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DIFFUSION APPROXIMATION TO RADIATION HEAT TRANSFER IN SEMITRANSPARENT MEDIUM

HARIHAR KHANAL[1] AND KEDAR NATH UPRETY[2]

[1] *Department of Mathematics, Embry-Riddle Aeronautical University, Daytona Beach, FL 32114-3900, U.S.A.*
harihar.khanal@erau.edu
[2] *Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal*

Abstract: Conduction-radiation heat transfer problem is usually formulated with a highly nonlinear integro-differential equation. In this paper we describe a simple technique for modeling radiation heat transfer in a semitransparent medium with arbitrary varying spectral absorption coefficient and purpose a computational model (described by a coupled system of elliptic-parabolic PDEs) based on flux limited diffusion theory. Numerical simulations of a glass cooling problem in one dimension show that the proposed method is comparable to a higher order discrete ordinate ($\mathbf{S_8}$) method.

Key Words: conduction-radiation transfer, glass cooling, diffusion approximation
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1. Introduction

Radiation transfer in semitransparent medium plays an important role in many industrial applications. For instance, quality of the products in glass industry depends on the proper control of temperature during the various fabricating processes. In many such industrial processes, temperature is generally very high resulting a strong contribution from the radiative transfer to the overall transport of energy within the system. For semitransparent media such as glass with very good infrared transmission, contribution of radiation should not be neglected for temperature higher than $400°C$ [14]. To perform analysis of energy transfer in such situations, it is often necessary to consider the effects of convection, conduction as well as radiative transfer of energy. Since the coupling between the energy equation and the equation of radiative transfer is highly nonlinear, it is of utmost importance to utilize efficient and accurate solution procedures to the radiation part of the overall problem. Consequently, great efforts have been expended in developing such methods. An excellent survey on radiation transfer in participating media with major events in the development of engineering treatment can be found in [3, 5, 12]. Mathematical modeling of coupled radiation - conduction heat transfer in semitransparent media (glass) can be found

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in [2, 10, 11, 14, 15]. The detail mathematical descriptions of the flux limited diffusion theory can be found in [1, 4, 7, 8, 9, 13].

In this paper a simple technique for modeling radiation heat transfer in participating media with an arbitrary varying spectral absorption coefficient is presented and some computational models based on diffusion approximation are developed. Boundary condition is derived using asymptotic matching of slow scale interior solution and fast scale boundary layer solution. Some simulations of combined conduction-radiation transfer in a semitransparent material (glass) with specularly emitting and reflecting bounding walls are presented. Initial simulation results in one dimensional case shows that the diffusion approximation in glass cooling problem is comparable to the high order discrete ordinate ($\mathbf{S_8}$) method [10].

Rest of the paper is organized as follows. Section §2 describes the mathematical modeling for conduction-radiation transfer in semitransparent medium. The formal solution of the radiative transfer equation is presented in §3. The computational models based on the diffusion approximation of the radiative transfer equation are derived in section §4. Some numerical simulation results are presented in §5, with conclusions in §6.

2. Mathematical Model

Let a beam of radiation of intensity I travel in the direction $\mathbf{\Omega}$ along a path s through an absorbing, emitting and scattering medium. The angular distribution of radiation intensity $I(\mathbf{r}(x,y,z),\nu,\mathbf{\Omega}(\theta,\phi),t)$ which is defined as the radiative energy passing through an area per unit solid angle, per unit area projected normal to the direction of passage, and per unit time satisfies the following general equation of transfer governing the radiation field.

$$\begin{aligned}\frac{1}{c_g}\frac{\partial I(\mathbf{r},\nu,\mathbf{\Omega},t)}{\partial t} + \mathbf{\Omega}\cdot\nabla I(\mathbf{r},\nu,\mathbf{\Omega},t) &= -\left[k(\nu)+\sigma(\nu)\right]I(\mathbf{r},\nu,\mathbf{\Omega},t) + j^e(\nu,\mathbf{r},t)\\ &\quad + \frac{1}{4\pi}\sigma(\nu)\int\limits_{\mathbf{\Omega}'=4\pi} P(\mathbf{\Omega}',\mathbf{\Omega})I(\mathbf{r},\nu,\mathbf{\Omega},t)d\Omega' \qquad (2.1)\end{aligned}$$

where c_g is the speed of propagation of light in the medium; ν is wavelength; κ is spectral absorption coefficient; σ is spectral scattering coefficient; $j^e(\nu,\mathbf{r},t)$ is radiation emitted by the matter per unit time, volume, solid angle and frequency; $P(\mathbf{\Omega}',\mathbf{\Omega})$ is the probability that incident radiation at $\mathbf{\Omega}$ will be scattered into an element of solid angle $d\Omega$ about the direction $\mathbf{\Omega}'$.

To derive a model for the semitransparent medium (glass), we make the following simplification assumptions.

(1) Due to the large magnitude of the speed propagation c, for most of engineering applications the first term of left hand side of equation (2.1) i.e., $\frac{1}{c_g}\frac{\partial I}{\partial t}$ can be neglected in comparison to other terms
(2) Since we are concerned with *semitransparent* medium, we set $\sigma(\nu) = 0$. (Purely absorbing and emitting medium)
(3) However, in equation (2.1) the intensity depends on time. Therefore we can assume time variable s a parameter. We also assume that a local thermodynamic equilibrium (LTE) is established and the Kirchhoff's law is valid. Then the emission of radiation

$j^e(\nu, \mathbf{r}, t)$ in any direction is related to the spectral absorption coefficient and the blackbody radiation intensity at the temperature $T(\mathbf{r})$ of the medium is given by

$$j^e(\nu, \mathbf{r}) = k(\nu) I_b(\nu, T(\mathbf{r})),$$

where $k(\nu) =$ absorption (= emissivity) coefficient and the spectral intensity of black body radiation is given by the Planck's function. i.e.,

$$I_b(\nu, T) = \frac{2h_p\nu^3}{c_g^2\left(\exp\left(\frac{h_p\nu}{k_bT} - 1\right)\right)} \tag{2.2}$$

(4) Finally we assume that the radiative properties changes with frequency of radiation and hence a nongray analysis of the problem is necessary. To this end, various models have been developed as a simplification of frequency dependent radiative properties [5]. In our present model we will adopt a multi-band model for absorption coefficient $k(\nu)$. i.e., we approximate the spectral absorption coefficient in each of the frequency band $\nu_k \leq \nu\nu_{k+1}$ by a piecewise constant function $k(\nu)$. Thus we have the following notational convention: For $\nu \in [\nu_k, \nu_{k+1}), k(\nu) := k(\nu_k) = k_k$ and

$$\mathbf{I^{(k)}}(\mathbf{r}, \mathbf{\Omega}) := \int_{v_k}^{v_{k+1}} I(\mathbf{r}, \nu, \mathbf{\Omega}) d\nu \text{ and } I_b^{(k)}(T(\mathbf{r})) := \int_{v_k}^{v_{k+1}} I_b(\nu, T(\mathbf{r})) d\nu$$

Radiative Transfer Equation

Under the above assumptions, the radiative heat transfer equation (2.1) reduces to

$$\mathbf{\Omega} \cdot \nabla I^{(k)}(\mathbf{r}, \mathbf{\Omega}) = k_k \left(I_b^{(k)}(T(\mathbf{r})) - I^{(k)}(\mathbf{r}, \mathbf{\Omega})\right) \tag{2.3}$$

The intensity of radiation $I^{(k)}$ leaving the boundary surface at $\boldsymbol{r_s}$ in the direction of $\mathbf{\Omega}$ is composed of two contributions - one is the transmitted into medium from the surrounding and other is the reflected internal intensity. We choose specularly emitting and specularly reflecting boundary model for our problem [5]. We assume that the surface is smooth and incident and reflected rays lie symmetrically with respect to the normal at the point of incidence and the reflected beam is contained within the solid angle $d\Omega$ equal to the solid angle of incidence $d\Omega'$. i.e., $d\Omega = d\Omega'$.

$$I^{(k)}(\boldsymbol{r_s}, \mathbf{\Omega}) = \underbrace{\epsilon\tau(\mathbf{\Omega}, \Omega'') I_b^{(k)}(T(\boldsymbol{r_s}))}_{\text{specular emitance}} + \underbrace{\rho(\mathbf{\Omega}, \mathbf{\Omega}') I^{(k)}(\boldsymbol{r_s}, \mathbf{\Omega}_s)}_{\text{specular reflectance}}, \quad \tau + \rho = 1 \tag{2.4}$$

where ϵ is the emissivity, $\rho(\mathbf{\Omega}, \mathbf{\Omega}')$ and $\tau(\mathbf{\Omega}, \mathbf{\Omega}'')$ are the boundary reflecivities from direction $\mathbf{\Omega}'$ to $\mathbf{\Omega}$ and from $\mathbf{\Omega}''$ to $\mathbf{\Omega}$ respectively, where $\mathbf{\Omega}'$ and $\mathbf{\Omega}''$ are the $s$pecular directions with which the beam of rays hit the surface in order to travel into the direction $\mathbf{\Omega}$ after a specular reflection.

Energy Equation

The equation for conservation of energy in an isotropic, homogeneous medium which participates in the radiative transfer can be obtained by making an energy balance on an arbitrary volume of matter as

$$\rho_g C(T) \frac{\partial T}{\partial t} = \nabla \cdot (K_c(T) \nabla T - \boldsymbol{q^r}) + \dot{q} \tag{2.5}$$

where ρ_g is the density of the material, $C(T)$ is the specific heat capacity of the medium, $K_c(T)$ is the thermal conductivity, $\boldsymbol{q^r}$ is the radiation heat flux vector and $\dot{q}$ is the internal heat generation rate.

Considering only a diffusion external radiation source, energy balance on the boundary gives the boundary conditions for the temperature as

$$K_c(T)\frac{\partial T}{\partial n}(\boldsymbol{r_s}(x,y,z),t) + h[T_{out}(\boldsymbol{r_s}(x,y,z)) - T(\boldsymbol{r_s}(x,y,z))] + g = 0 \tag{2.6}$$

where

$$g(\boldsymbol{r_s}(x,y,z)) = \pi\epsilon\sum_k [I_{bo}^{(k)}(T_{out}(\boldsymbol{r_s}(x,y,z))) - I_{bo}^{(k)}(T(\boldsymbol{r_s}(x,y,z)))]$$

where h is the convection film coefficient, ϵ is the boundary emissivity in opaque spectral region, $\boldsymbol{r_s}$ is the point on the boundary surface, T_{out} is the surrounding temperature, K_c is the thermal conductivity of the material and $I_{bo}^{(k)}(T)$ is the blackbody radiation intensity at temperature T for an opaque spectral band k.

Divergence of Radiative Heat Flux

Integrating equation (2), first over all solid angles and then over the entire spectrum, we get the divergence of radiative heat flux which is required in the total energy balance equation (2.4) as

$$\nabla\cdot\boldsymbol{q^r} = \sum_k \nabla\cdot\boldsymbol{q^r}^{(k)} = \sum_k k_k\left[4\pi I_b^{(k)}(T) - G^{(k)}\right] \tag{2.7}$$

where radiative heat flux $\boldsymbol{q^r}$ and incident radiation G for the k^{th} spectral band are

$$\boldsymbol{q^r}^{(k)} = \int\limits_{\Omega=4\pi} I^{(k)}\,\boldsymbol{\Omega}\,d\Omega, \qquad G^{(k)} = \int\limits_{\Omega=4\pi} I^{(k)}\,d\Omega. \tag{2.8}$$

The term $4\pi k_k I_b^{(k)}(T)$ and $k_k G^{(k)}$ in (2.7) respectively represent local rate of emission and the local rate of absorption per unit volume and frequency.

3. Formal Solution of Radiative Transfer Equation

Choose a length s as the distance back along $\boldsymbol{\Omega}$ from the point $\boldsymbol{r}$. Then equation (2.3) can be rewritten as

$$-\frac{\partial I^{(k)}(\boldsymbol{r}-s\boldsymbol{\Omega},\boldsymbol{\Omega})}{\partial s} + k_k I^{(k)}(\boldsymbol{r}-s\boldsymbol{\Omega},\boldsymbol{\Omega}) = k_k I_b^{(k)}(T(\boldsymbol{r}-s\boldsymbol{\Omega})) \tag{3.1}$$

which is a first order differential equation in $I^{(k)}$ whose solution can be easily found by taking $e^{(s_0-s)k_k}$ as an integrating factor, where s_0 is an arbitrary point along s.

Integrating equation (3.1) from s to s_0, we get

$$\begin{aligned} I^{(k)}(\boldsymbol{r}-s\boldsymbol{\Omega},\boldsymbol{\Omega}) &= I^{(k)}(\boldsymbol{r}-s_0\Omega,\Omega)e^{(s-s_0)k_k} \\ &\quad + k_k\int_s^{s_0} I_b^{(k)}(T(\boldsymbol{r}-s'\boldsymbol{\Omega}))e^{(s-s')k_k}ds' \end{aligned} \tag{3.2}$$

Set, $s = 0$, we get the specific intensity at the point $\boldsymbol{r}$ in the direction of $\boldsymbol{\Omega}$ as

$$I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = I^{(k)}(\boldsymbol{r} - s_0\boldsymbol{\Omega}, \boldsymbol{\Omega})e^{-s_0 k_k} + k_k \int_0^{s_0} I_b^{(k)}(T(\boldsymbol{r} - s'\boldsymbol{\Omega}))e^{-s' k_k} ds' \tag{3.3}$$

The constant of integration $I^{(k)}(\boldsymbol{r} - s_0\boldsymbol{\Omega}, \boldsymbol{\Omega})$ in equation (3.3) can be evaluated by using the formal boundary conditions.

$$I^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}) = \Gamma^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}), \boldsymbol{n} \cdot \Omega < 0 \tag{3.4}$$

where $\Gamma^{(k)}$ is the prescribed boundary data and $\boldsymbol{n}$ is a unit outward normal at the surface point $\boldsymbol{r_s}$.

Let s_0 be such that $\boldsymbol{r} - s_0\Omega = \boldsymbol{r_s}$ is a point on the boundary. i.e., $s_0 = |\boldsymbol{r} - \boldsymbol{r_s}|$ and from the boundary condition (3.4), we get

$$\left(I^{(k)}(\boldsymbol{r} - s_0\boldsymbol{\Omega}, \boldsymbol{\Omega})\right)_{s_0 = |\boldsymbol{r} - \boldsymbol{r_s}|} = \Gamma^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}) \tag{3.5}$$

replacing s_0 by $|\boldsymbol{r} - \boldsymbol{r_s}|$ in (3.5) we get

$$I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = \Gamma^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega})e^{-|\boldsymbol{r} - \boldsymbol{r_s}| k_k} + \int_0^{|\boldsymbol{r} - \boldsymbol{r_s}|} I_b^{(k)}(T(\boldsymbol{r} - s'\boldsymbol{\Omega}))e^{-s' k_k} ds' \tag{3.6}$$

Equation (3.5) is an explicit expression for the radiation intensity if the temperature field is known and the function $\Gamma^{(k)}$ is a specified function of $\boldsymbol{r_s}$ and $\boldsymbol{\Omega}$. However, generally the temperature field is not known and must be found in conjunction with the overall conservation of energy and the prescribed boundary data involve the radiation intensity from the interior of the medium that is reflected by the boundary surface and hence depends on an unknown quantity.

4. Diffusion Approximation to Radiative Transfer Equation

One of the earliest diffusion type approximations for the radiation transfer was due to Rosseland (1936) which expresses the radiative heat flux in terms of gradient of radiative intensity.

$$\boldsymbol{q^r} = -K^r(T)\nabla T, \quad K^r(T) = \frac{4\pi}{3k(\nu)} \frac{\partial I_b}{\partial T} \tag{4.1}$$

In the classical diffusion or Eddington approximation specific intensity of radiation is represented by the first two terms in a spherical harmonic expansion. The resulting expression has the form of the Ficks Law of diffusion which is usually expressed as

$$\boldsymbol{q^r} = -D_F \nabla G, \quad D_F = \frac{1}{3k(\nu)}. \tag{4.2}$$

In the presence of steep spatial gradients equations (4.1) and (4.2) predict radiative fluxes which are nonphysically large. But from (2.8), we have

$$|\boldsymbol{q}^{r(k)}| \leq G^{(k)} = c_g E^{(k)} \tag{4.3}$$

That is, the exact value of radiative heat flux cannot exceed the product of speed of propagation and energy density. Taking this into account, Levermore and Pomraning (1981) proposed the flux limited diffusion theory [4] which was based on Chapman-Enskog Asymptotic method, originally developed for treating the nonlinear Boltzman equation, to the radiative transfer equation. Sanchez and Pomraning (1990) generalized this and presented as a family of flux limiting diffusion theory with improved accuracy over the Eddington approximation and the earlier flux limited diffusion theory [8], which was further generalized by Szilard and Pomraning (1993) [13].

Derivation of Flux-limiting Diffusion Approximation

This method assumes the separability condition of the specific intensity $I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega})$ such that

$$I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = G^{(k)}(\boldsymbol{r})\psi^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) \tag{4.4}$$

where for consistency with an angular integration of (4.4) $\psi^{(k)}$ is normalized via

$$\int\limits_{\Omega=4\pi} \psi^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega})\, d\Omega = 1 \tag{4.5}$$

Using equation (4.4) in the transfer equation (2.3), we get

$$\Big(\boldsymbol{\Omega} \cdot \nabla G^{(k)}(\boldsymbol{r}) + k_k G^{(k)}(\boldsymbol{r})\Big)\, \psi^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = k_k I_b^{(k)}(T(\boldsymbol{r})) \tag{4.6}$$

We assume that the space dependence of $\psi^{(k)}$ is sufficiently weak and, in the lowest order, can be neglected i.e., $\nabla\psi^{(k)} = 0$. This assumption is justified for two extreme limiting cases: isotropic limit given by Eddignton approximation and the asymptotic limit for the case of collimated irradiation [3].

In Eddington approximation, from (21) and (6) we have

$$I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = \frac{1}{4\pi}\left(G^{(k)}(\boldsymbol{r}) - \boldsymbol{\Omega} \cdot \frac{1}{k_k}\nabla G^{(k)}(\boldsymbol{r})\right)$$
$$\text{i.e., } \psi^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = \frac{1}{4\pi}\left(-\boldsymbol{\Omega} \cdot \frac{1}{k_k G^{(k)}}\nabla G^{(k)}\right)$$

where $\nabla G^{(k)}$ is assumed to be small. If $\nabla G^{(k)} = 0$ (i.e. $\nabla\psi^{(k)} = 0$) we get the equilibrium approximation. This corresponds to the case when radiation field is local Plankian, only temperature dependent.

In the case of collimated irradiation, the solution (3.5) to the radiative transfer equation has the following form [3]

$$I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = (1 - \rho) q_c(\boldsymbol{r_s}) \delta(\boldsymbol{\Omega} - \boldsymbol{\Omega_c}(\boldsymbol{r_s})) e^{-\tau_c} \tag{4.7}$$

Integrating (4.7) over all solid angle, we obtain

$$G^{(k)}(\boldsymbol{r}) = (1 - \rho(\mu_c, \phi_c)) q_c(\boldsymbol{r_s}) e^{-\tau_c} \tag{4.8}$$
$$\text{i.e., } \psi^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = \delta(\boldsymbol{\Omega} - \boldsymbol{\Omega_c}(\boldsymbol{r_s}))$$

Integrating (4.6) over all solid angle and using (4.5) we get the corresponding conservation equation as

$$\boldsymbol{f}^{(k)} \cdot \nabla G^{(k)}(\boldsymbol{r}) = k_k \left(4\pi I_b^{(k)}(T(\boldsymbol{r})) - G^{(k)}(\boldsymbol{r})\right) \tag{4.9}$$

where we have defined the normalized flux $\boldsymbol{f}^{(k)}(\boldsymbol{r})$

$$f^{(k)}(\boldsymbol{r}) = \frac{q^{(k)}(\boldsymbol{r})}{G^{(k)}(\boldsymbol{r})} = \int\limits_{\Omega=4\pi} \psi^{(k)}(\boldsymbol{r},\boldsymbol{\Omega})\boldsymbol{\Omega} d\Omega \tag{4.10}$$

Subtracting (4.9) from (4.6), we get

$$k_k I_b^{(k)} = \left(\boldsymbol{\Omega}\cdot\nabla G^{(k)} - \boldsymbol{f}^{(k)}\cdot\nabla G^{(k)} + k_k 4\pi I_b^{(k)}\right)\psi^{(k)}$$

$$\Rightarrow \psi^{(k)} = \frac{1}{4\pi}\left(\frac{1}{1+\boldsymbol{f}^{(k)}\cdot\boldsymbol{R}-\boldsymbol{\Omega}\cdot\boldsymbol{R}}\right) \tag{4.11}$$

where

$$\boldsymbol{R} = -\frac{\nabla G^{(k)}}{4\pi k_k I_b^{(k)}} \tag{4.12}$$

The scalar quantity $R = |\boldsymbol{R}|$ is sometimes refered to the flux limiting parameter and it is a dimensionless measure of the strength of the spatial gradient of the incident intensity. We know from Fick's law of mass diffusion that heat flux vector and gradient of energy density are always (anti)parallel. From (4.10) and (4.12), we get $\boldsymbol{f}^{(k)}||\boldsymbol{q}^{\boldsymbol{r}(\boldsymbol{k})}||$ and $\boldsymbol{R}||\nabla G^{(k)}||$. Therefore, we conclude that

$$\boldsymbol{f}^{(\boldsymbol{k})} = \lambda\boldsymbol{R} \tag{4.13}$$

$$\Rightarrow \psi^{(k)} = \frac{1}{4\pi}\left(\frac{1}{1+\lambda R^2-\boldsymbol{\Omega}\cdot\boldsymbol{R}}\right) \tag{4.14}$$

To obtain the proportionality function $\lambda = \lambda(R)$ in (4.13) we use the normalization condition (4.5) in (4.14).

$$\begin{aligned}
1 =& \frac{1}{4\pi}\int\limits_{\Omega=4\pi}\frac{d\Omega}{1+\lambda R^2-\boldsymbol{\Omega}\cdot\boldsymbol{R}} = \frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^{\pi}\frac{\sin\theta d\theta\, d\phi}{1+\lambda R^2-\cos\theta R} \\
=& \frac{1}{2R}\int\limits_{-R}^{R}\frac{dt}{1+\lambda R^2+t} = \frac{1}{2R}\ln\left(\frac{1+\lambda R^2+R}{1+\lambda R^2-R}\right) \\
=& \frac{1}{R}\tanh^{-1}\left(\frac{R}{1+\lambda R^2}\right)
\end{aligned} \tag{4.15}$$

Solving this for λ, we get

$$\lambda(R) = \frac{1}{R}\left(\coth(R) - \frac{1}{R}\right) \tag{4.16}$$

$$\Rightarrow \psi^{(k)} = \frac{1}{4\pi}\left(\frac{1}{R\coth(R)-\boldsymbol{\Omega}\cdot\boldsymbol{R}}\right) \tag{4.17}$$

From (4.5), (4.14), (4.15), and (4.17) we can derive the Fick's Law of diffusion as

$$\boldsymbol{q}^{\boldsymbol{r}(\boldsymbol{k})} = -D_F^{(k)}\nabla G^{(k)} \tag{4.18}$$

where the dimensionless diffusion coefficient D_F is given by

$$D_F^{(k)}(\omega, R) = \frac{\lambda(R)}{\omega k_k}, \omega = \frac{4\pi I_b^{(k)}}{G^{(k)}} \tag{4.19}$$

Thus we get the diffusion equation for the incident radiation as

$$-\nabla \frac{\lambda(R)}{\omega k_k} \nabla G^{(k)} = k_k \left(4\pi I_b^{(k)} - G^{(k)}\right) \tag{4.20}$$

We derive an appropriate boundary conditions for the diffusion equation (4.20) based on the asymptotic matching between the slow scale interior solution governed by the flux limited diffusion equation and the fast scale boundary layer solution. The following description parallels to Pomraning [7] where he has used the case discrete normal mode to include more complicated transport phenomenon. Consider the equation of transfer (2.3) and the boundary condition (2.4)

$$\boldsymbol{\Omega} \cdot \nabla I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = k_k \left(I_b^{(k)}(T(\boldsymbol{r})) - I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega})\right) \tag{4.21}$$

$$I^{(k)}(\boldsymbol{r_s}, \Omega) = \epsilon\tau(\boldsymbol{\Omega}, \boldsymbol{\Omega}'') I_b^{(k)}(T(\boldsymbol{r_s})) + \rho(\boldsymbol{\Omega}, \boldsymbol{\Omega}') I^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}) \tag{4.22}$$

We decompose the specific intensity of radiation $I^{(k)}$ into the sum of two intensities $I_{in}^{(k)}$ and $I_{out}^{(k)}$ as

$$I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = \underbrace{I_{in}^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega})}_{\text{LP diffusion sol.}} + \underbrace{I_{out}^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega})}_{\text{Boundary Layer sol.}} \tag{4.23}$$

where $I_{in}^{(k)}$ is the interior solution presumed to be an accurate approximation of $I^{(k)}$ away from the boundary layer and $I_{out}^{(k)}$ is the contribution to the boundary (near surface) layer which is presumed to decay rapidly in space in a direction normal to the surface. Both the intensities $I_{in}^{(k)}$ and $I_{out}^{(k)}$ satisfy the transfer equation (4.21) i.e.,

$$\boldsymbol{\Omega} \cdot \nabla I_{in}^{(k)} + k_k I_{in}^{(k)} = k_k I_b^{(k)} \tag{4.24}$$

(4.25)

where $\mu = \boldsymbol{n} \cdot \boldsymbol{\Omega}$ (cosine of angle between the incident ray and the z-axis), $\boldsymbol{n}$ is unit outward normal at the surface point $\boldsymbol{r_s}$. Due to the assumed dominance of the normal spatial derivative, we have neglected the tangential derivative in equation (4.25). Since $I_{out}^{(k)}$ is assumed to decay rapidly with z, in particular faster than the exponential mean free path characteristic distance i.e., faster than $e^{-k_k z}$, this equation holds for $0 \le z < \infty$. Thus, the boundary condition for (4.25) can be given by combining (4.22) and (4.23) as

$$I_{in}^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}) + I_{out}^{(k)}((\boldsymbol{r_s})_{z=0}, \boldsymbol{\Omega}) = I^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}), \mu > 0 \tag{4.26}$$

To remove the ϕ dependence from equation (4.25) and (4.26), we define $\bar{h}(\mu)$ as

$$\bar{h}(\mu) = \int_0^{2\pi} h(\boldsymbol{\Omega}) d\Omega$$

for any function $h(\boldsymbol{\Omega})$ and integrate (4.25) and (4.26) over ϕ. This gives

$$\mu \frac{\partial \bar{I}_{out}^{(k)}}{\partial z} + k_k \bar{I}_{out}^{(k)} = 0, 0 \le z < \infty \tag{4.27}$$

$$\bar{I}_{out}^{(k)}((\boldsymbol{r_s})_{z=0}, \mu) = \bar{I}^{(k)}((\boldsymbol{r_s}), \mu) - \bar{I}_{in}^{(k)}(\boldsymbol{r_s}, \mu), \mu > 0 \tag{4.28}$$

(4.27) being a linear ordinary differential equation of first order, we can write its solution as

$$\bar{I}^{(k)}_{out} = \bar{I}^{(k)}_{out}((\boldsymbol{r_s})_{z=0}, \mu)e^{\frac{-k_k z}{\mu}} \tag{4.29}$$

Clearly, (4.29) vanished at $z = \infty$, but since $\mu > 0$, it must satisfy

$$\bar{I}^{(k)}_{out}((\boldsymbol{r_s})_{z=0}, \mu) = 0. \tag{4.30}$$

If it is to decay faster than $e^{-k_k z}$ i.e., for a small layer near the boundary, the incoming intensity right hand side of (4.28) can be neglected. Now, multiply (4.28) by some weight function $W(\mu)$ and integrating w.r. to μ and using (4.30) we get

$$\int_0^1 \left(\bar{I}^{(k)}((\boldsymbol{r_s}), \mu) - \bar{I}^{(k)}_{in}((\boldsymbol{r_s}), \mu\right) W(\mu)d\mu = 0 \tag{4.31}$$

Since $I^{(k)}_{in}$ is supposed to be accurately described by the LP flux limited diffusion theory, we have from (4.30)

$$\bar{I}^{(k)}(\boldsymbol{r_s}, \mu) = G^{(k)}(\boldsymbol{r_s})\bar{\psi}^{(k)}(\mu)$$

and from (4.13)

$$\bar{\psi}^{(k)}(\mu) = \int_0^{2\pi} \psi^{(k)}(\boldsymbol{\Omega})d\phi = \int_0^{2\pi} \frac{1}{4\pi}\left(\frac{1}{R\coth(R) - \boldsymbol{\Omega}\cdot\boldsymbol{R}}\right)d\phi$$

$$= \frac{1}{2R^2}\left(\frac{R\coth(R) + \mu R}{\coth^2(R) - \mu^2}\right) = \frac{1}{2R^2}\left(\frac{R\coth(R) - \mu\frac{\partial G^{(k)}}{\partial z}/k_k\omega}{\coth^2(R) - \mu^2}\right)$$

we get

$$\bar{I}^{(k)}_{in}(\boldsymbol{r_s}, \mu) = \frac{1}{2R^2}\left(\frac{R\coth RG^{(k)}(\boldsymbol{r_s}) + \frac{\mu}{k_k\omega}(\boldsymbol{n}\cdot\nabla G^{(k)}(\boldsymbol{r_s}))}{\coth^2(R) - \mu^2}\right) \tag{4.32}$$

Using (4.32) in (4.31) and choosing $W(\mu) = \mu$ (as usual Marshack's choice [5]) we obtain

$$\frac{1}{2}\int_0^1 \left(\frac{\coth RG^{(k)}(\boldsymbol{r_s})}{R(\coth^2 R - \mu^2)} + \frac{\boldsymbol{n}\cdot\nabla G^{(k)}(\boldsymbol{r_s})}{R^2(\coth^2 R - \mu^2)k_k\omega}\right)\mu d\mu = \int_0^1 \bar{I}^{(k)}(\boldsymbol{r_s}, \mu)\mu d\mu$$

$$\text{i.e., } \frac{1}{2}\left(\alpha G^{(k)}(\boldsymbol{r})\boldsymbol{s}) + \frac{\beta}{k_k\omega}\boldsymbol{n}\cdot\nabla G^{(k)}(\boldsymbol{r_s})\right) = \int_{\boldsymbol{n}\cdot\boldsymbol{\Omega}<0} |\boldsymbol{n}\cdot\boldsymbol{\Omega}|I^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega})d\Omega \tag{4.33}$$

where $\boldsymbol{n}$ is a unit outward normal vector at the surface point $\boldsymbol{r_s}$,

$$\alpha = \int_0^1 \frac{\coth R\mu d\mu}{R(\coth^2 R - \mu^2)} = \frac{\coth R}{R}\ln|\coth^2 R/(\coth^2 R - 1)| \tag{4.34}$$

and

$$\beta = \int_0^1 \frac{\mu d\mu}{R(\coth^2 R - \mu^2)} = \frac{1}{R}\ln|\coth^2 R/(\coth^2 R - 1)| \tag{4.35}$$

5. Numerical Computations

Now we present some numerical simulations employing a semi-implicit finite volume scheme for a glass cooling problem in one dimension. The glass has an initial temperature of $600°C$ uniformly distributed and start cooling down through radiation on the boundary surface which is directly exposed to the surrounding temperature of $T_{out} = 26.85°C$. The convection heat transfer coefficient (h) and the boundary emissivity (ϵ) both are chosen to be 0.89. But effect of convection in the total energy transport is found to be almost non-influencing. The refractive index for the glass is assumed to be constant $Rn_g = 1.46$. Thermal conductivity $K(T)$, specific heat capacity $C(T)$ and density of glass ρ_g are also taken to be constant throughout the computations. Absorption coefficient of glass k as a function of wave number for various spectral band widths is shown in the table 1. The

Table 1. Absorption coefficient for glass

η	1000-2000	2000-2500	2500-3500	3500-6000	6000-10000	10000-200000
k	80.0	35.0	10.0	1.0	0.01	0.02

boundary reflectivity ρ, τ are direction dependent and cannot be considered as constants. We assume that radiation on the boundary is unpolarized, obeys the laws of refraction and ρ is expressed in terms of incident angle θ as [5]

$$\rho = \frac{1}{2}\left\{\left(\frac{\sqrt{N-1+\mu^2}-\mu}{\sqrt{N-1+\mu^2}+\mu}\right)^2 + \left(\frac{N\mu-\sqrt{N-1+\mu^2}}{N\mu-\sqrt{N+1+\mu^2}}\right)^2\right\} \tag{5.1}$$

where

$$\mu = \cos\theta, N = 1/Rn_g^2.$$

The integrals involving blackbody radiation function (2.2) and the boundary reflectivity function (5.1) are numerically approximated using the 6-point Gaussian quadrature formula. Step sizes for spatial and temporal direction are taken to be 0.01 and 0.02 respectively and the simulations up to 100 seconds are reported. Since there is no exact solution, we use the results obtained by a higher order discrete ordinates ($\mathbf{S_8}$) method [15] as the basis for our comparison.

Temperature profiles at various times obtained from pure conduction and the Rosseland approximation are shown in figure 1. Here, by the pure conduction, we mean only the radiation term in the energy equation is neglected but only the surface radiation given by the boundary conditions is taken into account. Solution of the energy equation without consideration of all the radiation effect, being constant, is of no interest. Temperature distribution predicted by Eddington approximation and Levermore-Pomraning flux-limited diffusion approximation are presented in figure 2. In figure 3 the temperature profiles from Flux-limited diffusion and discrete ordinate methods are plotted together for comparison.

From figures 1 - 3, we see that the Rosseland approximation underestimates temperature profile where as the pure conduction overestimates the same. It is natural to expect higher temperature profile from the pure conduction for the case of glass which has a very

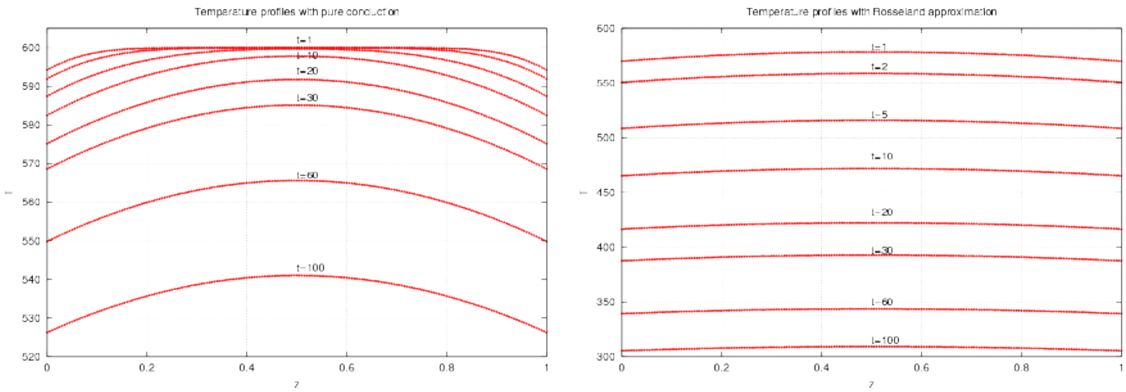

FIGURE 1. Temperature at t=1, 2, 5, 10, 20, 30, 60, 100s using pure conduction (left), Rosseland aproximation (right)

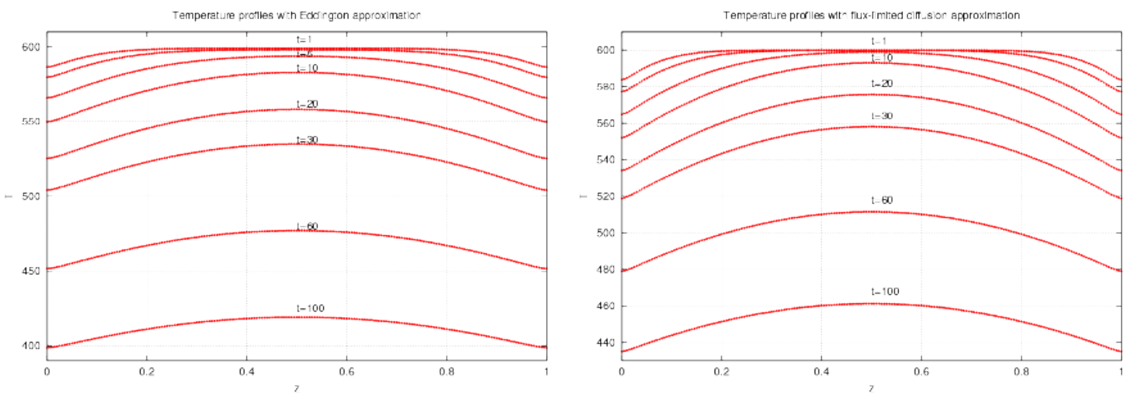

FIGURE 2. Temperature at t=1, 2, 5, 10, 20, 30, 60, 100s using Eddington approximation (left), flux-limited diffusion approximation (right)

good infrared transmission property for which at temperatures higher than $400°C$ radiation is a major influencing component in the energy transport. If we compare the Rosseland approximation with the Flux limited diffusion approximation, we observe an increasing discrepancies with the increase in time. This is because results Rosseland approximation is valid only for optically thick medium ($\tau_{l,k} >> 1$). For the characteristic length $l = 1$ cm and the absorption coefficient $\kappa = 0.01$, the optical dimension $\tau_{(l,k)} = 0.01 << 1$. Figure 3 shows that the Levermore-Pomraning flux limited diffusion approximation with appropriately chosen boundary condition is comparable with the higher order discrete ordinate ($\mathbf{S_8}$) method.

6. CONCLUSION

In this paper, a straightforward coupling of conduction and radiation using the diffusion approximation for the radiative transfer equation is presented. Based on the one dimensional simulations of a glass cooling problem, the flux limited diffusion descriptions with appropriately chosen boundary conditions is found to be comparable with the sophisticated discrete ordinate $\mathbf{S_8}$ method. The diffusion approximation method has several advantages.

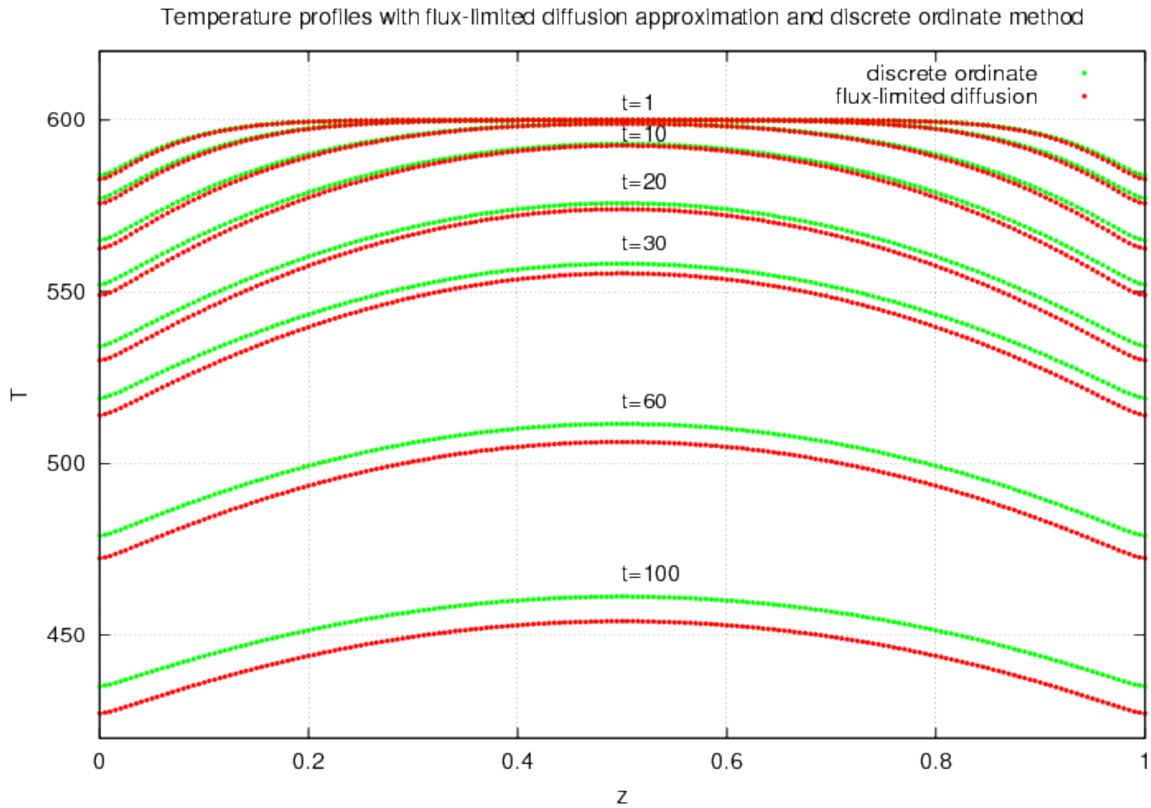

FIGURE 3. Temperature profiles at different times as predicted by discrete ordinate ($\mathbf{S_8}$) method (green) and flux-limited diffusion approximation (red)

It is easy to implement for any spectral variation of the absorption coefficients and for multidimensional cases with arbitrary geometry.

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