directions.

The main idea to establish the existence and regularity results consists of two steps. First step is to show local well-posedness and the second step is extending the local solution into a global one by obtaining global (in time) *a priori* bounds. The local well-posedness of the above system is well known. The main difficulty is global *a priori* bounds. Thus we mainly concentrate on the global bounds. The rest of this paper is divided into four sections. The third and final sections are devoted to the proof of the theorems 1.1 and 1.2.

2. Preliminaries

To simplify the notation, we will write $\|f\|_2$ for $\|f\|_{L^2}$, $\int f$ for $\int_{\mathbb{R}^2} f \, dxdy$ and $\frac{\partial}{\partial x} f$, $\partial_x f$ or f_x as the first partial derivative, and $\frac{\partial^2 f}{\partial x^2}$ or $\partial_{xx} f$ as the second partial throughout the rest of this paper.

The following anisotropic type Sobolev inequality will be frequently used. Its proof can be found in [3].

Lemma 2.1. *If* $f, g, h, \partial_y g, \partial_x h \in L^2(\mathbb{R}^2)$, *then*

$$\iint_{\mathbb{R}^2} |f \, g \, h| \, dxdy \le C \, \|f\|_2 \, \|g\|_2^{\frac{1}{2}} \, \|\partial_y g\|_2^{\frac{1}{2}} \, \|h\|_2^{\frac{1}{2}} \, \|\partial_x h\|_2^{\frac{1}{2}}, \tag{2.1}$$

where C *is a constant.*

The following simple fact on the boundedness of Riesz transforms will also be used. Its proof can be found in [4]

Lemma 2.2. *Let* f *be divergence-free vector field such that* $\nabla f \in L^p$ *for* $p \in (1, \infty)$. *Then there exists a pure constant* $C > 0$ *(independent of* p*) such that*

$$\|\nabla f\|_{L^p} \le \frac{C \, p^2}{p-1} \|\nabla \times f\|_{L^p}.$$

3. Global L^2-bound.

This section proves the global L^2-bound. More precisely, we prove the following theorem.

Theorem 3.1. *Assume that* (u_0, b_0, ω_0) *satisfies the condition stated in Theorem 1.2. Let* (u, b, ω) *be the corresponding solution of (1.5). Then for any* $T > 0$, (u, b, ω) *obeys the following global* L^2*-bound,*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \int_0^T \|\partial_x u_2\|_{L^2}^2 d\tau$$
$$+ \int_0^T \|\partial_y b_1(\tau), \partial_y b_2(\tau)\|_2^2 d\tau + \int_0^T (\|\partial_y \omega(\tau)\|_{L^2}^2) d\tau \le C(\|(u_0, b_0, \omega_0)\|_2^2)$$

Proof. Taking the L^2-inner product of (u, b, ω) with (1.5), respectively, yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u(t),\omega(t),b(t)\|_2^2) + \|\partial_x u_2(\tau)\|_2^2 + \|\partial_y b_1(\tau), \partial_y b_2(\tau)\|_2^2 \\ + \|\partial_y \omega(\tau)\|_2^2 + \|\omega(\tau)\|_2^2 = 2\left[\int (\partial_x u_2 - \partial_y u_1)\omega\, dx\, dy\right]\end{aligned} \tag{3.1}$$

Applying Hölder's inequality yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u(t),\omega(t),b(t)\|_2^2) + \|\partial_x u_2(\tau)\|_2^2 + \|\partial_y b_1(\tau), \partial_y b_2(\tau)\|_2^2 \\ + \|\partial_y \omega(\tau)\|_2^2 + \|\omega(\tau)\|_2^2 \le \frac{1}{2}(\|\partial_x u_2\|_2^2 + \|\partial_y \omega\|_2^2) + C\|\omega\|_2^2\end{aligned} \tag{3.2}$$

Gronwall's inequality then implies

$$\begin{aligned}\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \int_0^T \|\partial_x u_2\|_{L^2}^2 d\tau \\ + \int_0^T \|\partial_y b_1(\tau), \partial_y b_2(\tau)\|_2^2 d\tau + \int_0^T \|\partial_y \omega(\tau)\|_{L^2}^2 d\tau \le C(\|(u_0, b_0, \omega_0)\|_2^2)\end{aligned}$$

□

4. Proof of Theorem 1.1

In this section, we prove the theorem 1.1.

Proof of Theorem 1.1. To estimate the H^1-norm of (u, b, ω), we consider the equations of $\Omega = \nabla \times u$, $\nabla\omega$ and of the current density $j = \nabla \times b$,

$$\Omega_t + u \cdot \nabla\Omega = \partial_{xxx} u_2 + (b \cdot \nabla) j - \Delta\omega, \tag{4.1}$$

$$\begin{aligned} j_t + u \cdot \nabla j = -\partial_{yyy} b_1 + \partial_{xyy} b_2 + b \cdot \nabla\Omega \\ + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1),\end{aligned} \tag{4.2}$$

$$\partial_t \nabla\omega + \nabla(u \cdot \nabla\omega) + \nabla\omega = \nabla\omega_{xx} + \nabla\omega_{yy} + \nabla\Omega, \tag{4.3}$$

$$\nabla \cdot u = 0, \quad \nabla \cdot b = 0.$$

Dotting (4.1) by Ω, (4.2) by j, and (4.3) by $\nabla\omega$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\Omega\|_{L^2}^2 + \|j\|_{L^2}^2 + \|\nabla\omega\|_2^2\right) + \|\partial_{xx} u_1\|_2^2 + \|\partial_{xx} u_2\|_2^2 \\ &\quad + \|\partial_y j\|_{L^2}^2 + \|\nabla\omega_x\|_2^2 + \|\nabla\omega_y\|_2^2 + 2\|\nabla\omega\|_2^2 \\ &= 2\int \left[\partial_x b_1(\partial_x u_2 + \partial_y u_1)\, j - \partial_x u_1(\partial_x b_2 + \partial_y b_1)\right] j dx\, dy \\ &\quad - \int \nabla\omega \cdot \nabla u \cdot \nabla\omega + \int \nabla\omega \cdot \nabla\Omega \\ &\equiv J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned}$$

Invoking the divergence-free condition, we have

$$\|\nabla\partial_x u_2\|_2^2 = \|\partial_{xx} u_1\|_2^2 + \|\partial_{xx} u_2\|_2^2.$$

We now estimate the terms on the right. Since $j = \partial_x b_2 - \partial_y b_1$,

$$\begin{aligned} J_1 \quad &= 2\int \partial_x b_1 \partial_x u_2 \partial_x b_2 - 2\int \partial_x b_1 \partial_x u_2 \partial_y b_1 \\ &\equiv J_{11} + J_{12}. \end{aligned}$$

Applying Lemma 2.1, Young's inequality, and the simple fact that

$$\|\partial_x b_2\|_{L^2} \le \|j\|_{L^2}, \qquad \|\partial_{xy} b_1\|_{L^2} \le \|\partial_y j\|_{L^2},$$

Now we estimate J_{11}.

$$\begin{aligned} J_{11} \quad &\le 2\left|\int \partial_x b_1 \partial_x u_2 \partial_x b_2\right| \\ &\le C\|\partial_x b_2\|_2 \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_{xx} u_2\|_2^{\frac{1}{2}} \|\partial_x b_1\|_2^{\frac{1}{2}} \|\partial_{xy} b_1\|_2^{\frac{1}{2}} \\ &\le C\|j\|_2 \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_{xx} u_2\|_2^{\frac{1}{2}} \|\partial_y b_2\|_2^{\frac{1}{2}} \|\partial_y j\|_2^{\frac{1}{2}} \\ &\le \|\partial_{xx} u_2\|_2 \|\partial_y j\|_2 + C\|\partial_y b_2\|_2 \|\partial_x u_2\|_2 \|j\|_2^2 \\ &\le \frac{1}{48}(\|\partial_{xx} u_2\|_2^2 + \|\partial_y j\|_2^2) + C(\|\partial_y b\|_2^2 + \|\partial_x u_2\|_2^2)\|j\|_2^2. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} J_{12} \quad &\le \left|2\int \partial_x b_1 \partial_x u_2 \partial_y b_1\right| \\ &\le C\|\partial_y b_2\|_2 \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_{xx} u_2\|_2^{\frac{1}{2}} \|\partial_y b_1\|_2^{\frac{1}{2}} \|\partial_{yy} b_1\|_2^{\frac{1}{2}} \\ &\le \frac{1}{48}(\|\partial_{xx} u_2\|_2^2 + \|\partial_{yy} b_1\|_2^2) + C(\|\partial_x u_2\|_2^2 + \|\partial_y b_1\|_2^2)\|j\|_2^2. \end{aligned}$$

The dissipation is not sufficient to control J_2, this is the place where Lemma 2.1 cannot apply. J_2 can be bounded as

$$\begin{aligned} J_2 \quad &= \left|-\int u_1 \partial_{xy} b_1\, j - \int u_1 \partial_x b_1\, \partial_y j\right| \\ &\le \|u_1\|_\infty \|\partial_y j\|_2 \|j\|_2 + \|u_1\|_\infty \|j\| \|\partial_y j\|_2 \\ &\le \frac{1}{2}\|\partial_y j\|_2^2 + C\|u_1\|_\infty^2 \|j\|_2^2 \end{aligned}$$

J_3, and J_4 can be bounded by

$$\begin{aligned} J_3 \quad &\le \left|\int \partial_x u_1\, \partial_x b_2\, j\right| \le \int |(u_2 \partial_{xy} b_2 j + u_2 \partial_x b_2 \partial_y j)| \\ &\le C\|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|j\|_2^{\frac{1}{2}} \|\partial_y j\|_2^{\frac{1}{2}} \|\partial_{xy} b_2\|_2 \\ &\quad + C\|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_x b_2\|_2^{\frac{1}{2}} \|\partial_{xy} b_2\|_2^{\frac{1}{2}} \|\partial_y j\|_2 \\ &\le \frac{1}{48}\|\partial_y j\|_2^2 + C\|u_2\|_2^2 \|\partial_x u_2\|_2^2 \|j\|_2^2. \end{aligned}$$

Similarly,

$$J_4 \quad \le \frac{1}{48}\|\partial_y j\|_2^2 + C\|u_2\|_2^2 \|\partial_x u_2\|_2^2 \|j\|_2^2.$$

To bound J_5, we use $\nabla \cdot u = 0$ and integrate by parts to obtain

$$\begin{aligned} J_5 &= -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \\ &= -\int \partial_x u_1 \omega_x\, \omega_x - \int \partial_y u_2 \omega_y \omega_y - \int (\partial_x u_2 + \partial_y u_1)\omega_x \omega_y. \end{aligned}$$

The terms on the right can be bounded as

$$\begin{aligned} \int \partial_x u_1 \omega_x\, \omega_x \le & 2\left|\int u_2\, \omega_{xy}\, \omega_x\right| \\ & \le C\|\omega_{xy}\|_2 \|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\omega_x\|_2^{\frac{1}{2}} \|\omega_{xy}\|_2^{\frac{1}{2}} \\ & \le C\|\omega_{xy}\|_2^{\frac{3}{2}} \|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\omega_x\|_2^{\frac{1}{2}} \\ & \le \frac{1}{48}\|\omega_{xy}\|_2^2 + C\|u_2\|_2^2 \|\partial_x u_2\|_2^2 \|\nabla\omega\|_2^2. \end{aligned}$$

$$\begin{aligned} \left|\int \partial_y u_2 \omega_y \omega_y\right| = \left|2\int u_1 \partial_{xy}\omega \partial_y \omega\right| & \le \|u_1\|_\infty \|\partial_{xy}\omega\|_2 \|\partial_y \omega\|_2 \\ & \le \frac{1}{2}\|\partial_{xy}\omega\|_2^2 + \|u_1\|_\infty^2 \|\nabla\omega\|_2^2 \end{aligned}$$

$$\begin{aligned} \left|\int \partial_x u_2 \omega_x \omega_y\right| & \le \|\partial_x u_2\|_2 \|\omega_x\|_2^{\frac{1}{2}} \|\omega_{xy}\|_2^{\frac{1}{2}} \|\omega_y\|_2^{\frac{1}{2}} \|\omega_{xy}\|_2^{\frac{1}{2}} \\ & \le C\|\partial_x u_2\|_2 \|\nabla\omega\|_2 \|\nabla\omega_x\|_2 \\ & \le \frac{1}{48}\|\nabla\omega_x\|_2^2 + C\|\partial_x u_2\|_2^2 \|\nabla\omega\|_2^2. \end{aligned}$$

$$\begin{aligned} \left|\int \partial_y u_1 \omega_x \omega_y\right| &= \left|-\int u_1 \partial_{xy}\omega \partial_y \omega - \int u_1 \partial_x \omega \partial_{yy}\omega\right| \\ & \le \|u_1\|_\infty \|\partial_{xy}\omega\|_2 \|\partial_y \omega\|_2 + \|u_1\|_\infty \|\partial_x \omega\|_2 \|\partial_{yy}\omega\|_2 \\ & \le \frac{1}{2}(\|\nabla\omega_x\|_2^2 + \|\nabla\omega_y\|_2^2) + C\|u_1\|_\infty^2 \|\nabla\omega\|_2^2 \end{aligned}$$

Combining the estimates above, together with Gronwall's inequalities, we obtain

$$\begin{aligned} & \|\Omega\|_2^2 + \|j\|_2^2 + \|\nabla\omega\|_2^2 \\ & \quad + \int_0^t \left(\|\partial_{xx}u_1\|_2^2 + \|\partial_{xx}u_2\|_2^2 + \|\partial_y j\|_{L^2}^2 + \|\nabla\omega_x\|_2^2 + \|\nabla\omega_y\|_2^2 + \|\nabla\omega\|_2^2\right) d\tau \le C \end{aligned}$$

for any $t \le T$, where C depends on T and the initial H^1-norm. This completes the proof of theorem 1.1. □

5. Proof of theorem 1.2

5.1. **H^1-BOUND.** This section establishes the global H^1-bound. More precisely, we have the following theorem.

Theorem 5.1. *Assume* $(u_0, b_0, \omega_0) \in H^1(\mathbb{R}^2)$, *and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Then (1.5) has a global solution* (u, b, ω) *satisfying, for any* $T > 0$,

$$(u, b, \omega) \in L^\infty([0, T]; H^1(\mathbb{R}^2))$$

provided $\int_0^T \|\partial_y u_1, \partial_y u_2\|_\infty \, dt < \infty$.

Proof of Theorem 5.1. Dotting (4.1) by Ω, (4.2) by j, and (4.3) by $\nabla\omega$ we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Omega\|_{L^2}^2 + \|j\|_{L^2}^2 + \|\nabla\omega\|_2^2\right) + \|\partial_{xx}u_1\|_2^2 + \|\partial_{xx}u_2\|_2^2 \\
&\quad + \|\partial_y j\|_{L^2}^2 + \|\nabla\omega_y\|_2^2 + 2\|\nabla\omega\|_2^2 \\
&= 2\int \left[\partial_x b_1(\partial_x u_2 + \partial_y u_1)\, j - \partial_x u_1(\partial_x b_2 + \partial_y b_1)\right] j dx\, dy \\
&\quad - \int \nabla\omega \cdot \nabla u \cdot \nabla\omega + \int \nabla\omega \cdot \nabla\Omega \\
&\equiv K_1 + K_2 + K_3 + K_4 + K_5 + K_6.
\end{aligned}$$

K_1, K_3 and K_4 obey exactly the same bound as J_1, J_3 and J_4 of the previous section. The dissipation is not sufficient to apply anisotropic Sobolev inequality for K_2, so we bound it differently.

$$K_2 \qquad \le \left| \int \partial_x b_1\, \partial_y u_1\, j \right| \le \|\partial_y u_1\|_\infty \|\partial_x b_1\|_2 \|j\|_2.$$

To bound K_5, we use $\nabla \cdot u = 0$ and integrate by parts to obtain

$$\begin{aligned}
K_5 &= -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \\
&= -\int \partial_x u_1 \omega_x\, \omega_x - \int \partial_y u_2 \omega_y \omega_y - \int (\partial_x u_2 + \partial_y u_1)\omega_x \omega_y.
\end{aligned}$$

The following two terms on the right obey the similar bound as in the previous section.

$$\left| \int \partial_x u_1 \omega_x\, \omega_x \right| \le \frac{1}{48}\|\omega_{xy}\|_2^2 + C\|u_2\|_2^2 \|\partial_x u_2\|_2^2 \|\nabla\omega\|_2^2.$$

$$\left| \int \partial_x u_2 \omega_x \omega_y \right| \le \frac{1}{48}\|\nabla\omega_y\|_2^2 + C\|\partial_x u_2\|_2^2 \|\nabla\omega\|_2^2.$$

The following two terms can be bounded differently.

$$\left| \int \partial_y u_2 \omega_y \omega_y \right| \le \|\partial_y u_2\|_\infty \|\nabla\omega\|_2^2$$

$$\left| \int \partial_y u_1 \omega_x \omega_y \right| \le \|\partial_y u_1\|_\infty \|\nabla\omega\|_2^2$$

Combining the estimates above, together with Gronwall's inequalities, we obtain

$$\|\Omega\|_2^2 + \|j\|_2^2 + \|\nabla\omega\|_2^2 + \int_0^t \left(\|\partial_{xx}u_1\|_2^2 + \|\partial_{xx}u_2\|_2^2 + \|\partial_y j\|_{L^2}^2 + \|\nabla\omega_y\|_2^2 + \|\nabla\omega\|_2^2\right) d\tau \le C$$

for any $t \le T$, where C depends on T and the initial H^1-norm. This completes the proof of theorem 5.1. □

5.2. **Global H^2-bound and the proof of Theorem 1.2.** This subsection proves Theorem 1.2 by establishing the global H^2-bound for the solution.

Proof of Theorem 1.2. Taking the L^2 inner product of (4.1) with $\nabla\Omega$ and(4.2) with ∇j, and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2) + \|\Delta\partial_x u_2\|_2^2 + \|\nabla\partial_y j\|_2^2 = L_1 + L_2 + L_3 + L_4 + L_5 + L_6, \tag{5.1}$$

where

$$L_1 = -\int \nabla\Omega\cdot\nabla u\cdot\nabla\Omega\,dxdy, \qquad L_2 = -\int \nabla j\cdot\nabla u\cdot\nabla j\,dxdy,$$

$$L_3 = 2\int \nabla\Omega\cdot\nabla b\cdot\nabla j\,dxdy, \qquad L_4 = 2\int \nabla[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]\cdot\nabla j\,dxdy,$$

$$L_5 = -2\int \nabla[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\cdot\nabla j\,dxdy, \qquad L_6 = \int \Delta\Omega\,\Delta\omega\,dxdy.$$

Applying ∇ to (4.3) and taking the L^2-inner product with $\Delta\omega$, and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + 2\|\Delta\omega_x\|_2^2 + 2\|\Delta\omega_y\|_2^2 + \|\Delta\omega\|_2^2 = \int \Delta\Omega\,\Delta\omega - \int \Delta(u\cdot\nabla\omega)\Delta\omega \equiv L_6 + L_7. \tag{5.2}$$

Adding (5.1) and (5.2) yields

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2 + \|\Delta\omega\|_2^2) + \|\Delta\partial_x u_2\|_2^2 + \|\nabla\partial_y j\|_2^2 + 2\|\Delta\omega_x\|_2^2 + 2\|\Delta\omega_y\|_2^2 + \|\Delta\omega\|_2^2$$
$$= L_1 + L_2 + L_3 + L_4 + L_5 + 2L_6 + L_7.$$

We now estimate L_1 through L_7. We further split L_1 into 4 terms.

$$\begin{aligned} L_1 &= -\int \nabla\Omega\cdot\nabla u\cdot\nabla\Omega\,dxdy \\ &= -\int \left(\partial_x u_1(\partial_x\Omega)^2 + \partial_x u_2\partial_x\Omega\partial_y\Omega + \partial_y u_1\partial_x\Omega\partial_y\Omega + \partial_y u_2(\partial_y\Omega)^2\right) \\ &= L_{11} + L_{12} + L_{13} + L_{14}. \end{aligned}$$

Therefore,

$$L_{11} = -\int \partial_x u_1(\partial_{xx}u_2)^2 - \int \partial_x u_1(\partial_{xy}u_1)^2 + 2\int \partial_x u_1\partial_{xx}u_2\partial_{xy}u_1.$$

Integration by parts yields

$$\int \partial_x u_1 (\partial_{xx} u_2)^2 = -\int \partial_{xx} u_1 \partial_{xx} u_2 \partial_x u_2 - \int \partial_x u_1 \partial_{xxx} u_2 \partial_x u_2$$
$$\equiv L_{111} + L_{112},$$

L_{111} and L_{112} obey the exactly same bound as in [13].

$$L_{111} \le \frac{1}{48} \|\Delta \partial_x u_2\|_2^2 + C \|\partial_{xx} u_2\|_2^2 \|\partial_{xx} u_1\|_2^{\frac{2}{3}} \|\partial_x u_2\|_2^{\frac{2}{3}}.$$
$$L_{112} \le \frac{1}{48} \|\Delta \partial_x u_2\|_2^2 + C \|\Omega\|_2 \|\partial_x u_2\|_2 \|\nabla \Omega\|_2^2.$$
$$L_{12} \le \frac{1}{48} \|\partial_{xx} \Omega\|_2^2 + C \|\Omega\|_2^{\frac{2}{3}} \|\partial_{xx} u_1\|_2^{\frac{2}{3}} \|\nabla \Omega\|_2^2.$$

L_{13} and L_{14} can be bounded differently as

$$L_{13} \le \|\partial_y u_1\|_\infty \|\nabla \Omega\|_2^2$$
$$L_{14} \le \frac{1}{48} \|\partial_y u_2\|_\infty \|\nabla \Omega\|_2^2.$$

L_2 and L_3 admit the same bound as in th paper [13].

To estimate L_2, we write it out explicitly as

$$\begin{aligned} L_2 &= -\int \nabla j \cdot \nabla u \cdot \nabla j \, dxdy \\ &= -\int \big(\partial_x u_1 (\partial_x j)^2 + \partial_y u_1 \partial_x j \partial_y j + \partial_y u_2 (\partial_y j)^2 + \partial_x u_2 \partial_x j \partial_y j\big) \\ &= L_{21} + L_{22} + L_{23} + L_{24}. \end{aligned}$$

The terms on the right can be bounded as follows.

$$\begin{aligned} L_{21} &= \left| -2 \int u_2 \partial_x j \partial_{xy} j \right| \\ &\le C \|\partial_{xy} j\|_2 \|\partial_x j\|_2^{\frac{1}{2}} \|\partial_{xy} j\|_2^{\frac{1}{2}} \|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}}. \\ &\le \frac{1}{48} \|\nabla \partial_y j\|_2^2 + C \|u_2\|_2^2 \|\partial_x u_2\|_2^2 \|\nabla j\|_2^2. \end{aligned}$$

$$\begin{aligned} L_{22} &\le C \|\partial_y u_1\|_2 \|\partial_x j\|_2^{\frac{1}{2}} \|\partial_y j\|_2^{\frac{1}{2}} \|\partial_{xy} j\|_2 \\ &\le \frac{1}{48} \|\nabla \partial_y j\|_2^2 + C \|\partial_y u_1\|_2^2 \|\nabla j\|_2^2. \end{aligned}$$

Similarly,

$$L_{23} \le \frac{1}{48} \|\nabla \partial_y j\|_2^2 + C \|u_1\|_2^2 \|\Omega\|_2^2 \|\nabla j\|_2^2.$$

$$\begin{aligned} L_{24} &\le C \|\partial_x u_2\|_2 \|\partial_y j\|_2^{\frac{1}{2}} \|\partial_{xy} j\|_2^{\frac{1}{2}} \|\partial_x j\|_2^{\frac{1}{2}} \|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\le C \|\Omega\|_2 \|\nabla j\|_2 \|\nabla \partial_y j\|_2. \end{aligned}$$

We now turn to L_3. Observe that

$$\begin{aligned} L_3 &= \int \partial_x\Omega\partial_x b_1\partial_x j + \partial_x\Omega\partial_x b_2\partial_y j + \partial_y\Omega\partial_y b_1 j_x + \partial_y\Omega\partial_y b_2\partial_y j \\ &\equiv L_{31} + L_{32} + L_{33} + L_{34}. \end{aligned}$$

The terms on the right can be bounded as follows.

$$\begin{aligned} L_{31} &\leq \left|\int \partial_x\Omega\partial_x b_1\partial_x j\right| \\ &\leq C\|\partial_x\Omega\|_2\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xx} b_1\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\leq C\|\partial_x\Omega\|_2\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\leq C\|\partial_x j\|_2\|\nabla\partial_y j\|_2 + \|\partial_y j\|\|\partial_x b_1\|\|\partial_x\Omega\|_2^2 \\ &\leq \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_x b_1\|_2^2 + \|\partial_y j\|_2^2 + 1)(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2). \end{aligned}$$

The last three terms admit,

$$\begin{aligned} L_{32} &\leq \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_x b_2\|_2^2 + \|\partial_y j\|^2 + 1)(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2). \\ L_{33} &\leq \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_x b_1\|_2^2 + \|\partial_y j\|_2^2 + 1)(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2). \\ L_{34} &\leq \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_y j\|_2^2 + \|j\|_2 + 1)\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2). \end{aligned}$$

We now estimate L_4.

$$\begin{aligned} L_4 &= 2\int \nabla[\partial_x b_1(\partial_x u_2 + \partial_y u_1)] \cdot \nabla j \; dxdy \\ &= 2\int \partial_x[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]j_x + \partial_y[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]j_y \; dxdy \\ &\equiv L_{41} + L_{42}. \end{aligned}$$

We bound L_{41} and L_{42} as follows.

$$L_{41} = \int \partial_{xx} b_1\partial_x u_2\partial_x j + \partial_x b_1\partial_{xx} u_2\partial_x j + \partial_{xx} b_1\partial_y u_2\partial_x j + \partial_x b_1\partial_{xy} u_1\partial_x j$$

Now

$$\begin{aligned} &\left|\int \partial_{xx} b_1\partial_x u_2\partial_x j + \partial_{xx} b_1\partial_y u_2\partial_x j\right| \\ &\leq C\|\partial_{xy} b_2\|_2\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xx} u_2\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\qquad + C\|\partial_{xy} b_2\|_2\|\partial_y u_2\|_2^{\frac{1}{2}}\|\partial_{xy} u_2\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\leq C\|\partial_y j\|_2\|\nabla\partial_y j\|_2^{\frac{1}{2}}\|\partial_x u\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}} \\ &\leq \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_y j\|_2^2 + \|\partial_x u\|_2^2)(\|\nabla j\|_2^2 + \|\nabla\Omega\|_2^2). \end{aligned}$$

$$
\begin{aligned}
&\left|\int \partial_x b_1 \partial_{xx} u_2 \partial_x j + \partial_x b_1 \partial_{xy} u_1 \partial_x j\right| \\
&\le C\|\partial_{xx}u_2\|_2 \|\partial_x b_1\|_2^{\frac{1}{2}} \|\partial_{xx} b_1\|_2^{\frac{1}{2}} \|\partial_x j\|_2^{\frac{1}{2}} \|\partial_{xy} j\|_2^{\frac{1}{2}} \\
&\qquad + C\|\partial_{xy}u_1\|_2 \|\partial_x b_1\|_2^{\frac{1}{2}} \|\partial_{xx} b_1\|_2^{\frac{1}{2}} \|\partial_x j\|_2^{\frac{1}{2}} \|\partial_{xy} j\|_2^{\frac{1}{2}} \\
&\le \frac{1}{48}\|\nabla \partial_y j\|_2^2 + C(\|\partial_x u_1\|_2^2 + \|\partial_x b_1\|_2^2 + \|\partial_y j\|_2^2 + 1)(\|\nabla \Omega\|_2^2 + \|\nabla j\|_2^2).
\end{aligned}
$$

L_{42} can be written as

$$
\begin{aligned}
L_{42} &= 2\int (\partial_{xy} b_1 \partial_x u_2 + \partial_x b_1 \partial_{xy} u_2 + \partial_{xy} b_1 \partial_y u_1 + \partial_x b_1 \partial_{yy} u_1)\partial_y j \; dxdy \\
&\equiv L_{421} + L_{422} + L_{423} + L_{424}.
\end{aligned}
$$

The bounds for the terms on the right are given as follows.

$$
\begin{aligned}
L_{421} &\le C\|\partial_x u_2\|_2 \|\partial_{xy} b_1\|_2^{\frac{1}{2}} \|\partial_{xyy} b_1\|_2^{\frac{1}{2}} \|\partial_y j\|_2^{\frac{1}{2}} \|\partial_{xy} j\|_2^{\frac{1}{2}} \\
&\le C\|\Omega\|_2 \|\nabla j\|_2 \|\nabla \partial_y j\|_2 \\
&\le \frac{1}{48}\|\nabla \partial_y j\|_2^2 + C\|\Omega\|_2^2 \|\nabla j\|_2^2.
\end{aligned}
$$

$$
\begin{aligned}
L_{422} &\le C\|\partial_x b_1\|_2^{\frac{1}{2}} \|\partial_{xy} b_1\|_2^{\frac{1}{2}} \|\partial_{xy} u_2\|_2 \|\partial_y j\|_2^{\frac{1}{2}} \|\partial_{xy} j\|_2^{\frac{1}{2}} \\
&\le C\|j\|_2^{\frac{1}{2}} \|\partial_y j\|_2^{\frac{1}{2}} \|\partial_x \Omega\| \|\nabla j\|_2^{\frac{1}{2}} \|\nabla \partial_y j\|_2^{\frac{1}{2}} \\
&\le \frac{1}{48}\|\nabla \partial_y j\|_2^2 + C\|\partial_y j\|_2^2 \|\nabla j\|_2^2 + C\|j\|_2 \|\nabla \Omega\|_2^2.
\end{aligned}
$$

Other terms admit,

$$
\begin{aligned}
L_{423} &\le C\|\partial_{xy} b_1\|_2^{\frac{1}{2}} \|\partial_{xyy} b_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_{xy} u_1\|_2^{\frac{1}{2}} \|\partial_y j\|_2 \\
&\le \frac{1}{48}\|\nabla \partial_y j\|_2^2 + C\|\Omega\|_2 \|\nabla j\|_2^2 + C\|\partial_y j\|_2^2 \|\nabla \Omega\|_2^2.
\end{aligned}
$$

$$
L_{424} \le \|\partial_y u_1\|_\infty (\|\partial_{xy} b_1\|_2 \|\partial_y j\|_2 + \|\partial_x b_1\| \|\partial_{yy} j\|_2)
$$

L_5, L_6 and L_7 can be bounded exactly as in the paper [13]. We now estimate L_5. More explicitly, L_5 can be written as

$$
\begin{aligned}
L_5 &= -2\int \nabla[\partial_x u_1(\partial_x b_2 + \partial_y b_1)] \cdot \nabla j \; dxdy \\
&= -2\int \partial_x[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\partial_x j + \partial_y[(\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\partial_y j \; dxdy \\
&\equiv L_{51} + L_{52}.
\end{aligned}
$$

L_{52} is bounded as follows.

$$\begin{aligned} L_{52} &\leq C\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_x b_2\|_2^{\frac{1}{2}}\|\partial_{xx}b_2\|_2^{\frac{1}{2}}\|\partial_{yy}j\|_2 \\ &\quad +C\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_y b_1\|_2^{\frac{1}{2}}\|\partial_{xy}b_1\|_2^{\frac{1}{2}}\|\partial_{yy}j\|_2 \\ &\leq C\|\Omega\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla\partial_y j\|_2 \\ &\leq \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|\Omega\|_2\|j\|_2(\|\nabla\Omega\|_2^2+\|\nabla j\|_2^2). \end{aligned}$$

L_{51} contains four terms.

$$\begin{aligned} L_{52} &= -2\int(\partial_{xx}u_1\partial_x b_2 + \partial_x u_1\partial_{xx}b_2 + \partial_{xx}u_1\partial_y b_1 + \partial_x u_1\partial_{xy}b_1)\partial_x j\, dxdy \\ &\equiv L_{521}+L_{522}+L_{523}+L_{524}. \end{aligned}$$

These terms are estimated as follows.

$$L_{521} \leq \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_y j\|_2^2+\|j\|_2^2)(\|\nabla j\|_2^2+\|\nabla\Omega\|_2^2).$$
$$L_{522} \leq \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|\Omega\|_2^{\frac{2}{3}}\|\partial_{xy}u_1\|_2^{\frac{2}{3}}\|\nabla j\|_2^2.$$

$$\begin{aligned} L_{523} &\leq C\|\partial_{xx}u_1\|_2^{\frac{1}{2}}\|\partial_{xxy}u_1\|_2^{\frac{1}{2}}\|\partial_y b_1\|_2^{\frac{1}{2}}\|\partial_{xy}b_1\|_2^{\frac{1}{2}}\|\partial_x j\|_2 \\ &\leq C\|\Omega_{xy}\|_2^{\frac{1}{2}}\|\partial_y j\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2 \\ &\leq \frac{1}{48}\|\partial_{xy}\Omega\|_2^2 + C\|\partial_y j\|_2^2\|\nabla\Omega\|_2^2 + C\|j\|_2\|\nabla j\|_2^2. \end{aligned}$$

$$\begin{aligned} L_{524} &\leq C\|\partial_x j\|_2\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_{yy}b_1\|_2^{\frac{1}{2}}\|\partial_{xyy}b_1\|_2^{\frac{1}{2}} \\ &\leq C\|\partial_y j\|_2\|\Omega\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla\partial_y j\|_2^{\frac{1}{2}} \\ &\leq \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|\partial_y j\|_2^2\|\nabla\Omega\|_2^2 + \|\Omega\|_2\|\nabla j\|_2^2. \end{aligned}$$

L_6 can be easily bounded.

$$L_6 = \int \Delta\Omega\,\Delta\omega = \int \Delta(\partial_x u_2 - \partial_y u_1)\Delta\omega$$

with

$$\int \Delta\partial_x u_2\Delta\omega \leq \|\Delta\partial_x u_2\|_2\|\Delta\omega\|_2, \quad \left|\int \Delta\partial_y u_1\Delta\omega\right| \leq \|\Delta u\|_2\|\Delta\omega_y\|_2.$$

We now estimate the last term L_7.

$$\begin{aligned} L_7 &= -\int \Delta(u\cdot\nabla\omega)\Delta\omega = -\int\Delta(u_1\partial_1\omega + u_2\partial_2\omega)\Delta\omega \\ &= -\int \partial_{xx}u_1\partial_x\omega\Delta\omega - \int\partial_{xx}u_2\partial_y\omega\Delta\omega - \int\partial_{yy}u_1\partial_x\omega\Delta\omega - \int\partial_{yy}u_2\partial_y\omega\Delta\omega \\ &\quad -2\int\partial_x u_1\partial_{xx}\omega\Delta\omega - 2\int\partial_x u_2\partial_{xy}\omega\Delta\omega - 2\int\partial_y u_1\partial_{xy}\omega\Delta\omega - 2\int\partial_y u_2\partial_{yy}\omega\Delta\omega \\ &\equiv L_{71}+L_{72}+L_{73}+L_{74}+L_{75}+L_{76}+L_{77}+L_{78} \end{aligned}$$

Now

$$\begin{aligned} L_{71} & \left| \leq \int \partial_{xx} u_1 \partial_x \omega \Delta \omega \right| \\ & \leq C \|\partial_{xx} u_1\|_2 \|\partial_x \omega\|_2^{\frac{1}{2}} \|\partial_{xx} \omega\|_2^{\frac{1}{2}} \|\Delta \omega\|_2^{\frac{1}{2}} \|\Delta \partial_y \omega\|_2^{\frac{1}{2}} \\ & \leq \frac{1}{48} \|\Delta \partial_y \omega\|_2^2 + C(\|\nabla \omega\|_2^2 + \|\partial_{xx} u_1\|_2^2) \|\Delta \omega\|_2^2. \end{aligned}$$

Similarly, we obtain

$$L_{72} \leq \frac{1}{48}(\|\Delta \partial_y \omega\|_2^2 + \|\Delta \partial_x u_2\|_2^2) + C\|\nabla \omega\|_2^2 (\|\nabla \Omega\|_2^2 + \|\Delta \omega\|_2^2).$$

$$\begin{aligned} L_{73} = & \left| - \int \partial_u u_1 \partial_{xy} \omega \Delta \omega - \int \partial_y u_1 \partial_x \omega \Delta \partial_y \omega \right| \\ \leq & C \|\partial_{xy} \omega\|_2 \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_{xy} u_1\|_2^{\frac{1}{2}} \|\Delta \omega\|_2^{\frac{1}{2}} \|\Delta \partial_y \omega\|_2^{\frac{1}{2}} \\ & + C \|\Delta \partial_y \omega\|_2 \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_{xy} u_1\|_2^{\frac{1}{2}} \|\partial_x \omega\|_2^{\frac{1}{2}} \|\partial_{xy} \omega\|_2^{\frac{1}{2}} \\ \leq & C \|\nabla \partial_y \omega\|_2 \|\partial_y u_1\|_2^{\frac{1}{2}} \|\nabla \Omega\|_2^{\frac{1}{2}} \|\Delta \omega\|_2^{\frac{1}{2}} \|\Delta \partial_y \omega\|_2^{\frac{1}{2}} \\ & + C \|\Delta \partial_y \omega\|_2 \|\partial_y u_1\|_2^{\frac{1}{2}} \|\nabla \Omega\|_2^{\frac{1}{2}} \|\partial_x \omega\|_2^{\frac{1}{2}} \|\nabla \partial_y \omega\|_2^{\frac{1}{2}} \\ \leq & \frac{1}{48} \|\Delta \partial_y \omega\|_2^2 + C(\|\partial_y u_1\|_2^2 + \|\nabla \partial_y \omega\|_2^2 + 1)(\|\nabla \Omega\|_2^2 + \|\Delta \omega\|_2^2 + \|\nabla \omega\|_2^2). \end{aligned}$$

Similarly

$$\begin{aligned} L_{74} & \leq \frac{1}{48}(\|\Delta \partial_y \omega\|_2^2 + \|\Delta \partial_x u_2\|_2^2) + C\|\nabla \omega\|_2^2 (\|\nabla \Omega\|_2^2 + \|\Delta \omega\|_2^2). \\ L_{75} & \leq \frac{1}{48} \|\Delta \partial_y \omega\|_2^2 + C(\|\Omega\|_2^2 + \|\partial_{xx} u_1\|_2^2) \|\Delta \omega\|_2^2. \end{aligned}$$

$$\begin{aligned} L_{76} = & \left| -2 \int \partial_x u_2 \partial_{xy} \omega \Delta \omega \right| \\ & \leq C \|\partial_{xy} \omega\|_2 \|\Delta \omega\|_2^{\frac{1}{2}} \|\Delta \partial_y \omega\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_{xx} u_2\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla \partial_y \omega\|_2 \|\Delta \omega\|_2^{\frac{1}{2}} \|\Delta \partial_y \omega\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\nabla \Omega\|_2^{\frac{1}{2}} \\ & \leq \frac{1}{48} \|\Delta \partial_y \omega\|_2^2 + C\,(\|\nabla \partial_y \omega\|_2^2 + \|\partial_x u_2\|_2^2)(\|\nabla \Omega\|_2^2 + \|\Delta \omega\|_2^2). \end{aligned}$$

$$\begin{aligned} L_{77} = & \left| -2 \int \partial_y u_1 \partial_{xy} \omega \Delta \omega \right| \\ & \leq C \|\partial_{xy} \omega\|_2 \|\Delta \omega\|_2^{\frac{1}{2}} \|\Delta \partial_y \omega\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_{xy} u_1\|_2^{\frac{1}{2}} \\ & \leq C \|\nabla \partial_y \omega\|_2 \|\Delta \omega\|_2^{\frac{1}{2}} \|\Delta \partial_y \omega\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|\nabla \Omega\|_2^{\frac{1}{2}} \\ & \leq \frac{1}{48} \|\Delta \partial_y \omega\|_2^2 + C\,(\|\nabla \partial_y \omega\|_2^2 + \|\partial_y u_1\|_2^2)(\|\nabla \Omega\|_2^2 + \|\Delta \omega\|_2^2). \end{aligned}$$

$$L_{78} \leq \frac{1}{48} \|\Delta \partial_y \omega\|_2^2 + C(\|\Omega\|_2^2 + \|\partial_{xx} u_1\|_2^2) \|\Delta \omega\|_2^2.$$

Collecting the estimates above and applying Gronwall's inequality, we obtain the desired global H^2-bound.

$$\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2 + \|\Delta\omega\|_2^2 + \int_0^T \left(\|\Delta\partial_x u_2\|_2^2 + \|\nabla\partial_y j\|_2^2 + 2\|\Delta\omega_x\|_2^2 + 2\|\Delta\omega_y\|_2^2 + \|\Delta\omega\|_2^2\right) d\tau \leq C$$

for any $t \leq T$, where C depends on T and the initial data.

This completes the proof of the Theorem 1.2. □

6. Conclusion

We have investigated the global regularity issue concerning solutions of the 2D magneto-micropolar equations with partial dissipation. We established that if the velocity field u satisfies $(\partial_y u_1, \partial_y u_2) \in L^\infty\left([0,T],\mathbb{R}^2\right)$, then magneto-micropolar equations with partial dissipation have globally regular solution.

Acknowledgements

D. Regmi was partially supported by The Presidential Summer Incentive Award 2023-24, University of North Georgia. The authors are grateful to the referee for careful reading of the paper, valuable suggestions, and comments, which significantly contributed to the improvement and quality of this manuscript.

References

[1] G. Ahmadi and M. Shahinpoor, Universal stability of magneto-micropolar fluid motions, *Int. J. Engng. Sci.* 12 (1974), 657-663.

[2] C. Cao, D. Regmi and J. Wu, The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion, *J. Differential Equations* 254 (2013), 2661-2681.

[3] C. Cao and J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, *Adv. in Math.* 226 (2011), 1803-1822.

[4] J.-Y. Chemin, Perfect incompressible fluids, Oxford lecture series in mathematics and its applications, 14, *Oxford University Press Inc.*, New York (1998)

[5] J. Cheng and Y. Liu, Global regularity of the 2D magnetic micropolar fluid flows with mixed partial viscosity, *Computer and Mathematics with Applications* 70(2015), 66-72.

[6] S.C. Cowin, Polar fluids, *Phys. Fluids*11 (1968), 1919-1927.

[7] B. Dong and Z. Zhang, Global regularity of the 2D micropolar fluid flows with zero angular viscosity, *J. Differential Equations* 249(2010) 200-213.

[8] Y. Guo and H. Shang, Global well-posedness of two-dimensional magneto-micropolar equations with partial dissipation, *Applied Mathematics and computation* 313 (2017), 392-4-7

[9] H. Lin, S. Liu, H. Zhang, Global Regularity of 2D Incompressible Magneto-Micropolar Fluid Equations with Partial Viscosity, *Acta Math Sci* 43, 1275–1300 (2023).

[10] Yujun Liu, Global regularity and time decay for the 2D magneto-micropolar system with fractional dissipation and partial magnetic diffusion, *Commun. Math. Sci,* Vol 20, No.5, pp.1211-1241 (2022)

[11] G. Lukaszewick, Micropolar Fluids: Theory and Applications, Birkhauser, Boston, 1999.

[12] E.E. Ortega-Torres and M. A. Rojas-Medar, Magneto-micropolar fluid motion: global existence of strong solutions, *Abstr. Appl. Anal.* 4(1999), 109-125.

[13] D. Regmi, The 2D magneto-micropolar equations with partial dissipation, *Mathematical Method in Applied Sciences*, 42 (2019), Issue 12, 4305-4317

[14] D. Regmi and J. Wu, Global regularity for the 2D magneto-micropolar equations with partial dissipation, *J. Math. Study* 49 (2016), No.2, 169-194.

[15] M.A. Rojas-Medar, Magneto-micropolar fluuid motion: existence and uniqueness of strong solutions, *Math. Nachr.* 188 (1997), 301-319.

[16] H. Shang, C. Gu, Large time behavior for two-dimensional magneto-micropolar equations with only micro-rotational dissipation and magnetic diffusion, *Applied Mathematics Letters* 99, 85(2020).

[17] K. Yamazaki, Global regularity of the two-dimensional magneto-micropolar fluid system with zero angular viscosity, *Discrete Contin. Dyn. Syst.* 35 (2015), 2193-2207.

[18] B. Yuan, Regularity of weak solutions to magneto-micropolar equations, *Acta Math. Sci. Ser. B Engl. Ed.* 30B (2010), 1469-1480.