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# A COMPARISON OF CLASSICAL HOLOMORPHIC DYNAMICS AND HOLOMORPHIC SEMIGROUP DYNAMICS-I

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**Abstract:** *In this paper, we compare existing results of the classical holomorphic dynamics with some of the existing results of holomorphic semigroup dynamics. We also prove some results of holomorphic semigroup dynamics, and we see whether there is a connection or contrast with classical one. Also, we see how far the results generalize, and what new phenomena appear. .*

**Key Words**: Holomorphic semigroup, transcendental semigroup, Fatou set, Julia set, escaping set.
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## 1. Introduction

It is quite natural to extend Fatou-Julia-Eremenko theory of the iteration of a single holomorphic map in complex plane $\mathbb{C}$ or extended complex plane $\mathbb{C}_\infty$ (or certain subsets thereof) to the composite of the family of holomorphic maps. So, the purpose of this paper is to expose the theory of holomorphic dynamics not only for the iteration of single holomorphic map on $\mathbb{C}$ or $\mathbb{C}_\infty$ (or certain subsets thereof) but also for the composite of the family $\mathscr{F}$ of such maps. Let $\mathscr{F}$ be a set of holomorphic maps on $\mathbb{C}$ or $\mathbb{C}_\infty$ (or certain subsets thereof). For any map $\phi \in \mathscr{F}$, $\mathbb{C}$ or $\mathbb{C}_\infty$ is naturally partitioned into two subsets: the set of normality and its complement. We say that a family $\mathscr{F}$ is *normal* if each sequence from the family has a subsequence which either converges uniformly on compact subsets or diverges uniformly to $\infty$. The set of *normality* or *Fatou set* $F(\phi)$ of the map $\phi \in \mathscr{F}$ is the largest open set on which the iterates $\phi^n = \phi \circ \phi \circ \ldots \circ \phi$ (n-fold composition of $\phi$ with itself) is a normal family. The complement $J(\phi)$ is the *Julia set*. A maximally connected subset of the Fatou set $F(\phi)$ is a *Fatou component*. For $f \in \mathscr{F}$, define *escaping set* by $I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$ and any point $z \in I(f)$ is called *escaping point*. If $f$ is a polynomial of degree $\geq 2$, then $I(f)$ is a Fatou component containing $\infty$ and Julia set is equal to its boundary. For transcendental entire function $f$, $I(f)$ is non-empty proper subset of $\mathbb{C}$, its boundary is the Julia set and it always intersects the Julia set. The main concern of such an iteration theory is to describe the nature of the components of Fatou set and the structure and properties of Julia and escaping sets. For more information of these sets in classical holomorphic dynamics, we refer [2, 3, 6, 9].

Semigroup $S$ is a very classical algebraic structure with a binary composition that satisfies associative law. It naturally arose from the general mapping of a set into itself. So a set of holomorphic maps on $\mathbb{C}$ or $\mathbb{C}_\infty$ naturally forms a semigroup. We can construct a semigroup $S$ consisting of all elements that can be expressed as a finite composition of elements in $\mathscr{F}$. We say such a semigroup $S$ by *holomorphic semigroup* generated by the elements from $\mathscr{F}$. For a collection $\mathscr{F} = \{f_\alpha\}_{\alpha\in\Delta}$ of such maps, let $S = \langle f_\alpha \rangle$ be a *holomorphic semigroup* generated by them. We say $S$ is rational or transcendental semigroup depnding on whether $\mathscr{F}$ is a collection of rational maps or transcendental entire maps. The index set $\Delta$ to which $\alpha$ belongs is allowed to be infinite in general unless stated otherwise. If $f \in S$, then $f = f_{\alpha_1} \circ f_{\alpha_2} \circ f_{\alpha_3} \circ \cdots \circ f_{\alpha_m}$. The holomorphic semigroup $S$ is *abelian* if $f_i \circ f_j = f_j \circ f_i$ for all generators $f_i$ and $f_j$ of $S$. A semigroup generated by finitely many holomorphic functions $f_i, (i = 1, 2, \dots, n)$ is called *finitely generated holomorphic semigroup.* We write $S = \langle f_1, f_2, \dots, f_n\rangle$. If $S$ is generated by only one holomorphic function $f$, then $S$ is *cyclic semigroup.* We write $S = \langle f \rangle$. In this case, each $g \in S$ can be written as $g = f^n$, where $f^n$ is the nth iterates of $f$ with itself.

In this paper, classical holomorphic dynamics refers the iteration theory of single holomorphic map and holomorphic semigroup dynamics refers the dynamical theory generated by the composition of various classes of holomorphic maps. In holomorphic semigroup dynamics, algebraic structure of semigroup naturally attached to the dynamics and hence the situation is little bit complicated. For simplicity, we mention the existing results (of both classical and semigroup dynamics) in propositions and new results in lemmas and theorems.

## 2. The Fatou, Julia and escaping sets of Holomorphic Semigroups

Definitions of the Fatou, Julia, and escaping sets of classical holomorphic dynamics can be naturally extended in the settings of holomorphic semigroups. Note that a holomorphic semigroup $S$ forms a normal family in a domain $D$ if every sequence $(f_\alpha) \subseteq S$ has a subsequence $(f_{\alpha_k})$ which is uniformly convergent or divergent on all compact subsets of $D$. If there is a neighborhood $U$ of a point $z \in \mathbb{C}$ such that $S$ is a normal family in $U$, then we say that $S$ is normal at $z$. We say that a function $f \in S$ is *iteratively divergent* at $z \in \mathbb{C}$ if $f^n(z) \to \infty$ as $n \to \infty$. A semigroup $S$ is *iteratively divergent* at $z$ if every $f \in S$ is iteratively divergent at $z$. A semigroup $S$ is said to be *iteratively bounded* at $z$ if there is an element $f \in S$ which is not iteratively divergent at $z$.

Like in classical holomorphic dynamics (that is, based on the Fatou-Julia-Eremenko theory of a holomorphic function), the Fatou, Julia and escaping sets in the settings of a holomorphic semigroup are defined as follows:

**Definition 2.1** (**Fatou, Julia and escaping sets**). The *Fatou set* of a holomorphic semigroup $S$ is defined by $F(S) = \{z \in \mathbb{C} : S \text{ is normal at } z\}$, and the *Julia set* $J(S)$ of $S$ is the complement of $F(S)$. Let $S$ is a transcendental semigroup. Then the *escaping set* of $S$ is defined by $I(S) = \{z \in \mathbb{C} : S \text{ is iteratively divergent at } z\}$ We call each point of the set $I(S)$ an *escaping point*.

It is obvious that $F(S)$ is the largest open subset (of $\mathbb{C}$ or $\mathbb{C}_\infty$) on which the semigroup $S$ is normal. And its complement $J(S)$ is a closed set for any semigroup $S$. However, the escaping set $I(S)$ is neither an open nor a closed set (if it is non-empty) for any transcendental semigroup $S$. If $S = \langle f \rangle$, then the Fatou, Julia and escaping sets are respectively denoted by $F(f), J(f)$ and $I(f)$. Furthermore, let $f \in S$. Then $T = \langle f \rangle$ is a cyclic subsemigroup of $S$ generated by $f$. From these facts, we can say that the Fatou, Julia and escaping sets of the subsemigroup $T = \langle f \rangle$ are respectively the Fatou, Julia and escaping sets of the holomorphic function $f$. Hence, Definition 2.1 generalizes the Fatou, Julia and escaping sets of a holomorphic function of the introduction section.

Any maximally connected subset $U$ of the Fatou set $F(S)$ is called a *Fatou component*. As in classical holomorphic dynamics, a Fatou component $U$ of a holomorphic semigroup $S$ can be simply connected or multiply connected. Kumar and Kumar [8, Theorems 4.1, 4.2, 4.3, 4.5] extended some results of classical holomorphic dynamics related to simply and multiply connected Fatou components to holomorphic semigroup dynamics.

If $S$ is a polynomial semigroup, then $I(S)$ is a Fatou component containing $\infty$, and so it is an open subset of Fatou set $F(S)$. If $S$ is a transcendental semigroup, then escaping set $I(S)$ is neither an open nor a closed set (if it is non-empty). The following immediate result holds from Definition 2.1 of an escaping set.

**Theorem 2.1.** *Let $S$ be a holomorphic semigroup. Then $z \in \mathbb{C}$ is an escaping point under $S$ if and only if every sequence $(g_k)_{k\in\mathbb{N}}$ in $S$ is iteratively divergent at $z$.*

**Example 2.1.** For $k \in \mathbb{N}$, let us consider a semigroup $S = \langle \{f_k\} \rangle$, where $f_k(z) = e^{-1-\frac{1}{k}} e^z$. For $x > 1$, we have $f_k^n(x) \to \infty$ as $n \to \infty$. Furthermore, it is easy to see that $f^n(x) \to \infty$ as $n \to \infty$ for every $f \in S$. Therefore, by Theorem 2.1, $x > 1$ is in $I(S)$.

Definition 2.1 of escaping set is different than that of the definition given by Kumar and Kumar [7, Definition 2.1]. According to our definition, the escaping set of Example 2.1 is non-empty. However, according to Kumar and Kumar, the escaping set of this example is empty because the sequence $(f_k)_{k=1}^{\infty} \subset S$ does not contain subsequences tending to infinity. We can slightly generalize Theorem 2.1 to the following assertion which can be an alternative definition of an escaping set.

**Theorem 2.2** ([12, Lemma 3.3]). *Let a complex number $z \in \mathbb{C}$ is an escaping point of a holomorphic semigroup $S$. Then every non-convergent sequence in $S$ has a subsequence which diverges to $\infty$ at $z$.*

**Example 2.2.** Let us consider a semigroup $S = \langle f, g \rangle$, where $f(z) = \lambda e^z$ and $g(z) = \mu e^z$ with $\lambda < \mu < e^{-1}$. For $x > 1$, we have $h_n(x) \to \infty$ as $n \to \infty$ for every $(h_n) \subset S$. Therefore, by Theorem 2.2, every $x > 1$ is in $I(S)$.

It is noted that if for a sequence $(h_n) \subset S$ such that $h_n(z) \to \infty$ as $n \to \infty$, then we can not conclude always that there is an element $f \in S$ such that $h_n = f^k$ for some $k \in \mathbb{N}$ with $f^k(z) \to \infty$ as $k \to \infty$. For example, if we choose $h_n(z) = f \circ g \circ f \circ g^2 \circ \cdots \circ f \circ g^n$ in Example 2.2, then there does not exist an element $l \in S$ such that $l^k(z) \to \infty$ as $k \to \infty$.

## 3. Some Connection and Contrast between Classical and Holomorphic Semigroup Dynamics

The following immediate relations hold between the Fatou, Julia and escaping sets of a holomorphic semigroup $S$ and its cyclic subsemigroup from Definition 2.1. Indeed, this is a connection between classical and semigroup holomorphic dynamics.

**Theorem 3.1.** *Let $S$ be a holomorphic semigroup. Then*

(1) *$F(S) \subset F(f)$ for all $f \in S$ and hence $F(S) \subset \bigcap_{f \in S} F(f)$.*
(2) *$J(f) \subset J(S)$ for all $f \in S$.*
(3) *$I(S) \subset I(f)$ for all $f \in S$ and hence $I(S) \subset \bigcap_{f \in S} I(f)$.*

It is noted that we deal Theorem 3.1 (3) in the case of transcendental semigroups even though it holds for polynomial semigroups.

In analogy to classical rational dynamics ([2, Theorem 4.2.4]), Hinkkanen and Martin ([5, Lemma 3.1 and Corollary 3.1]) proved the following assertion.

**Proposition 3.1.** *Let $S$ be a rational semigroup. Then the Julia set $J(S)$ is perfect and $J(S) = \overline{\bigcup_{f \in S} J(f)}$.*

In analogy to classical transcendental dynamics ([6, Theorem 3.10]), Poon ([10, Theorems 4.1 and 4.2]) proved the following assertion.

**Proposition 3.2.** *Let $S$ be a transcendental semigroup. Then the Julia set $J(S)$ is perfect and $J(S) = \overline{\bigcup_{f \in S} J(f)}$.*

From Theorem 3.1 ((1) and (3)), we can say that the Fatou and escaping sets of a holomorphic semigroup may be empty. For example, the Fatou set of the semigroup $S = \langle f, g \rangle$ generated by the functions $f(z) = e^{\lambda z}$ and $g(z) = e^{s\lambda z} + 2\pi i/\lambda$ for all $s \in \mathbb{N}$ is empty for $\lambda > 1/e$ (and non-empty for $0 < \lambda < 1/e$ (see [10, Example 2.1])). The escaping set of the semigroup $S = \langle f, g \rangle$ generated by functions $f(z) = e^z$ and $g(z) = e^{-z}$ is empty (that is, the particular (say) $h = g \circ f^k \in S$ is iteratively bounded at any $z \in I(f)$). It is difficult to generalize holomorphic semigroups that can have empty escaping sets. We know that the Fatou set may be empty but the escaping set is non-empty in classical holomorphic dynamics. This is a contrast feature of the escaping set in classical and semigroup holomorphic dynamics. From the same Theorem part (2), and Propositions 3.1 and 3.2, we can say that, in classical and semigroup holomorphic dynamics, the Julia set is a closed, non-empty, unbounded and a perfect set. There are several transcendental semigroups whose Fatou and escaping sets are non-empty. From the following examples of Kumar and Kumar [8, Examples 3.2 and 3.3] and [7, Examples 2.6 and 2.7], we get non-empty Fatou and escaping sets.

**Example 3.1.** Let $S = \langle f, g \rangle$, where $f(z) = e^z + \lambda$ and $g(z) = e^z + \lambda + 2\pi i$ for all $\lambda \in \mathbb{C} - \{0\}$. Then $F(S) = F(f) \neq \emptyset$ and $I(S) = I(f) \neq \emptyset$.

**Example 3.2.** Let $S = \langle f, g \rangle$, where $f(z) = \lambda \sin z$ and $g(z) = \lambda \sin z + 2\pi$ for all $0 < |\lambda| < 1$. Then $F(S) = F(f) \neq \emptyset$ and $I(S) = I(f) \neq \emptyset$.

**Example 3.3.** Let $S = \langle f, g \rangle$, where $f(z) = e^{\lambda z}$ and $g(z) = e^{s\lambda z} + 2\pi i/\lambda$ for all $\lambda \in \mathbb{C} - \{0\}$ and $s \in \mathbb{N}$. Then $I(S) = I(f) \neq \emptyset$ and $F(S) = F(f) \neq \emptyset$ for $0 < \lambda < e^{-1}$.

**Example 3.4.** Let $S = \langle f, g \rangle$, where $f(z) = \lambda \sin z$ for all $\lambda \in \mathbb{C} - \{0\}$ and $g(z) = f^n + 2\pi$ for all $n \in \mathbb{N}$. Then $I(S) = I(f) \neq \emptyset$ and $F(S) = F(f) \neq \emptyset$.

Kumar and Kumar [7, Theorem 3.4] generalized these examples to the following result.

**Proposition 3.3.** *Let $S = \langle f, g \rangle$ be a transcendental semigroup generated by a periodic function $f$ with period $p$ and another function $g$ defined by $g = f^n + p$, $n \in \mathbb{N}$. Then $F(S) = F(f)$ and $I(S) = I(f)$.*

Next, we see series of results where classical transcendental dynamics (in particular, results related to escaping set) can be generalized to transcendental semigroup dynamics. If $I(S) \neq \emptyset$, then the statement $\partial I(f) = J(f)$ ([4, Statement 1, Page 339]) of classical holomorphic dynamics can be generalized to semigroup dynamics. The following results is due to Kumar and Kumar [7, Lemma 4.2 and Theorem 4.3] which yields a generalized answer in semigroup dynamics.

**Proposition 3.4.** *Let $S$ be a transcendental semigroup such that $I(S) \neq \emptyset$. Then*

(1) *$Int.(I(S)) \subset F(S)$ and $Ext.(I(S)) \subset F(S)$, where $Int.$ and $Ext.$ respectively denote the interior and exterior of $I(S)$.*
(2) *$\partial I(S) = J(S)$, where $\partial I(S)$ denotes the boundary of $I(S)$.*

*Proof.* (1) We refer, for instance, [7, Lemma 4.2 ].
(2) The facts $Int.(I(S)) \subset F(S)$ and $Ext.(I(S)) \subset F(S)$ yield $J(S) \subset \partial I(S)$. The fact $\partial I(S) \subset J(S)$ is obvious. □

From Proposition 3.4, the fact $J(S) \subset \overline{I(S)}$ follows trivially. If $I(S) \neq \emptyset$, then we prove the following result which is a generalization of Eremenko's result $I(f) \cap J(f) \neq \emptyset$ of classical transcendental dynamics to holomorphic semigroup dynamics.

**Theorem 3.2.** *Let $S$ be a transcendental semigroup such that $I(S) \neq \emptyset$. Then $I(S) \cap J(S) \neq \emptyset$*

**Lemma 3.1.** *Let $f$ be a transcendental entire function and $U$ be a multiply connected component of $F(f)$. Then $f^n(z) \to \infty$ locally uniformly on $U$.*

*Proof.* See, for instance, [1, Theorem 3.1]. □

*Proof of Theorem 3.2.* Case (1). Suppose $F(S)$ has a multiply connected component $U$. Then by Theorem 3.1 (1), $U$ is also a multiply connected component of $F(f)$ for each

$f \in S$. By Lemma 3.1, for each $f \in S$, $f^n(z) \to \infty$ locally uniformly on $U$ and on $\partial U$. It follows by normality (and also by Theorem 2.2) that every non-convergent sequence in $S$ has a subsequence which diverges to $\infty$ locally uniformly on $U$ and $\partial U$. This proves that $f^n(z) \to \infty$ for all $z \in U$, and $z \in \partial U$ for all $f \in S$. Again, by Theorem 3.1 (3) , $U \subset I(S)$. As $\partial U \subset J(f)$ for all $f \in S$, so, by Theorem 3.1 (2), $\partial U \subset J(S)$. This proves that $I(S) \cap J(S) \neq \emptyset$.

Case (2). Suppose that every components of $F(S)$ are simply connected. Let $U$ be an arbitrary simply connected Fatou component of $F(S)$. Then by Theorem 3.1 (1), $U$ is also a simply connected component of $F(f)$ for all $f \in S$. Then by hypothesis, we have $I(f) \cap J(f) \neq \emptyset$ for each $f \in S$. Again, by Theorem 3.1 (3) and Propositions 3.2 and 3.4 (2), we must conclude $I(S) \cap J(S) \neq \emptyset$. □

**Theorem 3.3.** *Let $S$ be a transcendental semigroup such that $I(S) \neq \emptyset$. Then $\overline{I(S)}$ has no bounded components.*

**Lemma 3.2.** *A multiply connected component of the Fatou set $F(S)$ of a transcendental semigroup $S$ lies in escaping set $I(S)$.*

*Proof.* Let $U$ be a multiply connected component of $F(S)$. Then it is a multiply connected component of $F(f)$ for all $f \in S$. Then by Lemma 3.1, $f^n \to \infty$ locally uniformally on $U$ for all $f \in S$. This proves $U \subset I(f)$ for all $f \in S$. Hence $U \subset I(S)$. □

*Proof of Theorem 3.3.* By Theorem 3.1(3), we can write $I(S) \subset I(f)$ for all $f \in S$. Therefore, $\overline{I(S)} \subset \overline{I(f)}$ for all $f \in S$. By Proposition **??**(3), $\overline{I(f)}$ has no bounded components. We prove that $\overline{I(S)}$ also has no bounded components. Suppose for the contrary that $A$ be a bounded component of $\overline{I(S)}$. Then it is bounded component of $\overline{I(f)}$ for each $f \in S$. In such a case, there is a domain $B$ (possibly homeomorphic to an annulus) which separates $A$ from $\infty$. Therefore, $B \cap I(f) = \emptyset$ for all $f \in S$. This shows that $B \subset F(f)$ for all $f \in S$. Hence, $B \subset F(S)$. Let $C$ be the bounded component of $\mathbb{C} - B$, then $C \cap J(f) \neq \emptyset$ for all $f \in S$. Therefore, $C \cap J(S) \neq \emptyset$. This proves that $A$ is contained in a multiply connected component of $F(S)$. Then by Lemma 3.2, $A \subset I(S)$, a contradiction. □

The one of the most important result of classical holomorphic dynamics is either $J(f) = \mathbb{C}$ or $\mathbb{C}_\infty$ or $J(f)$ has empty interior for any holomorphic function $f$ on $\mathbb{C}$ or $\mathbb{C}_\infty$ (see [2, Theorem 4.2.3] for rational function and [3, Lemma 3] for general holomorphic function). There are lot of examples of transcendental entire functions and rational functions whose Julia set is the entire complex plane or extended complex plane. For example,

(1) $J(\lambda z e^z) = \mathbb{C}$ for a suitable value of $\lambda$,
(2) $J(e^z) = \mathbb{C}$,
(3) $J(\lambda z e^z / z) = \mathbb{C}_\infty$ for a suitable value of $\lambda$,
(4) $J(\lambda \tan z) = \mathbb{C}_\infty$ for a suitable value of $\lambda$,
(5) $J((z-2)^2/z^2) = \mathbb{C}_\infty$,
(6) $J((z^2+1)^2/4z(z^2-1)) = \mathbb{C}_\infty$.

However, the analogous result is not hold in semigroup dynamics. Hinkkanen and Martin [5, Example-1] provided the following example that shows that Julia set of a rational semigroup $S$ may have non-empty interior even if $J(S) \neq \mathbb{C}_\infty$.

**Example 3.5.** Entire semigroup $S = \langle z^2, z^2/a \rangle$, where $a \in \mathbb{C}, |a| > 1$ has Fatou set $F(S) = \{z : |z| < 1 \text{ or } |z| > |a|\}$ and Julia set $J(S) = \{z : 1 \leq |z| \leq |a|\}$.

Let $U$ be a component of Fatou set $F(f)$. Then $f(U)$ is contained in some component $V$ of $F(f)$. It is noted that if $f$ is a rational function, then $V = f(U)$. If $f$ is a transcendental function, then it is possible that $V \neq f(U)$. Let us recall the following result of Bergweiler [3] of classical holomrphic dynamics.

**Proposition 3.5.** *If $f$ is entire, then $V - f(U)$ contains at most one point which is an asymptotic value of $f$.*

The following example of Huang [13, Example 2] shows that Proposition 3.5) can not be preserved for general semigroup dynamics. This is a contrast between classical holomorphic dynamics and semigroup dynamics.

**Example 3.6.** Let $S = \langle z^n, az^n \rangle$, where $n > 2$ and $|a| > 1$. The Fatou set $F(S)$ contains following components

$$U = \left\{ \sqrt[n]{\frac{1}{|a|}} < |z| < \sqrt[n]{\sqrt[n-1]{\frac{1}{|a|}}} \right\} \text{ and } V = \{|z| > 1\}.$$

For a function $f(z) = az^n$ in semigroup $S$, $f(U) \subset V$ and $V - f(U)$ is an unbounded domain.

**Definition 3.1** (**Backward orbit and exceptional set**)**.** Let $S$ be a holomorphic semigroup. We define the backward orbit of any $z \in \mathbb{C}$ (or $\mathbb{C}_\infty$) by

$$O^-(z) = \{w \in \mathbb{C}_\infty : \text{there exists } f \in S \text{ such that } f(w) = z\}$$

and the Fatou exceptional set of $S$ is defined by $E(S) = \{z \in \mathbb{C}_\infty : O^-(z) \text{ is finite}\}$. Any $z \in E(S)$ is called exceptional value.

It is noted that if $S$ finitely generated rational semigroup, then $E(S) \subset F(S)$, otherwise we can not assert it. For example ([13, example 1]), semigroup $S = \langle f_m \rangle$, where $f_m(z) = a^m z^n, m \in \mathbb{N}, n \geq 2$ and $|a| > 1$, is an infinitely generated polynomial semigroup. Then, $E(S) = \{0, \infty\}$. It is easy to see that 0 is a limit point of $J(f_m) = \{z : |z| = |a|^{\frac{-m}{n-1}}\}$, and hence $0 \in J(S)$. In the case of finitely generated rational semigroup $S$, we always have $E(S) \subset F(S) \subset F(f)$ for any $f \in S$. Hence $E(S)$ contains at most two points. However, if $S$ finitely generated transcendental semigroup, then we can not assert $E(S) \subset F(S)$ in general because for a transcendental function, it is difficult to determine whether Fatou exceptional value belongs Fatou set or Julia set. For example, 0 is the Fatou exceptional value of $f(z) = e^{\lambda z}$. It is known in classical holomorphic dynamics that $0 \in J(f)$ if $\lambda > 1/e$ and $0 \in F(f)$ if $\lambda < 1/e$. Poon and Yang [11] gave the following characterization whether a Fatou exceptional value belongs to the Fatou set or Julia set.

**Proposition 3.6.** *Let $f$ is transcendental entire function. If $F(f)$ has no unbounded component, then Fatou exceptional value always belongs to Julia set.*

In the case of finitely generated transcendental semigroup $S$, if $E(f) \subset F(f)$ for all $f \in S$, then we can say $E(S) \subset F(S) \subset F(f)$ for any $f \in S$. Hence $E(S)$ contains at most one point. This fact is a generalization of classical holomorphic dynamics to semigroup dynamics and so it is a nice connection between these two types of dynamics. Huang [13, Proposition 1] proved the following result which also shows a connection between classical holomorphic dynamics and semigroup dynamics.

**Proposition 3.7.** *Let $S$ be a holomorphic semigroup. If $z \notin E(S)$, then $J(S) \subseteq \overline{O^{-}(z)}$.*

If $z \in J(S)$ and $z \notin E(S)$, then $J(S) = \overline{O^{-}(z)}$ for any holomorphic semigroup $S$. This says that $\overline{O^{-}(z)}$ clusters at each point of $J(S)$. This result proved for rational semigroup by Hinkkanen and Martin [5, Lemma 3.2].

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