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ALMOST INCREASING SEQUENCE FOR ABSOLUTE RIESZ $|\overline{N}, p_n^{\alpha,\beta}|_q$ SUMMABLE FACTOR

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Abstract: In this paper, a result on absolute Riesz summability $|\overline{N}, p_n^{\alpha,\beta}|_q$ for an infinite series by Bor has been extended using more variables. Further, we develope some well known results from our main result.

Key Words: Absolute summability, Riesz mean, Hölder's inequality, $|\overline{N}, p_n^{\alpha,\beta}|_k$ summability.

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1. Introduction

Let partial sum's sequence of $\sum a_n$ be given by $\{s_n\}$ and n^{th} sequence to sequence transform of $\{s_n\}$ is given by u_n, where

(1.1) $$u_n = \sum_{k=0}^{\infty} u_{nk} s_k.$$

Definition 1: An infinite series $\sum a_n$ is absolute summable, if

$$\lim_{n\to\infty} u_n = s,$$

and

(1.2) $$\sum_{n=1}^{\infty} |u_n - u_{n-1}| < \infty.$$

Definition 2: Let $\{p_n\}$ be a sequence with $p_0 > 0$ and $p_n \geq 0$ for $n > 0$

(1.3) $$P_n = \sum_{v=0}^{n} p_v \to \infty.$$

For $\alpha > -1$, $0 < \beta \leq 1$, $\alpha + \beta > 0$, define:

(1.4) $$\in_0^{\alpha+\beta} = 1,\ \in_n^{\alpha+\beta} = \frac{(\alpha+\beta+1)(\alpha+\beta+2)....(\alpha+\beta+n)}{n!},\ (n = 1, 2, 3, ...)$$

(1.5) $$p_n^{\alpha,\beta} = \sum_{v=0}^{n} \in_{n-v}^{\alpha+\beta-1} p_v,$$

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$$P_n^{\alpha,\beta} = \sum_{v=0}^{n} p_n^{\alpha,\beta} \to \infty, \ n \to \infty \tag{1.6}$$

and

$$P_{-n}^{\alpha,\beta} = p_{-n}^{\alpha,\ \beta} = 0, \ n \geq 1.$$

Then, the sequence-to-sequence transformation t_n defines the $(\overline{N}, p_n^{\alpha,\beta})$ mean of series $\sum a_n$ and is given by:

$$t_n = \frac{1}{P_n^{\alpha,\beta}} \sum_{k=0}^{n} p_k^{\alpha,\beta} s_k, \ P_n^{\alpha,\beta} \neq 0, \ n \in N \tag{1.7}$$

and $\lim_{n\to\infty} t_n = s$, and the series is called $(\overline{N}, p_n^{\alpha,\beta})$, formed by sequence of coefficients $\{p_n^{\alpha,\beta}\}$.

Further, if sequences $\{t_n\}$ is of bounded variation with index $k \geq 1$ i.e.

$$\sum_{n=1}^{\infty} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}} \right)^{k-1} |\Delta t_{n-1}|^k < \infty, \tag{1.8}$$

then $\sum a_n$ is said to be absolutely $(R, p_n^{\alpha,\beta})_k$ summable with index k or $|\overline{N}, p_n^{\alpha,\beta}|_k$ summable to s, where

$$\Delta t_n = -\frac{p_n^{\alpha,\beta}}{P_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}} \sum_{v=1}^{n} P_{v-1}^{\alpha,\beta} a_v, \ \ n \geq 1. \tag{1.9}$$

Bor [1]-[3] generalized the result associated with Riesz summability factors. Bor and Özarslan [4], [5] established theorems using $|\overline{N}, p_n; \delta|$ summability factors. Özarslan [9], [10] used the definition of almost increasing sequence for absolute summability. Yildiz [17], [18] determined theorems on generalized absolute matrix summability factors. Mishra et al. [7], [8] provide interesting result on matrix summability and absolute summability. Sonker et al. [11] worked on absolute summability factors for n-tupled trianle matrices. Also, Sonker and Munjal [12]-[16] gave various useful results on summabilities. In this paper, we are going to prove the more generalized version of the result given by Bor [6], under the weaker conditions.

2. KNOWN-RESULT

By using $|\overline{N}, p_n^{\alpha}|_q$ summability, Bor [6] proved the following theorem.

2.1. **Theorem** [6]**:** Let $\{p_n\}$ be of +ive numbers s.t.:

$$P_n = O(np_n) \ \ as \ \ n \to \infty. \tag{2.1}$$

Let (χ_n) be an almost increasing sequence and assuming (ξ_n) and (λ_n) are s.t.:

$$|\Delta \lambda_n| \leq \xi_n, \tag{2.2}$$

$$\xi_n \to 0 \ \ as \ \ n \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} n|\Delta\xi_n|\chi_n \leq \infty, \tag{2.4}$$

$$|\lambda_n|X_n = O(1) \;\; as \;\; n \to \infty, \tag{2.5}$$

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta q-1} \frac{1}{P_{n-1}} = O\left(\left(\frac{P_v}{p_v}\right)^{\delta q} \frac{1}{P_v}\right), \tag{2.6}$$

and

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta q-1} |t_n|^q = O(\chi_n) \;\; as \;\; m \to \infty, \tag{2.7}$$

then $\sum a_n\lambda_n$ is $|\overline{N}, p_n; \delta|_q$ summable where, $q \geq 1$ and $0 \leq \delta \leq \frac{1}{q}$.

3. MAIN RESULT

A sequence is of bounded variation i.e. $(\lambda_n) \in BV$, if :

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty.$$

3.1. **Theorem:** Let (χ_n), (ξ_n) and (λ_n) be as defined in Theorem 2.1 and verify (2.2)-(2.5). If the following conditions also satisfy:

$$\sum_{n=v+1}^{\infty} \frac{1}{P_{n-1}^{\alpha,\beta}} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{-1} = O\left\{\frac{1}{P_v^{\alpha,\beta}}\right\}, \tag{3.1}$$

$$\sum_{n=1}^{m} \left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{-1} |t_n|^q = O(\chi_m), \tag{3.2}$$

and

$$\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1) \tag{3.3}$$

then, $\sum a_n\lambda_n$ is $|\overline{N}, p_n^{\alpha,\beta}|_q$ summable where $q \geq 1$.

Proof: Let Y_n denote the $(\overline{N}, p_n^{\alpha,\beta})$ mean of $\sum a_n\lambda_n$. We have:

$$Y_n = \frac{1}{P_n^{\alpha,\beta}} \sum_{v=0}^{n} p_v^{\alpha,\beta} \sum_{i=0}^{v} a_i\lambda_i. \tag{3.4}$$

For $n \geq 1$,

$$\Delta Y_n = \frac{p_n^{\alpha,\beta}}{P_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}} \sum_{v=1}^{n} P_{v-1}^{\alpha,\beta} a_v\lambda_v = \frac{p_n^{\alpha,\beta}}{P_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}} \sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} v a_v.$$

$$= \tfrac{n+1}{nP_n^{\alpha,\beta}}\, p_n^{\alpha,\beta} t_n\lambda_n$$

$$-\tfrac{p_n^{\alpha,\beta}}{P_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}} \sum_{v=1}^{n-1} p_v^{\alpha,\beta} t_v \lambda_v \tfrac{v+1}{v}$$

$$+\frac{p_n^{\alpha,\beta}}{P_n^{\alpha,\beta}P_{n-1}^{\alpha,\beta}}\sum_{v=1}^{n-1}P_v^{\alpha,\beta}t_v\Delta\lambda_v\frac{v+1}{v}$$

$$+\frac{p_n^{\alpha,\beta}}{P_n^{\alpha,\beta}P_{n-1}^{\alpha,\beta}}\sum_{v=1}^{n-1}P_v^{\alpha,\beta}t_v\lambda_{v+1}\frac{1}{v}$$

$$= Y_1 + Y_2 + Y_3 + Y_4. \tag{3.5}$$

To prove the main result, we prove that

$$\sum_{n=1}^{\infty}\left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{q-1}|\bar{\Delta}Y_n|^q < \infty. \tag{3.6}$$

Using Minkowski's inequality,

$$|Y_1+Y_2+Y_3+Y_4|^q \le 4^q(|Y_1|^q+|Y_2|^q+|Y_3|^q+|Y_4|^q)$$

then, equation (4.3) reduces to:

$$\sum_{n=1}^{\infty}\left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{q-1}|Y_r|^q = J_r < \infty \;\; for \;\; r=1,2,3,4. \tag{3.7}$$

Now the L.H.S. of equation (4.4) is given as:

$$J_1 = \sum_{n=1}^{m}\left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{q-1}\left|\frac{n+1}{nP_n^{\alpha,\beta}}\,p_n^{\alpha,\beta}t_n\lambda_n\right|^q$$

$$= \sum_{n=1}^{m}\left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{-1}|t_n|^q|\lambda_n|$$

$$= O(1)|\lambda_m|\sum_{n=1}^{m}\left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{-1}|t_n|^q$$

$$+O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}}\right)^{-1}|t_v|^q$$

$$= O(1)|\lambda_m|\chi_m + O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|\chi_n$$

$$= O(1) \;\; as \;\; m\to\infty, \tag{3.8}$$

$$J_2 = O(1)\sum_{n=2}^{m+1}\frac{1}{P_{n-1}^{\alpha,\beta}}\left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{-1} \times$$

$$\times\sum_{v=1}^{n-1}p_v^{\alpha,\beta}|t_v|^q|\lambda_v|\left(\frac{1}{P_{n-1}^{\alpha,\beta}}\sum_{v=1}^{n-1}p_v^{\alpha,\beta}\right)^{q-1}$$

$$= O(1)\sum_{v=1}^{m}p_v^{\alpha,\beta}|t_v|^q|\lambda_v|\frac{1}{P_v^{\alpha,\beta}}$$

$$= O(1)|\lambda_m|\sum_{n=1}^{m}\left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{-1}|t_n|^q$$

$$+O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\left(\frac{P_v^{\alpha,\beta}}{p_v^{\alpha,\beta}}\right)^{-1}|t_v|^q$$

$$= O(1) \ \ as \ \ m\to\infty, \tag{3.9}$$

$$J_3 = O(1)\sum_{n=2}^{m+1}\frac{1}{P_{n-1}^{\alpha,\beta}}\left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{-1}\times$$

$$\times\sum_{v=1}^{n-1}P_v^{\alpha,\beta}|t_v|^q\xi_v\left(\frac{1}{P_{n-1}^{\alpha,\beta}}\sum_{v=1}^{n-1}P_v^{\alpha,\beta}\xi_v\right)^{q-1}$$

$$= O(1)\sum_{v=1}^{m}P_v^{\alpha,\beta}\xi_v|t_v|^q\times$$

$$\times\sum_{n=v+1}^{m+1}\frac{1}{P_{n-1}^{\alpha,\beta}}\left(\frac{P_n^{\alpha,\beta}}{p_n^{\alpha,\beta}}\right)^{-1}$$

$$= O(1)\sum_{v=1}^{m}P_v^{\alpha,\beta}|t_v|^q\xi_v\frac{1}{P_v^{\alpha,\beta}}$$

$$= m\xi_m\sum_{v=1}^{m}\frac{1}{v}|t_v|^q + O(1)\sum_{v=1}^{m-1}\Delta(v\xi_v)\sum_{i=1}^{v}\frac{1}{i}|t_i|^q$$

$$= O(1)m\xi_m\chi_m + O(1)\sum_{v=1}^{m-1}|\Delta(v\xi_v)|\chi_v$$

$$= O(1) \ \ as \ \ m\to\infty, \tag{3.10}$$

and proceeding as in J_3, we get

$$J_4 = O(1) \ \ as \ \ m\to\infty. \tag{3.11}$$

Collecting (3.8)-(3.11), we get that the condition (3.6) holds.
Hence, the theorem is proved.

4. COROLLARIES

4.1. **Corollary:** Let (χ_n), (ξ_n) and (λ_n) are s.t. conditions (2.2)-(2.5) of Theorem 2.1, condition (3.3) of Theorem 3.1,

$$\sum_{n=v+1}^{\infty}\frac{p_n^{\alpha}}{P_n^{\alpha}P_{n-1}^{\alpha}} = O\left(\frac{1}{P_v^{\alpha}}\right), \tag{4.1}$$

$$\sum_{n=1}^{m}\frac{p_n^{\alpha}}{P_n^{\alpha}}|t_n|^q = O(\chi_m) \tag{4.2}$$

and

$$\sum_{n=1}^{m}\frac{1}{n}|t_n|^q = O(\chi_m) \ \ as \ \ m\to\infty \tag{4.3}$$

holds. Then, $\sum a_n\lambda_n$ is $|\overline{N}, p_n^{\alpha}|_q$ summable for $q\geq 1$.

Proof: By using $\beta = 1$ in main theorem, we will get (4.1), (4.2) and (4.3). The proof is same as the main theorem 3.1, but here we used equations (4.1), (4.2) and (4.3) instead of equations (3.1), (3.2) and (3.4).

4.2. **Corollary:** Let (X_n), (ξ_n) and (λ_n) are s.t. conditions (2.2)-(2.5) of Theorem 2.1, condition (3.3) of Theorem 3.1 and (4.1)-(4.3) holds. Then, $\sum a_n\lambda_n$ is $|\overline{N}, p_n^{\alpha}|$ summable.

Proof: By using $\beta = 1$ and $q = 1$ in main theorem and equations (4.1)-(4.3), we get this result.

5. CONCLUSION:

The negligeable set of conditions has been obtained for the infinite series in this paper. By the examination we may infer that our hypothesis is a summed up variant which can be diminished for a few notable summabilities as appeared in corollaries.

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