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DOUBLE SEQUENCE SPACE OF FUZZY REAL NUMBERS DEFINED BY ORLICZ FUNCTION

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Abstract: The theory of fuzzy logic and the fuzzy set has been successfully applied in various fields of research in social science, management science, and mathematics. In this paper, we use the concept of fuzzy real numbers to introduce and study the new double sequence spaces $l_\infty^{\mathcal{F}}(\mathcal{M}, \lambda, \rho)$, $C^{\mathcal{F}}(\mathcal{M}, \lambda, \rho)$ and $C_o^{\mathcal{F}}(\mathcal{M}, \lambda, \rho)$ of fuzzy real numbers defined by the Orlicz function and study some of their properties like linear space structure, completeness, and solidness.

Key Words: Fuzzy set and logic, Fuzzy real numbers, Orlicz function, Difference sequence space
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1. Introduction

Mathematics has traditionally been restricted to two conclusions: true and false. According to the traditional viewpoint, science should strive for certainty in all statements, and uncertainty is seen as unscientific[1]. However, American Mathematician L. A. Zadeh[2], initially introduced the concept of fuzzy set and fuzzy logic in 1965 to deal with difficulties seen in mathematics that have no clear answer. Since then, a number of authors have researched various aspects of its theory and applications. A large number of authors have used the fuzzy set and fuzzy numbers in different classes of sequence spaces. Motloka [3] has studied the boundedness and convergent sequence of fuzzy numbers and has shown that every convergent sequence of fuzzy numbers is bounded. The notion of the fuzzy set has been successfully applied in studying the double sequence of fuzzy real numbers by researchers. Hardy [4] developed the concept of regular convergence for double sequences in the sense that the double sequence has a limit in the Pringsheim sense and has one-sided limits. In 2005, Altay and Basar[5] defined the double sequence spaces, looked at some of their properties, and demonstrated that they are fully paranormed or normed spaces under certain conditions. In order to study the double statistical convergence of a sequence of fuzzy numbers, Savas[7] introduced some new double sequence spaces of fuzzy numbers in

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2010. Das, M.[6] introduced certain vector-valued difference double sequences defined by the Orlicz function in 2012 to investigate their various properties. Similarly, Savas and Patterson [8] developed some new double sequence spaces and investigated some of their features in 2007. Tripathy and Sarma [9] investigated various aspects of converging, null, and bounded sequence spaces of fuzzy real numbers defined by an Orlicz function. In 2009, Tripathy and Sarma [10] introduced double sequence spaces and examined some topological and algebraic properties of these spaces. Tripathy and Sarma [11] proposed certain double sequence spaces of fuzzy real numbers defined by the Orlicz function in 2011 in order to investigate their various features. In 2010, Savas [12] used the Orlicz function to develop a novel concept for double lacunary sequence spaces of fuzzy numbers and investigate whether a sequence of fuzzy numbers is double strongly P- convergent with respect to an Orlicz function. Kılınç, G., and Solak, İ.[13] studied the topological and algebraic characteristics of new modulus-defined double sequence spaces they established in 2014. Also, Pahari [14] created and studied a new null class of normed space using the Orlicz function and investigated its linear topological properties. Similarly, Habil [15] in 2016, made an effort to provide such an informative overview, summarizing key points from the foundational theory of double sequences and double series and providing thorough justifications for each. Basu [16] defined a new class of fuzzy number sequences using Orlicz functions in 2018 and derived numerous useful classes with rich structural features. Likewise in 2018, Sarma [17] investigated various properties of double sequence spaces of fuzzy real numbers defined by an Orlicz function. The concepts of the limit superior and limit inferior of a double sequence of fuzzy numbers were presented by Talo [18], and a number of properties were obtained for these concepts. The properties of convergent, null and limited double sequence spaces defined by the double Orlicz function were investigated by Dabbas and Battor [19] in 2020. In 2021, Mansoor and Battor [20] developed some new double sequence spaces based on the double Orlicz function and fuzzy metric to test some basic properties of the new double sequence spaces. Paudel and Pahari[21] studied the fundamental notion of fuzzy metric space of fuzzy real numbers with some topological properties in 2021. Also Paudel and et.al [22] in 2022 introduced generalized from of p- bounded variation of difference sequence space of fuzzy real numbers to studied different properties of it. Wladyslaw Orlicz firstly introduced Orlicz space in 1932. Later on, Lindenstrauss and Tzafriri [23] used the idea of the Orlicz function to construct the sequence space $l_{\mathcal{M}} = \{x \in \omega : \sum_{k=1}^{\infty} \mathcal{M}(\frac{|x|}{\rho}) < \infty\}$.

The space $l_{\mathcal{M}}$ with the norm $||x||$ defined by

$$||x|| = \inf \{\rho > 0 : \sum_{k=1}^{\infty} \mathcal{M}(\frac{|x_k|}{\rho}) \leq 1\}$$

becomes a Banach space and it is called Orlicz sequence space.

This paper is organized as follows: in Section 2, we examine some definitions that are essential for the paperwork. Section 3 will go over the double sequence spaces of fuzzy real numbers defined by the Orlicz function. Section 4 will offer some results on the double sequence space of fuzzy real numbers defined by the Orlicz function. Section 5 concludes the paper.

2. Preliminaries and definition:

Let $\mathcal{D}$ be the set of all bounded intervals $\mathcal{A}$ = [a,b] on the real line $\mathbb{R}$. For any $\mathcal{A}, \mathcal{B} \in D$ with $\mathcal{A} = [a_1, b_1]$ and $\mathcal{B} = [a_2, b_2]$, $\mathcal{A} \leq \mathcal{B}$ if $a_2 \leq a_1$ and $b_1 \leq b_2$. Define a relation d on $\mathcal{D}$ by $d(\mathcal{A}, \mathcal{B}) = max\{\mid a_1 - a_2 \mid, \mid b_1 - b_2 \mid\}$
Then clearly, d defines a metric on $\mathcal{D}$ and obviously $(\mathcal{D}, d)$ is a complete metric space.

Definition 2.1. [24] A fuzzy real number is a fuzzy set i.e a mapping $\mathcal{X} : \mathbb{R} \to I = [0, 1]$ associating each real number $t \in \mathbb{R}$ with its membership value $\mathcal{X}(t)$ satisfying that $\mathcal{X}$ is

i. normal if there exists a real number t such that $\mathcal{X}(t) = 1$
ii. convex if for $t, s \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $\mathcal{X}(\lambda t + (1 - \lambda)s) \geq \min \{\mathcal{X}(t), \mathcal{X}(s)\}$.
iii upper semi continuous if for each $\varepsilon > 0$,$\mathcal{X}^{-1}([0, a + \varepsilon))$ is open for all $a \in I$ in the usual topology of $\mathbb{R}$.

Definition 2.2. [25] The α- level set on a fuzzy set $\mathcal{X}$ is denoted by $\mathcal{X}^\alpha$ and defined by $\mathcal{X}^\alpha = \{t \in \mathbb{R} : \mathcal{X}(t) \geq \alpha\}$.
The *support* of a fuzzy number is the set of all those elements of the fuzzy number having membership value greater than zero.
Suppose $\mathbb{R}(I)$ denotes the set of all real fuzzy numbers which are upper semi-continuous and have compact support. In other words , if $\mathcal{X} \in \mathbb{R}(I)$ then for any $\alpha \in [0, 1]$,

$$\mathcal{X}^\alpha = \begin{cases} t : \mathcal{X}(\mathrm{t}) \geq \alpha & \text{for } \alpha \in (0, 1] \\ t : \mathcal{X}(\mathrm{t}) > \alpha & \text{for } \alpha = 0 \end{cases} \tag{2.1}$$

The addition and scalar multiplication on $\mathbb{R}$(I) are defined as

$$[\mathcal{X} + \mathcal{Y}]^\alpha = \mathcal{X}^\alpha + \mathcal{X}^\alpha \text{ and } (\alpha\mathcal{X})^\alpha = \alpha(\mathcal{X})^\alpha \text{ for all } \alpha \in [0, 1].$$

Consider a mapping $\bar{d}$ **:** $\mathbb{R}(I) \times \mathbb{R}(I) \longrightarrow \mathbb{R}(I)$ by the relation

$$\bar{d}(\mathcal{X}, \mathcal{Y}) = \sup d(\mathcal{X}^\alpha, \mathcal{Y}^\alpha) \text{ for } 0 \leq \alpha \leq 1.$$

Then $\bar{d}$ defines a metric on $\mathbb{R}(I)$ and $(\mathbb{R}(I), \bar{d})$ forms a complete metric space. Also , for any $\mathcal{X}, \mathcal{Y} \in \mathbb{R}(I), \mathcal{X} \leq \mathcal{Y}$ if and only if $[X^\alpha] \leq [Y^\alpha]$ for $\alpha \in [0, 1]$ and $X^\alpha = [x_1^\alpha, x_2^\alpha]$ and $Y^\alpha = [y_1^\alpha, y_2^\alpha]$.
Let $\lambda : \mathbb{R}(I) \times \mathbb{R}(I) \longrightarrow \mathbb{R}$ be defined by $\lambda(X, Y) = \sup_{0 \leq \alpha \leq 1} \lambda_\alpha(X^\alpha, Y^\alpha)$
where, $\lambda_\alpha : \mathbb{R}(I) \times \mathbb{R}(I) \longrightarrow \mathbb{R}$ is defined by $\lambda_\alpha(X^\alpha, Y^\alpha) = min\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.
Similarly, $\rho : \mathbb{R}(I) \times \mathbb{R}(I) \longrightarrow \mathbb{R}$ be defied by $\rho(X, Y) = \sup_{0 \leq \alpha \leq 1} \rho_\alpha(X^\alpha, Y^\alpha)$
where, $\rho_\alpha : \mathbb{R}(I) \times \mathbb{R}(I) \longrightarrow \mathbb{R}$ is defined by $\rho_\alpha(X^\alpha, Y^\alpha) = max\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.
A sequence of fuzzy numbers $\mathcal{X} = (\mathcal{X}_k)$ is a function $\mathcal{X} : \mathbb{N} \longrightarrow \mathbb{R}(I)$, where $\mathbb{N} = 0, 1, 2, . \quad . \quad .$
The number $\mathcal{X}_k$ is the kth value of the function at $k \in \mathbb{N}$ and is the kth term of the sequence.

Definition 2.3. Let ω be the set of all real or complex-valued double sequences which is a vector space with coordinate wise addition and scalar multiplication. Then any vector subs-pace of ω is called sequence space.

3. Double Sequence Space of Fuzzy Real Numbers And Orlicz Function

A double sequence of fuzzy numbers $\mathcal{X} = (\mathcal{X}_{nk})$ is a function form $\mathcal{X} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}(I)$, the set of fuzzy real numbers. The fuzzy number $\mathcal{X}_{nk}$ is the value of the function at the point $(n, k) \in N \times N$ and is called (n, k)-term of the double sequence.

Example 3.1. The function $\mathcal{X} : \mathbb{N} \times \mathbb{N} \longrightarrow$ defined by $\mathcal{X}(n, k) = \frac{n}{n+k}$ is a double sequence. A double sequence $\mathcal{X} = (\mathcal{X}_{nk})$ of fuzzy numbers is said to be bounded if there exist fuzzy numbers M and m such that $m \leq \mathcal{X}_{nk} \leq M$ for all $n, k \in \mathbb{N}$.

A double sequence $\mathcal{X} = (\mathcal{X}_{nk})$ of fuzzy numbers is said to be Cauchy double sequence if $\forall \epsilon > 0 \exists n_o \in \mathbb{N} : d(\mathcal{X}^i_{nk}, \mathcal{X}^j_{nk}) < \varepsilon$ for $min(i, j) \geq n_o$.
We also say that the double sequence $\mathcal{X} = (\mathcal{X}_{nk})$ of fuzzy numbers converges to a fuzzy number $\mathcal{X}_o$ if $\mathcal{X}_{nk}$ tends to $\mathcal{X}_o$ as both n and k tend to ∞ independently of one another.

A double sequence space $\omega^{\mathcal{F}}$ of fuzzy numbers is said to be solid if $(\mathcal{Y}_{nk}) \in \omega^{\mathcal{F}}$, whenever $|\mathcal{Y}_{nk}| \leq |\mathcal{X}_{nk}|$ for all $n, k \in \mathbb{N}$ for some $(\mathcal{X}_{nk}) \in \omega^{\mathcal{F}}$.

Tripathy and Sarma [10] in 2011 defined classes of double sequence spaces of fuzzy numbers defined by Orlicz function as follow :

$$(2^{\ell}\infty)_{\mathcal{F}}(\mathcal{M}) = \{\mathcal{X} = (\mathcal{X}_{nk}) \in \mathbb{R}(I) : sup_{n,k}\mathcal{M}(\tfrac{\bar{d}(\mathcal{X}_{nk},\bar{0})}{\rho}) < \infty, \quad for\ some \quad \rho > 0\}.$$

$$2^{C}\mathcal{F}(\mathcal{M}) = \{\mathcal{X} = (\mathcal{X}_{nk}) \in \mathbb{R}(I) : \lim_{n,k} \mathcal{M}(\tfrac{\bar{d}(\mathcal{X}_{nk},\mathcal{X})}{\rho}) = 0, \quad for\ some \quad \rho > 0\}.$$

$$(2^{C}0)_{\mathcal{F}}(\mathcal{M}) = \{\mathcal{X} = (\mathcal{X}_{nk}) \in \mathbb{R}(I) : \lim_{n,k} \mathcal{M}(\tfrac{\bar{d}(\mathcal{X}_{nk},\bar{0})}{\rho}) = 0, \quad for\ some \quad \rho > 0\}.$$

In 2020, Dabbas and Battor[19] defined the classes of sequences of fuzzy real numbers using double Orlicz functions as follows:

$$\ell^{\mathcal{F}}_{\infty}(\mathcal{X}, \mathcal{P}) = \{(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) \in \omega_{\mathcal{F}} : \sup_{n,k}\{(\lambda(\tfrac{\bar{d}(\mathcal{X}_{nk},\bar{0})}{r}))^{\mathcal{P}_{nk}} \vee (\rho(\tfrac{\bar{d}(\mathcal{Y}_{nk},\bar{0})}{r}))^{\mathcal{P}_{nk}} < \infty\} \quad for\ some \quad r > 0\}.$$

$$C^{\mathcal{F}}(\mathcal{X}, \mathcal{P}) = \{(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) \in \omega_{\mathcal{F}} : \lim_{n,k}\{(\lambda(\tfrac{\bar{d}(\mathcal{X}_{nk},X_1)}{r}))^{\mathcal{P}_{nk}} \vee (\rho(\tfrac{\bar{d}(\mathcal{Y}_{nk},\mathcal{X}_2)}{r}))^{\mathcal{P}_{nk}} = 0\} \quad for\ some \quad r > 0\}.$$

$$C^{\mathcal{F}}_{o}(\mathcal{X}, \mathcal{P}) = \{(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) \in \omega_{\mathcal{F}} : \lim_{n,k}\{(\lambda(\tfrac{\bar{d}(\mathcal{X}_{nk},\bar{0})}{r}))^{\mathcal{P}_{nk}} \vee (\rho(\tfrac{\bar{d}(\mathcal{Y}_{nk},\bar{0})}{r}))^{\mathcal{P}_{nk}} = 0\} \quad for\ some \quad r > 0\}.$$

and studies different properties of the classes.

In this paper, we define the sequence classes using the concept of Orlicz function as follows:

$$\ell^{\mathcal{F}}_{\infty}(\mathcal{M}, \lambda, \rho) = \{(\mathcal{X}_{nk}) \in \omega^{\mathcal{F}} : \sup_{n,k} \mathcal{M}(\tfrac{\lambda(\mathcal{X}_{nk},\bar{0})}{r}) < \infty; \sup_{n,k} \mathcal{M}(\tfrac{\rho(\mathcal{X}_{nk},\bar{0})}{r}) < \infty \quad forsome \quad r > 0\}.$$

$$C^{\mathcal{F}}(\mathcal{M}, \lambda, \rho) = \{(\mathcal{X}_{nk}) \in \omega^{\mathcal{F}} : \lim_{n,k} \mathcal{M}(\tfrac{\lambda(\mathcal{X}_{nk},\mathcal{L})}{r}) = 0; \lim_{n,k} \mathcal{M}(\tfrac{\rho(\mathcal{X}_{nk},\mathcal{L})}{r}) = 0 \quad forsome \quad r > 0\}.$$

$$C^{\mathcal{F}}_{o}(\mathcal{M}, \lambda, \rho) = \{(\mathcal{X}_{nk}) \in \omega^{\mathcal{F}} : \lim_{n,k} \mathcal{M}(\tfrac{\lambda(\mathcal{X}_{nk},\bar{0})}{r}) = 0; \lim_{n,k} \mathcal{M}(\tfrac{\rho(\mathcal{X}_{nk},\bar{0})}{r}) = 0 \quad forsome \quad r > 0\}.$$

And, we study different properties like linearity, completeness and solidity of these spaces.

4. Main Results

In this section, we shall investigate some topological properties of the classes $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$, $C^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and $C_o^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$.

Theorem 4.1. *The classes $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$, $C^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and $C_o^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ are liner space.*

Proof. Firstly, we show the class $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ is linear. For, let $\mathcal{X}=(\mathcal{X}_{nk})$ and $\mathcal{Y}=(\mathcal{Y}_{nk})$ be to elements of $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ then $\exists\ r_1>0, r_2>0$ such that

$\sup_{n,k}\mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\bar{0})}{r_1})<\infty;\ \sup_{n,k}\mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\bar{0})}{r_1})<\infty$

$\sup_{n,k}\mathcal{M}(\frac{\lambda(\mathcal{Y}_{nk},\bar{0})}{r_2})<\infty;\ \sup_{n,k}\mathcal{M}(\frac{\rho(\mathcal{Y}_{nk},\bar{0})}{r_2})<\infty$

Then for any α,β and $r=max\{2\alpha r_1, 2\beta r_2\}$,

$$\begin{aligned}\sup_{n,k}\mathcal{M}(\tfrac{\lambda(\alpha\mathcal{X}_{nk}+\beta\mathcal{Y}_{nk},\bar{0})}{r}) &\le \sup_{n,k}\mathcal{M}(\tfrac{\lambda(\alpha\mathcal{X}_{nk},\bar{0})+\lambda(\beta\mathcal{Y}_{nk},\bar{0})}{r})\\ &= \sup_{n,k}[\mathcal{M}\{\tfrac{\alpha}{r}\lambda(\mathcal{X}_{nk},\bar{0})+\tfrac{\beta}{r}\lambda(\mathcal{Y}_{nk},\bar{0})\}]\\ &\le \sup_{n,k}[\mathcal{M}\{\tfrac{1}{2}\lambda(\mathcal{X}_{nk},\bar{0})+\tfrac{1}{2}\lambda(\mathcal{Y}_{nk},\bar{0})\}]\\ &\le \tfrac{1}{2}\sup_{n,k}[\mathcal{M}(\tfrac{\lambda(\mathcal{X}_{nk},\bar{0})}{r_1})]+\tfrac{1}{2}\sup_{n,k}[\mathcal{M}(\tfrac{\lambda(\mathcal{Y}_{nk},\bar{0})}{r_2})]<\infty\end{aligned}$$

Thus, $\sup_{n,k}\mathcal{M}(\frac{\lambda(\alpha\mathcal{X}_{nk}+\beta\mathcal{Y}_{nk},\bar{0})}{r})\le\frac{1}{2}\sup_{n,k}[\mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\bar{0})}{r_1})]+\frac{1}{2}\sup_{n,k}[\mathcal{M}(\frac{\lambda(\mathcal{Y}_{nk},\bar{0})}{r_2})]<\infty$.

Similarly, we can show that,

$\sup_{n,k}\mathcal{M}(\frac{\rho(\alpha\mathcal{X}_{nk}+\beta\mathcal{Y}_{nk},\bar{0})}{r})\le\frac{1}{2}\sup_{n,k}[\mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\bar{0})}{r_1})]+\frac{1}{2}\sup_{n,k}[\mathcal{M}(\frac{\rho(\mathcal{Y}_{nk},\bar{0})}{r_2})]<\infty$

So that, $\sup_{n,k}\mathcal{M}(\frac{\lambda(\alpha\mathcal{X}_{nk}+\beta\mathcal{Y}_{nk},\bar{0})}{r})<\infty$, $\sup_{n,k}\mathcal{M}(\frac{\rho(\alpha\mathcal{X}_{nk}+\beta\mathcal{Y}_{nk},\bar{0})}{r})<\infty$

$\implies \alpha\mathcal{X}_{nk}+\beta\mathcal{Y}_{nk}\in\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and hence $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ is linear space.

Similarly, we can show that the spaces $C^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and $C_o^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ are also linear spaces. □

Theorem 4.2. *The class $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ is a metric space with the relation*

$\bar{d}(\mathcal{X},\mathcal{Y})\ = inf\{r>0:\sup_{n,k}\mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r})<1;\sup_{n,k}\mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r})<1\}$

for $\mathcal{X}=(\mathcal{X}_{nk}),\mathcal{Y}=(\mathcal{Y}_{nk})\in\mathcal{Z}^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and $r>0$ for $n,k\in\mathbb{N}$.

Proof. Let $\mathcal{X},\mathcal{Y},\mathcal{Z}\in\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ where $\mathcal{X}=(\mathcal{X}_{nk}),\mathcal{Y}=(\mathcal{Y}_{nk}),\mathcal{Z}=(\mathcal{Z}_{nk})$.Then

$\bar{d}(\mathcal{X},\mathcal{Y})=0$

$\implies inf\{r>0:\sup_{n,k}\mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r})<1;\sup_{n,k}\mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r})<1\}=0$

$\implies \mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r})$=0 and $\mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r})=0$

$\implies \lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk}))$=0 and $\rho(\mathcal{X}_{nk},\mathcal{X}_{nk})=0$

So that,

$$min\{|\mathcal{X}_{ni}^{\alpha}-\mathcal{Y}_{ni}^{\alpha}|,|\mathcal{X}_{jk}^{\alpha}-\mathcal{Y}_{jk}^{\alpha}|\}=0 \tag{4.1}$$

$$max\{|\mathcal{X}_{ni}^{\alpha}-\mathcal{Y}_{ni}^{\alpha}|,|\mathcal{X}_{jk}^{\alpha}-\mathcal{Y}_{jk}^{\alpha}|\}=0 \tag{4.2}$$

From these two relation, we see that $\mathcal{X}_{nk}=\mathcal{X}_{nk}\implies\mathcal{X}=\mathcal{Y}$.

Thus, $\bar{d}(\mathcal{X},\mathcal{Y})=0\implies\mathcal{X}=\mathcal{Y}$.

Conversely, we assume that $\mathcal{X} = \mathcal{Y}$. Then by the definition of λ_α and ρ_α, we have

$\lambda_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk}) = 0$ and $\rho_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk}) = 0$, $\forall n, k \in \mathbb{N}, \alpha \in (0,1)$

$\implies \sup_{n,k} \lambda_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk}) = 0$ and $\sup_{n,k} \rho_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk}) = 0$

$\implies \lambda(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) = 0$ and $\rho(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) = 0$.

By the definition of Orlicz function, we have

$\mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) = 0$ and $\mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) = 0$,$\forall n, k \in \mathbb{N}$ and $r > 0$

$\implies inf\{r > 0 : \sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1; \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1\} = 0$

$\therefore \bar{d}(\mathcal{X}, \mathcal{Y}) = 0$

We have, $\bar{d}(\mathcal{X}, \mathcal{Y})= inf\{r > 0 : \sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1; \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1\}$.

Here, $\lambda(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) = \sup_{\lambda\in(0,1)} \lambda_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk})$

$$= \sup_{\alpha\in(0,1)} min[\{|\mathcal{X}^\alpha_{ni} - \mathcal{Y}^\alpha_{ni}|, |\mathcal{X}^\alpha_{jk} - \mathcal{Y}^\alpha_{jk}|\}]$$
$$= \sup_{\alpha\in(0,1)} min[\{|\mathcal{Y}^\alpha_{ni} - \mathcal{X}^\alpha_{ni}|, |\mathcal{Y}^\alpha_{jk} - \mathcal{X}^\alpha_{jk}|\}]$$
$$= \sup_{\alpha\in(0,1)} \lambda_\alpha(\mathcal{Y}^\alpha_{nk}, \mathcal{X}^\alpha_{nk})$$
$$=\lambda(\mathcal{Y}_{nk}, \mathcal{X}_{nk})$$

Similarly, we can show that $\rho(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) = \rho(\mathcal{Y}_{nk}, \mathcal{X}_{nk})$.

Now, $\bar{d}(\mathcal{X}, \mathcal{Y}) = inf\{r > 0 : \sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1; \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1\}$.

$$= inf\{r > 0 : \sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{Y}_{nk},\mathcal{X}_{nk})}{r}) < 1; \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{Y}_{nk},\mathcal{X}_{nk})}{r}) < 1\}.$$
$$= \bar{d}(\mathcal{Y}, \mathcal{X})$$

Let r_1 and r_2 such that, $\sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{Z}_{nk},\mathcal{Y}_{nk})}{r_1}) \le 1$ and $\sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{Z}_{nk},\mathcal{Y}_{nk})}{r_2}) \le 1$.

Let $r = r_1 + r_2$ then $r > 0$ and by the definition of λ we have,

$\lambda(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) = \sup_{\lambda\in(0,1)} \lambda_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk}) = \sup_{\alpha\in(0,1)} min[\{|\mathcal{X}^\alpha_{ni} - \mathcal{Y}^\alpha_{ni}|, |\mathcal{X}^\alpha_{jk} - \mathcal{Y}^\alpha_{jk}|\}]$.

Using the definition of λ_α we have,

$\lambda_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk}) \le \lambda_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Z}^\alpha_{nk}) + \lambda_\alpha(\mathcal{Z}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk})$

$\implies \sup_{\alpha\in(0,1)} \lambda_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk}) \le \sup_{\alpha\in(0,1)} \lambda_\alpha(\mathcal{X}^\alpha_{nk}, \mathcal{Z}^\alpha_{nk}) + \sup_{\alpha\in(0,1)} \lambda_\alpha(\mathcal{Z}^\alpha_{nk}, \mathcal{Y}^\alpha_{nk})$.

$$\therefore \quad \lambda(\mathcal{X}_{nk}, \mathcal{Y}_{nk}) \le \lambda(\mathcal{X}_{nk}, \mathcal{Z}_{nk}) + \lambda(\mathcal{Z}_{nk}, \mathcal{Y}_{nk}) \tag{4.3}$$

Using the continuity of $\mathcal{M}$ we get,

$$\sup_{n,k} \mathcal{M}\{\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}\} \le \sup_{n,k} \mathcal{M}\{\frac{\lambda(\mathcal{X}_{nk},\mathcal{Z}_{nk})}{r_1+r_2} + \frac{\lambda(\mathcal{Z}_{nk},\mathcal{Y}_{nk})}{r_1+r_2}\}.$$
$$\le \sup_{n,k} \mathcal{M}\{\frac{r_1}{r_1+r_2}\frac{\lambda(\mathcal{X}_{nk},\mathcal{Z}_{nk})}{r_1}\} + \sup_{n,k} \mathcal{M}\{\frac{r_1}{r_1+r_2}\frac{\lambda(\mathcal{Z}_{nk},\mathcal{Y}_{nk})}{r_1+r_2}\}$$
$$\le \sup_{n,k} \frac{r_1}{r_1+r_2}\mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\mathcal{Z}_{nk})}{r_1}) + \sup_{n,k} \frac{r_1}{r_1+r_2}\mathcal{M}(\frac{\lambda(\mathcal{Z}_{nk},\mathcal{Y}_{nk})}{r_1+r_2})$$
$$\le \frac{r_1}{r_1+r_2}.1 + \frac{r_1}{r_1+r_2}.1$$

$\therefore \sup_{n,k} \mathcal{M}\{\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}\} \le 1$.

This relation is true for all $r > 0$, so that

$\inf\{r > 0 : \sup_{n,k} \mathcal{M}\{\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}\} \le 1\}$.

Then from (4.3), we have

$$\inf\{r > 0 : \sup_{n,k} \mathcal{M}\{\frac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}\} \le 1\} \le \inf\{r_1 > 0 : \sup_{n,k} \mathcal{M}\{\frac{\lambda(\mathcal{X}_{nk},\mathcal{Z}_{nk})}{r_1}\} \le 1\}$$
$$+ \inf\{r_2 > 0 : \sup_{n,k} \mathcal{M}\{\frac{\lambda(\mathcal{Z}_{nk},\mathcal{Y}_{nk})}{r_2}\} \le 1\}$$

Similarly, we can show that,

$$\inf\{r > 0 : \sup_{n,k} \mathcal{M}\{\tfrac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}\} \le 1\} \le \inf\{r_1 > 0 : \sup_{n,k} \mathcal{M}\{\tfrac{\rho(\mathcal{X}_{nk},\mathcal{Z}_{nk})}{r_1}\} \le 1\} + \inf\{r_2 > 0 : \sup_{n,k} \mathcal{M}\{\tfrac{\rho(\mathcal{Z}_{nk},\mathcal{Y}_{nk})}{r_2}\} \le 1\}$$

Thus we have,

$$\bar{d}(\mathcal{X},\mathcal{Y}) = inf\{r > 0 : \sup_{n,k} \mathcal{M}(\tfrac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1; \sup_{n,k} \mathcal{M}(\tfrac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1\}$$

$$\le inf\{r_1 > 0 : \sup_{n,k} \mathcal{M}(\tfrac{\lambda(\mathcal{X}_{nk},\mathcal{Z}_{nk})}{r_1}) < 1\} + inf\{r_2 > 0 : \sup_{n,k} \mathcal{M}(\tfrac{\lambda(\mathcal{Z}_{nk},\mathcal{Y}_{nk})}{r_2}) < 1\}$$
$$+ inf\{r_1 > 0 : \sup_{n,k} \mathcal{M}(\tfrac{\rho(\mathcal{X}_{nk},\mathcal{Z}_{nk})}{r_1}) < 1\} + inf\{r_2 > 0 : \sup_{n,k} \mathcal{M}(\tfrac{\rho(\mathcal{Z}_{nk},\mathcal{Y}_{nk})}{r_2}) < 1\}$$

$$= \bar{d}(\mathcal{X},\mathcal{Z}) + \bar{d}(\mathcal{Z},\mathcal{Y})$$

$\therefore \bar{d}(\mathcal{X},\mathcal{Y}) \le \bar{d}(\mathcal{X},\mathcal{Z}) + \bar{d}(\mathcal{Z},\mathcal{Y})$.

Hence, $\bar{d}(\mathcal{X},\mathcal{Y}) = inf\{r > 0 : \sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\bar{0})}{r}) < 1; \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\bar{0})}{r}) < 1\}$
for $\mathcal{X} = (\mathcal{X}_{nk})$, $\mathcal{Y} = (\mathcal{Y}_{nk}) \in \ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and $r > 0$ for $n,k \in \mathbb{N}$ is a metric and hence $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$is a metric space with the metric $\bar{d}(\mathcal{X},\mathcal{Y})$.

Similarly we can show that, the spaces $C^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and $C_o^{\mathcal{F}}((\mathcal{M},\lambda,\rho)$ are metric spaces.

□

Theorem 4.3. *The class of sequence spaces $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$, $C\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ $C_o^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ are complete metric spaces with the metric*

$$\bar{d}(\mathcal{X},\mathcal{Y}) = inf\{r > 0 : \sup_{n,k} \mathcal{M}(\tfrac{\lambda(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1; \sup_{n,k} \mathcal{M}(\tfrac{\rho(\mathcal{X}_{nk},\mathcal{Y}_{nk})}{r}) < 1\}$$

for $\mathcal{X} = (\mathcal{X}_{nk}), \mathcal{Y} = (\mathcal{Y}_{nk}) \in \mathcal{Z}^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and $r > 0$ for $n,k \in \mathbb{N}$.

Proof. Firstly, we show that the space $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ is complete. Then by similar procedure we can show for rest spaces are complete.

For, let $\{\mathcal{X}^i\} = \{\mathcal{X}^i_{nk}\}$ be Cauchy sequence in $\ell_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and let $\varepsilon > 0$ be given, then there exists $i_o \in \mathbb{N}$ such that, $\bar{d}(\mathcal{X}^i_{nk},\mathcal{X}^j_{nk}) < \varepsilon, \forall i,j \ge i_o, n,k \in \mathbb{N}$.

$$\Longrightarrow inf\{r > 0 : \sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk})}{r}) \le 1; \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk})}{r}) \le 1\} < \varepsilon \tag{4.4}$$

$$\Longrightarrow \sup_{n,k} \mathcal{M}(\tfrac{\lambda(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk})}{r}) \le 1 \text{ and } \mathcal{M}(\tfrac{\rho(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk})}{r}) \le 1$$

Here,

$$sup_{n,k}\mathcal{M}(\frac{\lambda(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk})}{r}) \le 1) \tag{4.5}$$

For $\varepsilon > 0$, let us choose $\mu,\eta > 0$ such that $\mathcal{M}(\frac{\mu\eta}{2}) \ge 1$.
Then from(4.4), $\sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk})}{r}) \le 1 \le \mathcal{M}(\frac{\mu\eta}{2})$.
Since, $\mathcal{M}$ is non-decreasing function, we have

$$\tfrac{\lambda(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk})}{r} \le \tfrac{\mu\eta}{2} \implies \lambda(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk}) \le \tfrac{\mu\eta}{2}.r$$

For,$r > 0$, let us choose $r = \frac{\varepsilon}{\mu\eta}$ then we have $\lambda(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk}) \le \frac{\varepsilon}{2} < \varepsilon$.
Thus $\lambda(\mathcal{X}^i_{nk},\mathcal{Y}^j_{nk}) < \varepsilon, \forall \quad i,j \ge i_o$.

This shows that $\{\mathcal{X}^i_{nk}\}$ is a Cauchy sequence in $\mathbb{R}(I)$. Since, $\mathbb{R}(I)$ is complete, so we suppose that, $\mathcal{X}_{nk} \in \mathbb{R}(I)$ such that $\lim_{i\to\infty} \mathcal{X}^i_{nk} = \mathcal{X}_{nk}$ for $n, k \in \mathbb{N}$. To complete the proof, we show that $\mathcal{X}_{nk} \in \ell^{\mathcal{F}}_\infty(\mathcal{M}, \lambda, \rho)$.

We have, $\sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}^i_{nk},\mathcal{X}^j_{nk})}{r}) \leq 1$ for, $r > 0$.

Let us fix i and taking $j \longrightarrow \infty$ then we get,

$$\sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}^i_{nk},\mathcal{X}_{nk})}{r}) \leq 1 \text{ ,for } r > 0 \text{ and } i \geq i_o.$$

By the similar process, we can show that

$$\sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}^i_{nk},\mathcal{X}_{nk})}{r}) \leq 1 \text{ for, } r > 0 \text{ and } i \geq i_o.$$

Thus, $\sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}^i_{nk},\ \mathcal{X}_{nk})}{r}) \leq 1$; $\sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}^i_{nk},\mathcal{X}_{nk})}{r}) \leq 1$, for all $r > 0$ and $i \geq i_o$.

Taking infimum over such $r > 0$, we get using (3) we have

$$inf\{r > 0 : \sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}^i_{nk},\mathcal{X}^j_{nk})}{r}) \leq 1 \ ; \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}^i_{nk},\mathcal{X}^j_{nk})}{r}) \leq 1\} < \varepsilon.$$

$$\implies \bar{d}(\mathcal{X}^i_{nk}, \mathcal{X}^j_{nk}) < \varepsilon, \forall i, j \geq i_o, n, k \in \mathbb{N}.$$

By triangle inequality of the metric $\bar{d}$

$$\bar{d}(\mathcal{X}_{nk}, \bar{0}) \leq \bar{d}(\mathcal{X}_{nk}, \mathcal{X}^i_{nk}) + \bar{d}(\mathcal{X}^i_{nk}, \bar{0}) < \varepsilon + \bar{d}(\mathcal{X}^i_{nk}, \bar{0}) < \infty$$

$\implies \inf\{r > 0 : \sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\bar{0})}{r} \leq 1); \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}^i_{nk},\bar{0})}{r} \leq 1)\} < \infty$

$\implies \mathcal{X}_{nk} \in \ell^{\mathcal{F}}_\infty(\mathcal{M}, \lambda, \rho)$ and hence $\ell^{\mathcal{F}}_\infty(\mathcal{M}, \lambda, \rho)$ is complete. This completes the proof

□

Theorem 4.4. *The spaces* $\ell^{\mathcal{F}}_\infty(\mathcal{M}, \lambda, \rho)$, $C^{\mathcal{F}}(\mathcal{M}, \lambda, \rho)$ *and* $C^{\mathcal{F}}_o(\mathcal{M}, \lambda, \rho)$ *are solid.*

Proof. Suppose $\mathcal{X} = (\mathcal{X}_{nk}) \in \ell^{\mathcal{F}}_\infty(\mathcal{M}, \lambda, \rho)$ and let $\mathcal{Y} = (\mathcal{Y}_{nk})$ be double sequence of fuzzy numbers such that

$$\lambda(\mathcal{Y}_{nk}, \bar{0}) \leq \lambda(\mathcal{X}_{nk}, \bar{0}) \text{ and } \rho(\mathcal{Y}_{nk}, \bar{0}) \leq \rho(\mathcal{X}_{nk}, \bar{0}) \text{ for } n, k \in \mathbb{N}.$$

Let, $r > 0$ then we have $\frac{\lambda(\mathcal{Y}_{nk},\bar{0})}{r} \leq \frac{\lambda(\mathcal{X}_{nk},\bar{0})}{r}$ and $\frac{\rho(\mathcal{Y}_{nk},\bar{0})}{r} \leq \frac{\rho(\mathcal{X}_{nk},\bar{0})}{r}$.

$\mathcal{M}$ is non-decreasing function, we have

$$\mathcal{M}(\frac{\lambda(\mathcal{Y}_{nk},\bar{0})}{r}) \leq \mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\bar{0})}{r}) \text{ and } \mathcal{M}(\frac{\rho(\mathcal{Y}_{nk},\bar{0})}{r}) \leq \mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\bar{0})}{r}) \text{ for all } n, k \in \mathbb{N}.$$

This relation is true for all $n, k \in \mathbb{N}$, so

$$\sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{Y}_{nk},\bar{0})}{r}) \leq \sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\bar{0})}{r}) \text{ and } \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{Y}_{nk},\bar{0})}{r}) \leq \sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\bar{0})}{r}).$$

Since $\mathcal{X} = (\mathcal{X}_{nk}) \in \ell^{\mathcal{F}}_\infty(\mathcal{M}, \lambda, \rho)$ we have, $\sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{X}_{nk},\bar{0})}{r}) < \infty$ $\sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{X}_{nk},\bar{0})}{r}) < \infty$.

Thus, $\sup_{n,k} \mathcal{M}(\frac{\lambda(\mathcal{Y}_{nk},\bar{0})}{r}) < \infty$ and $\sup_{n,k} \mathcal{M}(\frac{\rho(\mathcal{Y}_{nk},\bar{0})}{r}) < \infty$ for $r > 0$

$\implies \mathcal{Y} = (\mathcal{Y}_{nk}) \in \ell^{\mathcal{F}}_\infty(\mathcal{M}, \lambda, \rho)$. Thus $\ell^{\mathcal{F}}_\infty(\mathcal{M}, \lambda, \rho)$ is solid.

Similarly, other spaces can be shown solid.

□

5. Conclusion

The concepts of fuzzy logic and fuzzy sets have been applied in the study of various sequence spaces. In this paper, we have used the Orlicz function to define new double sequence spaces $\mathcal{Z}^F(\mathcal{M}, \lambda, \rho)$ of fuzzy real numbers and study various properties like linearity, completeness, and solidity. It can be applied in the study of various sequence spaces.6mm

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References

[1] B. Yuan, and J. KGeorge, *Fuzzy sets and fuzzy logic: theory and applications*, , 443-455,PHI New Delhi,(1995)

[2] L. A. Zadeh, *Is there a need for fuzzy logic?*, Information Sciences, 178, 2751–2779, 2008.

[3] M. Matloka, *Sequence of fuzzy numbers*, Busefal, 28, 28-37, 1986.

[4] G.H Hardy, *On the Convergence of Certain Multiple Series Proc*, London Math. Soc, 1, 124-128, 1904

[5] Altay, B.,Başar, F., *Some new spaces of double sequences*, Journal of Mathematical Analysis and Applications, 309(1), 70-90, 2005.

[6] Das, M. , *Difference Operator on Double Sequences With an Orlicz Function*, International Journal of Science, Environment and Technology,1(1) 1-6, 2012.

[7] Savaş, E., *New double sequence spaces of fuzzy numbers*, Quaestiones Mathematicae, 33(4), 449-456, 2010.

[8] Savas, E. and Patterson, R. F., *Double sequence spaces defined by Orlicz functions*,Iranian Journal of Science and Technology (Sciences), 31(2), 183-188, 2007.

[9] Tripathy, B.C. and Sharma B. , *Sequences of fuzzy numbers defined by orlicz function*, MathematicSlovaca, 58, 621-628, 2008.

[10] Tripathy, B. C. and Sarma, B., *Some double sequence spaces of fuzzy numbers defined by Orlicz functions*, Acta Mathematica Scientia, 31(1), 134-140, 2011.

[11] Tripathy, B., and Sarma, B., *Vector-valued double sequence spaces defined by Orlicz function*, Mathematica Slovaca, 59(6), 767-776, 2009.

[12] Savas, E., *On some double lacunary sequence spaces of fuzzy numbers*, Mathematical and Computational Applications, 15(3), 439-448, 2010.

[13] Kılınç, G., and Solak, İ., *Some double sequence spaces defined by a modulus function*, Gen. Math. Notes, 52(2), 19-30, 2014

[14] Pahari, N. P., *On normed space valued total paranormed Orlicz space of null sequences and its topological structures*, Int. Journal of Mathematics Trends and Technology, 6 , 105-112, 2014.

[15] Habil, E. D., *Double sequences and double series*, IUG Journal of Natural Studies, 14(1), 2016.

[16] Basu, A., *On new fuzzy metric orlicz sequence space*, International Journal of Pure and Applied Mathematics, 118(4), 931-947, 2018.

[17] Sarma, B., *Double sequence spaces of fuzzy real numbers of paranormed type under an orlicz function*, Mathematica Sciences a Springer Journal, 8(7),46- 50, 2016

[18] Talo, Ö., *The core of a double sequence of fuzzy numbers*, Fuzzy Sets and Systems, 424, 132-154, 2021.

[19] Dabbas, A. F.,and Battor, A. H, *Double sequence space of fuzzy real numbers of paranormed type defined by double orilcz functions*, International Journal of Academic and Applied Research, 4(6),22-31, 2020.

[20] Mansoor, M. M., and Battor, A. H., 2021 *Some New Double Sequence Spaces of Fuzzy Numbers Defined by Double Orlicz Functions Using A Fuzzy Metric*, In Journal of Physics: Conference Series, 1818(1), 012092, 2021.

[21] Paudel, G.P. and Pahari, N.P., *On Fundamental Properties in Fuzzy Metric Space*, Academic Journal of Mathematics Education, 4(1), 20-25,2021.

[22] Paudel G. P., Pahari N. P., Kumar S., *Generalized form of p-bounded variation of sequences of fuzzy real numbers*, Pure and Applied Mathematics Journal, 11 (3), 47-50,2022

[23] Lindenstrauss, J.and,Tzafriri, L., *On Orlicz sequence spaces II*, Israel Journal of Mathematics, 11(4), 355-379,1972.

[24] Esi, A., Acikgoz, M. *Some new classes of sequences of fuzzy numbers*, International Journal of Fuzzy Systems, 13(3), 218-224,2006.

[25] Parashar, S. D., and Choudhary, B., *Sequence spaces defined by Orlicz functions*, Indian Journal of Pure and Applied Mathematics, 25(4), 419-419, 1994