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APPROXIMATION OF A FUNCTION BELONGING TO *LIP* (α, θ, ω) BY $(C, 1)(E, q)$ MEAN

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Abstract: A result has been established for a conjugate trigonometric Fourier series in which degree of approximation determine in class *Lip* (α, θ, w) by a product $(C,\ 1)(E,\ q)$ mean (sequence to sequence transformation). This result is more applicable by using the product mean as it is used as double filter.
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1. Introduction

Let $\{s_n\}$ represents the partial sum sequence of series $\sum a_n$. The $(C, 1)$ transformed mean of $\{s_n\}$ is

$$C_n^1 = \frac{1}{n+1}\sum_{k=0}^{n} s_k, \quad n = 0, 1, 2, ... \tag{1.1}$$

It is said to be Cesàro summable of order one (C, 1) to s if $\lim_{n\to\infty} C_n^1 = s$. Euler transformed means of $q > 0$ of $\{s_n\}$ is

$$E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k, \quad q > 0,\ n = 0, 1, 2, ... \tag{1.2}$$

Let the product $(C, 1)(E, q)$ transformed mean denoted by $C_n^1 E_n^q$ and given by

$$C_n^1 E_n^q = \frac{1}{n+1}\sum_{k=0}^{n} E_k^q = (1+n)^{-1}\sum_{k=0}^{n}(1+q)^{-k}\sum_{v=0}^{k}\binom{k}{v} q^{k-v} s_v. \tag{1.3}$$

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Let f is 2π-periodic function and belongs to $L_\theta[0, 2\pi] = L_\theta$ for $\theta \geq 1$ (bounded, integrable). Then,

$$s_n(f;x) = \frac{a_0}{2} + \sum_{l=1}^{n}(a_l \cos lx + b_l \sin lx) \tag{1.4}$$

and modulus of continuity of function f is

$$\Omega_\theta(\delta; f) = \sup_{0<|h|\leq\delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^\theta dx \right\}^{1/\theta}. \tag{1.5}$$

If, for $\alpha > 0$, then condition for $f \in Lip(\alpha, \theta)$, $\theta \geq 1$ is

$$\Omega_\theta(\delta; f) = O(\delta^\alpha). \tag{1.6}$$

The L_θ-norm of the function is

$$\|f\|_\theta := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^\theta dx \right\}^{1/\theta}. \tag{1.7}$$

For the function f, conjugate Fourier series is

$$\sum_{l=1}^{\infty}(-a_l \sin lx + b_l \cos lx) \tag{1.8}$$

and

$$\tilde{s}_n(f;x) = \sum_{l=1}^{n}(-a_l \sin lx + b_l \cos lx) \tag{1.9}$$

represents the partial sum. If $\psi(t) = f(x+t) - f(x-t)$ and conjugate $\tilde{f}$ is

$$2\pi \tilde{f}(x) = -\lim_{\varepsilon \to 0} \int_\varepsilon^\pi \cot(t/2)\psi(t)dt. \tag{1.10}$$

If $f \in Lip\,\alpha$, then condition is

$$|f(x+t) - f(x)| = O(t^\alpha). \tag{1.11}$$

If $0 < \alpha \leq 1$ and $\theta \geq 1$, then $\left(\int_0^{2\pi} |f(x+t) - f(x)|^\theta dx\right)^{1/\theta} = O(t^\alpha)$ is condition for $f \in Lip(\alpha, \theta)$. For $r \geq 1$, the $L_\theta-$norm for $f \in L_\theta[-\pi, \pi]$ is

$$\|f\|_\theta = \left(\int_0^{2\pi} |f(x)|^\theta dx \right)^{1/\theta}. \tag{1.12}$$

For $\theta \geq 1$, the condition for the function $f \in L_w^\theta[0, 2\pi] = L_w^\theta$ is

$$\|f\|_{\theta,w} = \left(\int_0^{2\pi} w(x)|f(x)|^\theta dx \right)^{1/\theta} < \infty. \tag{1.13}$$

If

$$\sup_I \left(\frac{1}{|I|} \int_I [w(x)]^{-1/(\theta-1)} dx \right)^{\theta-1} \left(\frac{1}{|I|} \int_I w(x)dx \right) < \infty, \quad \theta > 1. \tag{1.14}$$

Then, function belongs to A_θ i.e. Muckenhoupt class, where the supremum is in $|I| \leq 2\pi$. Let $w \in A_\theta$ and $f \in L_w^\theta$. The condition

$$\Omega_\theta(f; \delta) = \sup_{0<|h|\leq\delta} \|\Delta_h(f)\|_{\theta,w} \tag{1.15}$$

represents the modulus of continuity, where

$$\Delta_h(f)(x) = \frac{1}{h}\int_0^h |f(x+t) - f(x)|\,dt. \tag{1.16}$$

The $Lip(\alpha,\ \theta,\ w)$ for $0 < \alpha \leq 1$ and $\delta > 0$ is

$$Lip(\alpha,\ \theta,\ w) = \{f \in L_w^\theta : \Omega(f,\delta)_{\theta,w} = O(\delta^\alpha)\}. \tag{1.17}$$

The problem in which approximation of a signal can done solved by selecting a function closely matches (approximates) a target function among a well-defined class. Quade [17] solved a problem related with approximation by trigonometric polynomials. Many research articles have been devoted to the study of summability of infinite series due to its wide range of applications. After this, several mathematicians determined various results on approximation by summability techniques of a signal which belongs to various classes like Chandra [1-3], Khan [4, 5], Leindler [6], Mittal et al. [8] and Mishra et al. [9-16]. Recently, Sonker and Munjal [18-25] gave a number of theorems exploring summability of the Orthogonal, Fourier series and infinite series. In the present result, summability method has been used for sharper estimations.

2. Known Results

A result on $(C,\,1)(E,\,1)$ transformed means has been established by Lal and Singh [7].

Theorem 1 [7]: A function belonging to $Lip(\alpha,\,r)$ and 2π-periodic ($f : R \to R$), then by the $(C,\,1)$ $(E,\,1)$ transformed mean of conjugate Fourier series, the degree of approximation (doa) of $\tilde{f}(x)$ of f satisfies,

$$M_n(f) = Min\left\|(CE)_n^1 - \tilde{f}\right\|_r = O(n^{1/r-\alpha}), \quad n = 0, 1, 2, \ldots \tag{2.1}$$

and $(C,1)(E,1)$ transformed mean is

$$(CE)_n^1 = (n+1)^{-1}\sum_{k=0}^{n}\left(2^{-k}\sum_{i=0}^{k}\begin{pmatrix} k \\ i \end{pmatrix} S_i\right). \tag{2.2}$$

3. Main Result

In this paper, we establish a result for a function belonging to $Lip(\alpha,\ \theta,\ w)$ by $(C,\,1)(E,\,q)$ mean with the help of conjugate trigonometric Fourier series.

Theorem 2: Let function $f \in Lip(\alpha,\ \theta,\ w)$-class with $\theta \geq 1,\ \alpha\theta \geq 1$ which is Lebesgue integrable 2π-periodic, then degree of approximation of $\tilde{f}(x)$ with $(C,\,1)(E,\,q)$ transformed means of conjugate Fourier series of f satisfies,

$$M_n(f) = Min\left\|\left(C_n^1 E_n^q\right) - \tilde{f}\right\|_{\theta,w} = O\left(n^{\frac{1}{\theta}-\alpha}\right), \text{ where } n = 0, 1, 2, \ldots, \tag{3.1}$$

provided

$$\left(\int_0^{\pi/(n+1)} (|\psi(t)|t^{-\alpha})^\theta\right)^{1/\theta} = O\left(\frac{1}{n+1}\right) \tag{3.2}$$

$$\left(\int_0^{\pi/(n+1)} \left(|\psi(t)|t^{-\delta-\alpha}\right)^\theta\right)^{1/\theta} = O(n+1)^\delta), \tag{3.3}$$

where δ is an arbitrary number with $1/\theta + 1/\varphi = 1$ and $(\alpha+\delta)\varphi + 1 < 0$ for $\theta > 1$.

4. Lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 1 [7]: $|K_n(t)| = O((n+1)t) + O(1/t)$ for $\pi/(n+1) \geq t \geq 0$.

Lemma 2 [7]: $|K_n(t)| = O(1) + O(1/t)$ for $\pi/(n+1) \leq t \leq \pi$.

5. Proof of Main Theorem

The $\tilde{s}_n(f;x)$ is represented by

$$\tilde{s}_n(f;x) = \frac{1}{\pi}\int_0^{\pi} \frac{\cos(n+1/2)t - \cos(t/2)}{2\sin(t/2)}\psi(t)dt. \tag{5.1}$$

Therefore,

$$\tilde{f}(x) - \tilde{s}_n(f;x) = -\frac{1}{2\pi}\int_0^{\pi} \frac{\cos(n+1/2)t}{\sin(t/2)}\psi(t)dt. \tag{5.2}$$

The $(C,1)(E,q)$ transform

$$\tilde{f} - C_n^1 E_n^q = -\frac{1}{2\pi(n+1)}\left[\sum_{k=0}^{n}(1+q)^{-k}\int_0^{\pi}\frac{\psi(t)}{sin(\ t/2)}\sum_{v=0}^{k}\begin{pmatrix} k \\ v \end{pmatrix} q^{k-v} cos[\ (v+1/2)t]dt\right]$$

$$= \left[\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}\right]\psi(t)K_n(t)\,dt = I_1 + I_2 \qquad (say). \tag{5.3}$$

Using condition (3.2) and Lemma 1, we have

$$|I_1| = \int_0^{\pi/(n+1)} |K_n(t)|\left|\psi(t)\right|dt \leq \left[\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)} (|K_n(t)|\,t^{\alpha})^{\varphi}dt\right]^{1/\varphi}\left[\int_0^{\pi/(n+1)}\left(\frac{\left|\psi(t)\right|}{t^{\alpha}}\right)^{\theta}dt\right]^{1/\theta}$$

$$= O((n{+}1)^{-1})\left[\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}\left((n+1)t^{\alpha+1} + t^{\alpha-1}\right)^{\varphi}dt\right]^{1/\varphi}$$

$$= O((n{+}1)^{-1})\left[\left(\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}(n+1)t^{(\alpha+1)\varphi}dt\right)^{1/\varphi} + \left(\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}t^{(\alpha-1)\varphi}dt\right)^{1/\varphi}\right]$$

$$= O((n{+}1)^{-1})\left[(n+1)(n+1)^{-\alpha-1-1/\varphi} + (n+1)^{-\alpha+1-1/\varphi}\right]$$

$$= O((n{+}1)^{-1})\left[(n+1)(n+1)^{-\alpha-1-1+\frac{1}{\theta}} + (n+1)^{-\alpha+\frac{1}{\theta}}\right]$$

$$= O\left[(n+1)(n+1)^{-\alpha-2+1/\theta} + (n+1)^{-\alpha-1+1/\theta}\right]$$

$$= O\left((n+1)^{-\alpha-1+1/\theta}\right). \tag{5.4}$$

Now, we consider and using condition (3.3) and

$$= O((n+1)^{\delta})\left[\left(\int_{\pi/(n+1)}^{\pi} t^{(\alpha+\delta)\varphi}dt\right)^{1/\varphi} + \left(\int_{\pi/(n+1)}^{\pi} t^{(\alpha+\delta-1)\varphi}dt\right)^{1/\varphi}\right]$$

$$= O((n+1)^{\delta})\left[(n+1)^{-\alpha-\delta-1/\varphi} + (n+1)^{-\alpha-\delta+1-1/\varphi}\right]$$

$$= O\left[(n+1)^{-\alpha-1/\varphi} + (n+1)^{-\alpha+1-1/\varphi}\right]$$

$$= O\left[(n+1)^{-\alpha-1+1/\theta} + (n+1)^{-\alpha+1/\theta}\right]$$

$$= O\left((n+1)^{-\alpha+1/\theta}\right). \tag{5.5}$$

Combining (5.1)-(5.5), we have

$$\left|C_n^1 E_n^q - \tilde{f}\right| = O\left((n+1)^{1/\theta-\alpha}\right) \tag{5.6}$$

Hence,

$$\left\|C_n^1 E_n^q - \tilde{f}\right\|_{\theta,\, w} = O\left(\int_0^{2\pi} \left|C_n^1 E_n^q - \tilde{f}(x)\right|^{\theta} w(x)dx\right)^{1/\theta} = O\left((n+1)^{1/\theta-\alpha}\right) \tag{5.7}$$

This completes the proof.

Remark: The result will be for (C, 1) (E, 1) means, if the result can be found by using q=1 in (C, 1)(E, q) transformation.

6.Corollaries

Corollary 1: If the function belong to Lipα, $0 < \alpha \leq 1$ class which is Lebesgue integrable function $f : R \to R$ and 2π-periodic, then the degree of approximation of $\tilde{f}(x)$ of conjugate Fourier series of f by the $(C, 1)(E, q)$ transformed means satisfies,

$$\left\|(C_n^1 E_n^q) - \tilde{f}\right\|_{\infty} = O\left(\frac{1}{n^{\alpha}}\right), \quad n = 0, 1, 2, \ldots$$

Proof. For proof we take w(x)=1 and $\theta \to \infty$ in the main theorem which reduces to the corollary 1.

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