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A REVIEW OF THE TSIRELSON'S SPACE NORM

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Abstract: In most cases, the space of all sequences converging to zero or the space of bounded sequences is always embedded in complete normed linear spaces. This concept, however, was modified by B.S. Tsirelson by constructing reflexive complete normed linear spaces with monotone unconditional Schauder basis without embedded copies of sequences converging to zero or the space of bounded sequence. In this article, a relation with any four non-negative integers has been proved, and this concept is used to prove the triangle inequality of a slightly different Tsirelson's type of norm in the space of all real sequences with finite support. Furthermore, all properties of the norm have been studied for a different type of norm function in the space of real sequences with finite support.

Key Words: Normed Linear Space, Banach Space, Tsirelson Space
AMS (MOS) Subject Classification: 46Bxx, 46Exx.

1. Introduction

The concept of Tsirelson's space was developed in 1970 by the contribution of B.S. Tsirlenson in the paper [1], without isomorphic copies of c_0 or l_p, $(1 \leq p < \infty)$. Among the different properties of the unit ball, certain saturation properties have been used to derive the special properties of Tsirlenson's space. As the dual of Tsirelson's space, Figiel and Johnson in the paper [2] gave the different implicit expression for the norm of dual of Tsirelson space and the detailed modification can be found in the paper ([3], [4]). Tsirelson spaces neither contain copies of the space of all sequences converging to zero nor the space of bounded sequences having a 1-conditional basis.

A related space of holomorphic functions as a generalized Tsirelsion's space is discussed in the paper ([4], [5]), however, its basic properties like Tsirelson space provided a single Banach space with a complex array of properties against which one can nullify many conjuctors and spaces which are non-separable and without sub-symmetric basic sequence can be found in [3]. Alencar et al. ([4], [5]) studied the tensor product of Tsirelson space and proved that " if the completion of n-fold invective tensor product of the dual space of Tsirelson space lacks embedded copies of the space of bounded sequence, then it has no unconditional basis and it fails local unconditional structure".

Petter and Thaddeus in [4] discussed the behavior of Block basic sequences in Tsireln's

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space by proving that every bounded block basic sequence $\{s_n\}_{n=1}^{\infty}$ spans a complemented sub-sequence of Tsirel'son space, which is equivalent to sub-sequence of $\{s_n\}_{n=1}^{\infty}$. They also studied Tsirelson's space in terms of bounded linear operators and the blocking principle by giving characterizations of the isometry of the space along with the description of the action of the shift operator. It is well-known fact that Tsirelson space does not contain sub-symmetric sequences. That is every sub-sequence $\{s_{n_j}\}_{n=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ has a permutation, however it is not equivalent to $\{s_{n_j}\}_{n=1}^{\infty}$.

Based on that concept, Petter and Thaddeus in [4] established a special condition on which $\{s_{n_j}\}_{n=1}^{\infty}$ is equivalent to a particular permutation of itself. P. Gilles gives [6] an example of a weak Hilbert space H which is not isomorphic to H and proved that " there is a Banach space with the 1-unconditional basis which is a weak Hilbert space which is not isomorphic to l_2". D. Ojeda in the paper [7] contracted an expression for the norm of the space given by Tsirelson by proving that the norm of the dual of the Tsirelson space constructed by Figiel and Johnson is dual to the implicit equation of this norm. Ojeda [7] also defines a norm on space of finitely supported sequences of real numbers with the supremum norm and Tsirelson space is complete concerning this norm. O. Kurka [8] proved that the classes of all complete normed linear spaces which are isomorphic to a subspace of the space of all sequences converging to zero are a complete analytic set concerning the Effros Borel structure of separable complete normed linear spaces, by making use of Tsirelson's like spaces as a consequence of Argyros and Deliyanni.

2. Definitions and Preliminary results

Definition 2.1. [4] A set $M = \{m_1, m_2, m_3, ..., m_k\}$ of natural numbers, where $m_1 < m_2 < m_3 <, ..., < m_k$ is said to be admissible if it is finite and $k \leq m_1$.

Definition 2.2. [4] A basic sequence $\{x_m\}_{m=1}^{\infty}$ is said to be monotonically unconditional or 1-unconditional if

$$\| \sum_i b_i x_i \| \leq \| \sum_i c_i x_i \| \quad \text{for every scalar sequence} \quad | b_i | \leq | c_i |, (1 \leq i < \infty).$$

Definition 2.3. [4] Two bases $\{y_m\}_{m=1}^{\infty}$ and $\{z_m\}_{m=1}^{\infty}$ are said to be permutatively equivalent if there is a perrmutaton of ρ of the set of natural numbers so that $\{y_m\}_{m=1}^{\infty}$ is equivalent to $\{z_{\rho(m)}\}_{m=1}^{\infty}$.

Definition 2.4. [4] Let X be an infinite-dimensional complete normed linear space. Then X is said to be minimal if it embeds into each of its infinite-dimensional subspaces.

Definition 2.5. [4] Let X_1 and X_2 be infinite-dimensional complete normed linear spaces. Then X_1 and X_2 are said to be totally incomparable if neither of X_1 and X_2 contains an infinite-dimensional subspace that is isometric to a subspace of the other.

Theorem 2.6. [4] *Tsirelson's space contains no sub-symmetric basic sequences.*

Corollary 2.7. [4] *For any $1 < p < \infty$, Tsirelson's space X does not contains isomorphic copies of c_0 or l_p and does not contains infinite dimensional uniformly convexible sub-spaces.*

Theorem 2.8. [4] *Tsirelson space X is reflexive with 1-unconditional basis and not containing isomorphic copies of* c_0 *or* $l_p(1 \leq p < \infty)$, *not containing subsymmetric basis sequences, and not containing uniformly convexible subspace of infinite dimension.*

Theorem 2.9. [8] *The classes of all complete normed linear spaces which can be embeded isomorphically into the space of sequences convergingto zero is complete analytic. Particularly it is not Borel.*

Theorem 2.10. [8] *A complete normed linear space with a basis* $\{y_j\}_{j=1}^{\infty}$ *is reflexive if and only if* $\{y_j\}_{j=1}^{\infty}$ *is shrinking and boundedly complete.*

The set of all vectors of the form $y = \sum_n b_n s_n$ for which the finite norm defined in the following ways:

$$\|y\|_{X^2} = \max\{\sup \mid b_n \mid, \frac{1}{\sqrt{2}} \sup(\sum_{i=1}^{k} \|E_j y\|^2)^{\frac{1}{2}}\}, \tag{2.1}$$

where the second supremem is taken over all choices of $k \leq E_1 < E_2 < \ldots\ldots < E_k$ and $E_y = \sum_{n \in E} b_n t_n$ is known as convexied Tsirelson space on X^2 [9].

Theorem 2.11. [9] *The set of all unit vectors* $\{e_n\}$ *from a 1-unconditional basis for*X^2 *and the space* X^2 *is of type II and weak co-type-II, however, it does not contain a Hilbert space.*

3. Main Result

We prove the following lemma and use this lemma to prove the result on (3.3).

Lemma 3.1. *For any non-negative real numbers p, q,r, s*

$$\max\{p+q, r+s\} \leq \max\{p, r\} + \max\{q, s\}$$

Proof. If$p \leq r$ and $q \leq s$. Then we have

$$\begin{aligned}\max\{p, r\} + \max\{q, s\} &= r + s \\ &= \max\{p+q, r+s\}\end{aligned}$$

If $p \leq r$ and $q \geq s$. Then there may arise the following two cases:
Case-I: $p + q \leq r + s$. In that case

$$\begin{aligned}\max\{p+q, r+s\} =& r+s \\ \leq& r+q \\ =& \max\{p, r\} + \max\{q, s\}\end{aligned}$$

Case-II: $p + q \geq r + s$. In that case

$$\begin{aligned}\max\{p+q, r+s\} =& p+q \\ \leq& r+q \\ =& \max\{p, r\} + \max\{q, s\}\end{aligned}$$

If $p \geq r$ and $q \leq s$. Then there may arise the following two cases:
Case-I: $P + q \leq r + s$. In that case

$$\begin{aligned}\max\{p+q, r+s\} &= r+s \\ &\leq p+s \\ &= \max\{p,r\} + \max\{q,s\}\end{aligned}$$

Case-II: If $p + q \geq r + s$, then

$$\begin{aligned}\max\{p+q, r+s\} &= p+q \\ &\leq p+s \\ &= \max\{p,r\} + \max\{q,s\}\end{aligned}$$

Hence in either cases

$$\max\{p+q, r+s\} \leq \max\{p,r\} + \max\{q,s\}$$

□

Corollary 3.2. *For any non-negative real numbers p, q,r, s*

$$\min\{p+q, r+s\} \geq \min\{p,r\} + \min\{q,s\}$$

Construction of Tsirelson norm: Let $\mathbb{R}^{\mathbb{N}}$ be the space of all real sequence with finite support and $\{e_n\}$ be the standard unit vectors basis for $\mathbb{R}^{\mathbb{N}}$. Let $x \in \mathbb{R}^{\mathbb{N}}$ be an arbitrary sequence of real numbers with finite support. Then there exists $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that $x = \sum_{n=1}^{\infty} a_n e_n$. For any admissible subset E of natural numbers, we define

$$\|x_0\| = \max_n \mid a_n \mid$$

$$\|x\|_{m+1} = \max\{\|x_m\|, \frac{1}{2}[\sum_{j=1}^{k} \|E_j x\|_m]\}, \quad m \geq 0 \tag{3.1}$$

Lemma 3.3. *For all $m \geq 0$, $\|x\|_m$ defines a norm on $\mathbb{R}^{\mathbb{N}}$.*

Proof. We use the mathematical induction to prove the above lemma.
$\|x\|_0 = \max_n \mid a_n \mid = \|x\|_\infty$, which is clearly norm on l_∞. Suppose $\|x\|_m$ defines a norm on $\mathbb{R}^{\mathbb{N}}$.
For $\|x\|_{m+1}$, $\|x\|_{m+1} \geq 0$ is obvious.

$$\begin{aligned}\|x\|_{m+1} = 0 &\Leftrightarrow \max\{\|x\|_m, \frac{1}{2}[\sum_{j=1}^{k} \|E_j x\|_m]\} = 0 \\ &\Leftrightarrow \|x\|_m = 0 \& \sum_{j=1}^{k} \|E_j x\| = 0 \\ &\Leftrightarrow x = 0\end{aligned}$$

Let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\|\alpha x\| &= \max\{\|\alpha x\|_m, \frac{1}{2}\max[\sum_{k=1}^{k}\|E_j \alpha x\|]\}\\ &= \max\{\mid \alpha \mid \|x\|_m, \frac{1}{2}\max[\sum_{k=1}^{k}\mid \alpha \mid \|E_j x\|]\}\\ &= \mid \alpha \mid \max\{\|x\|_m, \frac{1}{2}\max[\sum_{k=1}^{k}\|E_j x\|]\}\\ &= \mid \alpha \mid \|x\|_{m+1}\end{aligned}$$

For triangle inequality, let $x, y \in \mathbb{R}^{\mathbb{N}}$. Then there exists $a_n, b_n \in \mathbb{R}$ such that $x = \sum_{n=1}^{\infty} a_n e_n$ and $x = \sum_{n=1}^{\infty} b_n e_n$. Now

$$\|x\|_{m+1} = \max\{\|x+y\|_m, \frac{1}{2}\max[\sum_{j=1}^{k}\|E_j(x+y)\|_m]\}$$

But

$$\begin{aligned}\sum_{j=1}^{k}\|E_j(x+y)\|_m &= \sum_{j=1}^{k}\|\sum_{n\in E_j}(a_n+b_n)e_n\|_m\\ &= \sum_{j=1}^{k}\|\sum_{n\in E_j}a_n e_n + \sum_{n\in E_j}b_n e_n\|_m\\ &= \sum_{j=1}^{k}\|E_j x + E_j y\|_m\\ &\leq \sum_{j=1}^{k}\|E_j x\|_m + \sum_{j=1}^{k}\|E_j y\|_m\end{aligned}$$

Also we have

$$\|x+y\|_m \leq \|x\|_m + \|y\|_m$$

Using the lemma (3.1), we have

$$\begin{aligned}\|x+y\|_{m+1} &\leq \max\{\|x\|_m + \|y\|_m, \frac{1}{2}\max[\sum_{j=1}^{k}\|E_k x\|_m] + \frac{1}{2}\max[\sum_{J=1}^{k}\|E_j y\|_m]\}\\ &\leq \max\{\|x\|_m, \frac{1}{2}\max[\sum_{j=1}^{k}\|E_j x\|_m]\} + \max\{\|y\|_m, \frac{1}{2}\max[\sum_{j=1}^{k}\|E_j y\|_m]\}\\ &= \|x\|_{m+1} + \|y\|_{m+1}\end{aligned}$$

Therefore $\|x+y\|_{m+1} \leq \|x\|_{m+1} + \|y\|_{m+1}$, showing that $\|x\|_{m=1}$ Is a norm on $\mathbb{R}^{\mathbb{N}}$. Hence by mathematical induction, $\|x\|_m$ is a norm on $\mathbb{R}^{\mathbb{N}}$ for all $m \geq 0$. □

Lemma 3.4. *Let* $\|.\|$ *denotes the Tsirelson's norm on* $\mathbb{R}^{\mathbb{N}}$. *Then the function defined by* $\|x\| = \lim_{n\to\infty}\|x\|_n$ *for all* $n \in \mathbb{R}^{\mathbb{N}}$ *is indeed a norm.*

Proof. Let $x = \sum_{n=1}^{\infty} a_n e_n \in \mathbb{R}^{\mathbb{N}}$.

(1) Since $\|x\|_m \geq 0 \Rightarrow \lim\limits_{n\to\infty} \|x\|_m \geq 0 \Rightarrow \|x\| \geq 0$.
(2) Assume $\|x\| = 0$. Using the lemma (3.4), we have $\|x\|_m \leq \|x\|_{m+1}$ for all $m \geq 0$ implying that $\|x\| \geq \|x\|_m \geq \|x\|_0 = \max_n \mid a_n \mid$. This implies that

$$\begin{aligned} \forall n \in \mathbb{N} \quad \max_n \mid a_n \mid \leq 0 &\Rightarrow a_n = 0 \\ &\Rightarrow x = 0 \end{aligned}$$

Conversely assume that $x = 0$ for all $m \geq 0$. Then $\|x\|_m = 0 \Rightarrow \|x\| = 0$.
(3) Let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \|ax\| &= \lim_{n\to\infty} \|ax\|_n \\ &= \mid a \mid \lim_{n\to\infty} \|x\|_m \\ &= \mid a \mid \|x\| \end{aligned}$$

(4) Let $x = \sum_{n=1}^{\infty} a_n e_n \in \mathbb{R}^{\mathbb{N}}$ and $y = \sum_{n=1}^{\infty} b_n e_n \in \mathbb{R}^{\mathbb{N}}$. Then

$$\begin{aligned} \|x+y\| &= \lim_{n\to\infty} \|x+y\|_m \\ &= \lim_{n\to\infty} \|x+y\| \\ &\leq \lim_{n\to\infty} \|x\|_m + \lim_{n\to\infty} \|y\|_m \\ &= \|x\| + \|y\| \end{aligned}$$

Hence $\|.\|$ is a norm on $\mathbb{R}^{\mathbb{N}}$.

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