INVARIANCE OF DOMAIN FOR OPERATORS
OF CLASS $\mathcal{A}_G(S_+)$

DHRUBA R. ADHIKARI

Department of Mathematics
Kennesaw State University, Gerogia 30060, USA
dadhikar@kennesaw.edu

Abstract: Let X be a real reflexive Banach space and X^* its dual space. Let $T : X \supset D(T) \to 2^{X^*}$ an operator of class $\mathcal{A}_G(S_+)$, where $G \subset X$ is open. An invariance of domain result for T is established. This result extends a similar result of Park for single-valued operators of type (S_+). The Skrypnik's topological degree theory is used, utilizing approximating schemes of mappings of class $\mathcal{A}_G(S_+)$, along with the methodology of a recent invariance of domain result by Kartsatos and the author.

Key Words: Browder and Skrypnik degree theory, invariance of domain, bounded demicontinuous operator of type (S_+)

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1. Introduction–Preliminaries

Let X be a real reflexive Banach space with X^* its dual space. The norms of X, X^* will be denoted by $\|\cdot\|$ and will be understood from the context in which the symbol is used. We denote by $\langle x^*, x \rangle$ the value of the functional $x^* \in X^*$ at $x \in X$. The symbols ∂D, $\overline{D}$ denote the strong boundary and closure of the set D, respectively. The symbol $B(x_0, r)$ denotes the open ball of radius r with center at x_0.

If $\{x_n\}$ is a sequence in X, we denote its strong convergence to x_0 in X by $x_n \to x_0$ and its weak convergence to x_0 in X by $x_n \rightharpoonup x_0$. An operator $T : X \supset D(T) \to Y$ is said to be "bounded" if it maps bounded subsets of the domain $D(T)$ onto bounded subsets of Y. The operator T is said to be "compact" if it maps bounded subsets of $D(T)$ onto relatively compact subsets in Y. It is said to be "demicontinuous" if it is strong-weak continuous on $D(T)$. The symbols $\mathbf{R}$ and $\mathbf{R}_+$ denote $(-\infty, \infty)$ and $[0, \infty)$, respectively. The normalized duality mapping $J : X \supset D(J) \to 2^{X^*}$ is defined by

$$Jx = \{x^* \in X^* : \langle x^*, x \rangle = \|x\|^2, \ \|x^*\| = \|x\|\}, \ x \in X.$$

The Hahn-Banach theorem ensures that $D(J) = X$, and therefore $J : X \to 2^{X^*}$ is a multi-valued mapping defined on the whole space X. By a well-known renorming theorem due to Trojanski [15], one can always renorm the reflexive Banach space X with an equivalent norm with respect to which both X and X^* become locally uniformly convex (therefore strictly

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convex). Henceforth, we assume that X is a locally uniformly convex reflexive Banach space. With this setting, the normalized duality mapping J is single-valued homeomorphism from X onto X^*.

For a multivalued operator T from X to X^*, we write $T : X \supset D(T) \to 2^{X^*}$, where $D(T) = \{x \in X \ : \ Tx \neq \emptyset\}$ is the effective domain of T. We denote by $Gr(T)$ the graph of T, i.e., $Gr(T) = \{(x, y) \ : \ x \in D(T), \ y \in Tx\}$.

An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "monotone" if for every $x, \ y \in D(T)$ and every $u \in Tx, \ v \in Ty$ we have

$$\langle u - v, x - y \rangle \geq 0.$$

A monotone operator T is said to be "maximal monotone" if $Gr(T)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by the set inclusion. In our setting, a monotone operator T is maximal if and only if $R(T + \lambda J) = X^*$ for all $\lambda \in (0, \infty)$.

Definition 1.1. An operator $C : X \supset D(C) \to X^*$ is said to be of type (S_+) if for every sequence $\{x_n\} \subset D(C)$ with $x_n \rightharpoonup x_0$ in X and

$$\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0 \in \overline{D(C)}$ in X.

Definition 1.2. The family $C(t) : X \supset D \to X^*, t \in [0, 1]$, of operators is said to be a "homotopy of type (S_+)" if for any sequences $\{x_n\} \subset D$ with $x_n \rightharpoonup x_0$ in X and $\{t_n\} \subset [0, 1]$ with $t_n \to t_0$ and

$$\limsup_{n \to \infty} \langle C(t_n)x_n, x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0$ in X, $x_0 \in D$ and $C(t_n)x_n \rightharpoonup C(t_0)x_0$ in X^*. A homotopy of type (S_+) is "bounded" if the set

$$\{C(t)x \mid t \in [0, 1], \ x \in D\}$$

is bounded.

We now define the class $\mathcal{A}_G(S_+)$ of multivalued operators, where is introduced by Kittila in [8].

Definition 1.3. Let G be an open subset of X. An operator $T : X \supset D(T) \to 2^{X^*}$ is of class $\mathcal{A}_G(S_+)$ if there exists a sequence (T_n) (called an approximating sequence of T) of bounded demicontinuous mappings of type (S_+) from $\overline{G}$ to X^* with the following properties.

(A1) For each $C > 0$ there exists $K \geq 0$ such that $\langle T_n(x), x \rangle \geq -K$ for all $u \in \overline{G}, \|x\| \leq C$ and for all $n \in \mathbb{N}$.

(A2) Let $\{t_n\} \subset [0, 1]$, $(u_n) \subset \overline{G}$ and let $\{T_{m_n}\}$ be any subsequence of $\{T_n\}$. If $t_n \to 0$, $x_n \to x$ in X and $t_n T_{m_n}(x_n) \rightharpoonup z$ in X^*, then $z = 0$.

(A3) Let $\{x_n\} \subset \overline{G}$ and $\{T_{m_n}\}$ be any subsequence of $\{T_n\}$. If $x_n \rightharpoonup x$ in X, $T_{m_n}(x_n) \rightharpoonup w$ in X^* and

$$\limsup_{n \to \infty} \langle T_{m_n} x_n), x_n \rangle \leq \langle w, x \rangle,$$

then $x_n \to x$ in X, $x \in D(T)$ and $w \in T(x)$.

The main purpose of this paper is to give an invariance of domain result for operators of class $\mathcal{A}_G(S_+)$. These operators are generalizations of operators of type (S_+) and have an approximating scheme in terms of bounded demicontinuous (S_+)-operators as given in Definition 1.3, and therefore this paper generalizes results of Park in [10]. A multivalued degree for operators in $\mathcal{A}_G(S_+)$ is developed by Kittila in [8] via the Skyrpnik's degrees (cf. [14]). The methodologies of [8] and recent papers of the author and Kartsatos [1] and Kartsatos and Skrypnik [7] along with properties of the Skrypnik's degree in [14] have been utilized.

Invariance of domain results date as far back as Brouwer [3] for continuous injection in $\mathbf{R}^n$. Schauder [13] extended the Brouwer's invariance of domain result to infinite dimensional Banach spaces for compact displacements of the identity, i.e. for operators of the form $I + C$ with C compact. Tromba [16] extended the Schauder's result to Fredholm maps of index zero. For other results on invariance of domain under continuity or demicontinuity assumptions on the main operators, the reader is referred to Berkovits [2], Deimling [4], Kartsatos [5], Nagumo [9], Petryshyn [11, 12] (for A-proper mappings), Skrypnik [14, p.59] and the references therein. For the existence of pathwise connected set in the ranges of certain operators, the reader is referred to [6] and the references therein.

2. INVARIANCE OF DOMAIN RESULT FOR OPERATORS OF CLASS $\mathcal{A}_G(S_+)$

In this section, we first prove a result that has to do with placing a pathwise connected set in the range of the operators of class $\mathcal{A}_G(S_+)$. As a consequence of this result, we then obtain an invariance of domain result for operators in class $\mathcal{A}_G(S_+)$. It is well-known (cf. Kittila [8, p.13]) that a densely defined maximal monotone operator $A : X \supset D(T) \to 2^{X^*}$, $0 \in D(A)$, $0 \in A(0)$ satisfies $A + T$ is in $\mathcal{A}_G(S_+)$ whenever $T \in \mathcal{A}_G(S_+)$. In particular, $T + J \in \mathcal{A}_G(S_+)$.

Proposition 2.1. Let $T : X \supset D(T) \to 2^{X^*}$ be of class $\mathcal{A}_G(S_+)$ with an approximating sequence T_n, where $G \subset X$ is open and bounded. Assume that $T + \epsilon J(\cdot - x_0)$ is injective on G for each $\epsilon \geq 0$ and for every $x_0 \in D(T)$. Moreover, assume that for each $x_0 \in D(T)$ and for each $r > 0$, there exists a bounded $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\langle T_n x, x_0 \rangle \leq \phi(\|x\|)$ for all $x \in \overline{G} \cap \partial B(0, r)$ and for all large n. For a pathwise connected set $M \subset X^*$, assume that $T(D(T) \cap G) \cap M \neq \emptyset$ and $T(D(T) \cap \partial G) \cap M = \emptyset$. Then $M \subset T(D(T) \cap G)$.

Proof: Let $y_0 \in T(D(T) \cap G) \cap M$. Then there exists $x_0 \in D(T) \cap G$ such that $y_0 \in T(x_0)$. Let $p \in M$. Take $f : [0, 1] \to M$ be a path in M such that $f(0) = y_0$ and $f(1) = p$. We now claim that there exist $n_0 \in \mathbb{N}$ such that

$$(2.1) \qquad T_n x + \frac{1}{n} J(x - x_0) = f(t)$$

has no solution $x \in \partial G$ for any $t \in [0, 1]$ and for all $n \geq n_0$. Assuming on the contrary and without loosing the generality, let $\{x_n\} \subset \partial G$ with $x_n \rightharpoonup x$ and $\{t_n\} \subset [0, 1]$ with $t_n \to t_0$ be such that

$$T_n x_n + \frac{1}{n} J(x_n - x_0) = f(t_n).$$

This implies that $T_n x_n \to f(t_0)$. Since $x_n \rightharpoonup x$, we have

$$\limsup_{n\to\infty} \langle T_n x_n, x_n \rangle \leq \langle f(t_0), x \rangle,$$

and then by the condition (A_3) of Definition 1.1, we have $x_n \to x$, $x \in D(T)$ and $f(t_0) \in Tx$. Since $f(t_0) \in M$ and $x \in D(T) \cap \partial G$, we have a contradiction to $T(D(T) \cap \partial G) \cap M = \emptyset$.

We now consider the following homotopy equation:

$$(2.2) \qquad H_n(x,t) \equiv T_n x + \frac{1}{n} J(x - x_0) - f(t) = 0.$$

We have already established that this equation has no solution on ∂G for sufficiently large n and for any $t \in [0,1]$, and therefore this is an admissible homotopy of type (S_+).

We next consider the homotopy equation

$$(2.3) \qquad G_n(x,t) \equiv (1-t)\left(T_n x + \frac{1}{n} J(x - x_0) - y_0 \right) + t J(x - x_0) = 0.$$

We show that (2.3) has no solution on ∂G for any $t \in [0,1]$ and for all $n \geq n_0$. If not, let $\{x_n\} \subset \partial B(x_0, r)$ with $x_n \rightharpoonup x$ and $\{t_n\} \subset [0,1]$ with $t_n \to t_0$ such that

$$(2.4) \qquad (1 - t_n)\left(T_n x_n + \frac{1}{n} J(x_n - x_0) - y_0 \right) + t_n J(x_n - x_0) = 0.$$

Since (2.1) has no solution on ∂G for any $n \geq n_0$ and $t \in [0,1]$, we have that $t_n = 0$ is impossible for all large n. Since J is injective, $t_n = 1$ is also imposssible. Suppose $t_0 = 1$. The equation (2.4) implies

$$(2.5) \qquad (1 - t_n)\langle T_n x_n, x_n - x_0 \rangle + a_n \|x_n - x_0\|^2 - (1 - t_n)\langle y_0, x_n - x_0 \rangle) = 0,$$

where

$$a_n = \frac{1 - t_n}{n} + t_n.$$

We may assume that $x_n \rightharpoonup x$ and let $C > 0$ be such that $\|x_n\| \leq C$ for all n. By the condition (A_1), there exists a $K > 0$ such that $\langle T_n x_n, x_n \rangle \geq -K$ for all n. Also, by the hypothesis, there exists a bounded function $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\langle T_n x_n, x_0 \rangle \leq \phi(\|x_n\|)$ for all n. Then from (2.4), we have

$$(2.6) \qquad -(1 - t_n)K - (1 - t_n)\phi(\|x_n\|) + a_n \|x_n - x_0\|^2 - (1 - t_n)\langle y_0, x_n - x_0 \rangle \leq 0.$$

Since $t_n \to 1$, $a_n \to 1$ and ϕ is bounded, letting $n \to \infty$ in (2.6), we obtain $x_n \to x_0 \in \partial G$, which is a contradiction.

Next, we assume that $t_0 \in [0,1)$. If $t_0 = 0$, define $\alpha_n = \dfrac{t_n}{1 - t_n}$. Then $\alpha_n \downarrow 0$ and

$$T_n x_n + \left(\frac{1}{n} + \alpha_n \right) J(x_n - x_0) = y_0.$$

This equation is like (2.1) for which we have already established the impossibility of solutions on ∂G with $f(t) \equiv y_0$. For the remaining case, $t_0 \in (0,1)$, we define

$$\beta_n = \frac{1}{n} + \frac{t_n}{1 - t_n}.$$

Then $\beta_n \to \beta_0 := \dfrac{t_0}{1 - t_0} > 0$. Then the equation becomes

$$(2.7) \qquad T_n x_n + \beta_n J(x_n - x_0) = y_0.$$

If
$$\limsup_{n\to\infty}\langle T_n x_n, x_n - x\rangle > 0,$$
then, by passing to a subsequence, let
$$q := \lim_{n\to\infty}\langle T_n x_n, x_n - x\rangle > 0.$$
In view of (2.7), this yields
$$\limsup_{n\to\infty}\langle \beta_n J(x_n - x_0), (x_n - x_0) - (x - x_0)\rangle = -q < 0.$$

Since $\beta_n \to \beta_0 > 0$ and J is of type (S_+), we obtain $x_n \to x \in \partial G$. From this and (2.7), we get $T_n x_n \to w := -\beta_0 J(x - x_0) + y_0$. By the condition (A_3), we obtain $x \in D(T)$ and $w \in T(x)$, i.e. $y_0 \in T(x) + \beta_0 J(x - x_0)$. This leads to a contradiction to the injectivity of $T + \epsilon J(\cdot - x_0)$ because $x \neq x_0$.

Thus, $H_n(x, t)$ and $G_n(x, t)$ are admissible homotopies for the Skrypnik's degree, $d_{(S_+)}$, for the mappings of type (S_+). By the invariance of the degree under these homotopies, we obtain

$$
\begin{aligned}
d_{(S_+)}(T_n + \tfrac{1}{n}J(\cdot - x_0) - p, G, 0) &= d_{(S_+)}(H_n(\cdot, 1), G, 0)\\
&= d_{(S_+)}(H_n(\cdot, 0), G, 0)\\
&= d_{(S_+)}(G_n(\cdot, 1), G, 0)\\
&= d_{(S_+)}(G_n(\cdot, 0), G, 0)\\
&= d_{(S_+)}(J(\cdot - x_0), G, 0) = 1.
\end{aligned}
$$

Here, the last equality follows by considering the (S_+)-homotopy

$$Q(x, t) = (1 - t)J(x - x_0) + tJx$$

with a continuous curve $y(t) = tJx_0$ so that

$$
\begin{aligned}
d_{(S_+)}(J(\cdot - x_0), G, 0) &= d_{(S_+)}(Q(\cdot, 0), G, 0)\\
&= d_{(S_+)}(Q(\cdot, 1), G, Jx_0)\\
&= d_{(S_+)}(J, G, Jx_0) = 1.
\end{aligned}
$$

Therefore, for every n, there is $x_n \in G$ such that

$$H_n(x_n, 1) = 0,$$

i.e.

$$T_n x_n + \frac{1}{n}J(x_n - x_0) = p,$$

which implies $T_n x_n \to p$. By the condition (A_3), we deduce that $x_n \to x \in \overline{G}$, $x \in D(T)$, and $p \in T(x)$. Since $T(D(T) \cap \partial G) \cap M = \emptyset$, we can only have $x \in G$. Since p was an arbitrary point in M, we obtain $M \subset T(D(T) \cap G)$. $\qquad\square$

We now apply Proposition 2.1 to obtain the following invariance of domain result.

Theorem 2.2 (Invariance of Domain)**.** *Let* $T : X \supset D(T) \to 2^{X^*}$ *be of class* $\mathcal{A}_G(S_+)$ *with an approximating sequence* T_n, *where* $G \subset X$ *is open. Assume that* $T + \epsilon J(\cdot - x_0)$ *is locally injective on* G *for each* $\epsilon \geq 0$ *and for every* $x_0 \in D(T)$. *Moreover, assume that for each* $x_0 \in D(T)$ *and for each* $r > 0$, *there exists a bounded* $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $\langle T_n x, x_0 \rangle \leq \phi(\|x\|)$ *for all* $x \in \overline{G} \cap \partial B(0, r)$ *and for all large* n. *Then* $T(D(T) \cap G)$ *is open.*

Proof: Let $y_0 \in T(D(T) \cap G)$. Then there exists $x_0 \in G$ such that $y_0 \in T(x_0)$. Since T is locally injective on G, there is $r > 0$ such that T is injective on $\overline{B(x_0, r)} \cap D(T)$, where $\overline{B(x_0, r)} \subset G$. It is then clear that $y_0 \notin T(D(T) \cap \partial B(x_0, r))$. By Lemma 3.2 in [8], there exits $n_0 \in \mathbb{N}$ such that $y_0 \notin T_n(\partial B(x_0, r))$ for all $n \geq n_0$, where $\{T_n\}$ is an approximating sequence of T as in the hypothesis of the theorem.

We claim that there exists $\delta > 0$ such that $B(y_0, \delta) \cap T(D(T) \cap \partial B(x_0, r)) = \emptyset$. Assume the contrary, and let $w_n \in B(y_0, 1/n) \cap T(D(T) \cap \partial B(x_0, r))$. Then $y_n \to y_0$ and $w_n \in T(x_n)$ with $x_n \in \partial B(x_0, r)$. Now the condition (A3) applies since $x_n \rightharpoonup x$ (up to subsequence) and $y_n \to y_0$. We then obtain $x_n \to x \in \partial B(x_0, r)$, $x \in D(T)$ and $y_0 \in T(x)$. This contradicts $y_0 \notin T(D(T) \cap \partial B(x_0, r))$.

Since $B(y_0, \delta) \cap T(D(T) \cap \partial B(x_0, r)) = \emptyset$, $y_0 \in T(D(T) \cap B(x_0, r))$ and the ball $B(y_0, \delta)$ is pathwise connected, we can apply Proposition 2.1 to obtain $B(y_0, \delta) \subset T(D(T) \cap B(x_0, r))$.□

It would be interesting to establish analogous results via degree theory for operators of the form $A + T$, where $A : X \supset D(A) \to 2^{X^*}$ is maximal monotone and T is in $\mathcal{A}_G(S_+)$. Similar results are expected for the sum $L + A + T$ in the spirit of results in [1], where L is densely defined linear maximal monotone operator and T in class $A_G(S_+)$ with respect to $D(L)$.

References

[1] D.R. Adhikari and A.G. Kartsatos, *Invariance of domain and eigenvalues for perturbations of densely defined linear maximal monotone operators*, Applicable Analysis **95** (2016), no. 1, 24–43.

[2] J. Berkovits, *On the degree theory for nonlinear mappings of monotone type*, Ann. Acad. Sci. Fenn. Ser. A I, Math. Dissertationes **58** (1986).

[3] L.E.J. Brouwer, *Über Abbildungen von Mannigfaltigkeiten*, Math. Ann. **71** (1912), 97–115.

[4] K. Deimling, *Zeros of accretive operators*, Manuscr. Math. **13** (1974),), 365–374.

[5] A.G. Kartsatos, *Zeros of demicontinuous accretive operators in reflexive banach spaces*, J. Integral Equations **8** (1985), 175–184.

[6] ———, *Sets in the ranges of nonlinear accretive operators in banach spaces*, Studia Math. **114** (1995), 261–273.

[7] A.G. Kartsatos and I.V. Skrypnik, *Degree theories and invariance of domain for perturbed maximal monotone operators in Banach spaces*, Adv. Differential Equations **12** (2007), 1275–1320.

[8] A. Kittilä, *On the topological degree for a class of mappings of monotone type and applications to strongly nonlinear elliptic problems*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertations **91** (1994), 48pp.

[9] M. Nagumo, *Degree of mapping in convex linear topological spaces*, Amer. J. Math. **73** (1951), 497–511.

[10] J.A. Park, *Invariance of domain theorem for demicontinuous mappings of type* (S_+), Bull. Korean Math. Soc. **29** (1992), no. 1, 81–87.

[11] W. V. Petryshyn, *Invariance of domain theorem for locally A-proper mappings and its implications*, J. Funct. Anal. **5** (1970), 137–159.

[12] ———, *Approximation-solvability of nonlinear functional and differential equations*, Marcel Dekker, New York, 1993.

[13] J. Schauder, *Invarianz des gebietes in funktional raümen*, Studia Math. **1** (1929), 123–130.

[14] I.V. Skrypnik, *Methods for analysis of nonlinear elliptic boundary value problems*, Ser. II, vol. 139, Amer. Math. Soc. Transl., Providence, Rhode Island, 1994.

[15] S.L. Trojanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. **37** (1971), 173–180.

[16] A.J. Tromba, *Some theorems on fredholm maps*, Proc. Amer. Math. Soc. **34** (1972), 578–585.