The Nepal Math. Sc. Report
Vol. 34, No.1 and 2, 2016

A NOTE ON VECTOR-VALUED DISCRETE SCHRÖDINGER OPERATORS

KESHAV RAJ ACHARYA

Department of Mathematics, Embry–Riddle Aeronautical University
Daytona Beach, FL 32114-3900, U.S.A.
acharyak@erau.edu

Abstract: The main purpose of this paper is to extend some theory of Schrödinger operators from one dimension to higher dimension. In particular, we will give systematic operator theoretic analysis for the Schrödinger equations in multidimensional space. To this end, we will provide the detail proves of some basic results that are necessary for further studies in these areas. In addition, we will introduce Titchmarsh-Weyl $m-$ function of these equations and express $m-$ function in term of the resolvent operators.

Key Words: Discrete Schrödinger equation, Titchmarsh-Weyl m-function
AMS (MOS) Subject Classification. 39A70, 47A05, 34B20.

1. Introcuction

The Jacobi and Schrödinger equations are the basic equations in Mathematical physics. These equations are used to describe quantum mechanical particles. In one dimensional space, the theory of Schrödinger equations

$$-y'' + V(x)y = zy(x),\ x \in \mathbb{R}, z \in \mathbb{C}$$

and the Jacobi equations

$$a(n)y(n+1) + a(n-1)y(n-1) + b(n)y(n) = zy(n), z \in \mathbb{C},$$

where $a(n), b(n)$ are bounded sequences, are well developed. For a few references see [2, 8, 9, 11]. However, much less is known about the theory in multidimensional space. In this paper, we attempt to extend some theory of Schrödinger equations from one dimensional space to multidimensional space. So we consider discrete Schrödinger equations whose solutions are vector valued functions and extend some basic results of Jacobi and Schrödinger equations from one dimensional space. There has been some research work in higher dimension for example, see [3, 4, 7]. However, we did not find a detail presentations of the basic theory, required for further studies in this area, which we discussed in section 2.

In addition, we also discuss about the Titchmarsh-Weyl m function. These $m-$ functions play very important role in the spectral theory of Jacobi and Schrödinger operators.

Received July 16, 2016 .

The spectrum of these operators can be described by these $m-$ function. Some of the analogous theory in one dimensional space can be found also in above mentioned references. We discuss about this in Section 3.

The extension of the theory to multidimensional space is not obvious in most of the cases. The situation becomes completely different and a careful analysis is required to study such phenomena.

We consider a multidimensional discrete Schrödinger equation of the form

$$y(n+1) + y(n-1) + B(n)y(n) = zy(n), \ z \in \mathbb{C} \tag{1.1}$$

where $y(n) = [y_1(n) \ y_2(n), \dots \ y_d(n)]^t$ (t stands for a transpose), is a vector valued sequence in $l^2(I, \mathbb{C}^d)$. Here $l^2(I, \mathbb{C}^d)$ is a Hilbert space of square summable vector valued sequences with the inner product

$$\langle u, v\rangle = \sum_{n \in I} u(n)^* v(n),$$

where " $*$ " stands for conjugate transpose and $B(n) \in \mathbb{C}^{d\times d}$ is a symmetric $d \times d$ matrix. Usually $I = \mathbb{Z}$ or $I = \mathbb{N}$. The equation (1.1) can be generalized to a multidimensional Jacobi equation of the form

$$A(n)y(n+1) + A(n-1)y(n-1) + B(n)y(n) = zy(n), \ z \in \mathbb{C} \tag{1.2}$$

with $A(n), B(n)$ are sequences of $d \times d$ matrices. If $I = N$ The equation (1.2) can be written in the form:

$$\begin{pmatrix} B(1) & A(1) & 0 & \\ A(1) & B(2) & A(2) & \ddots \\ 0 & A(2) & B(3) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = z \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}.$$

The matrix

$$J = \begin{pmatrix} B(1) & A(1) & 0 & \\ A(1) & B(2) & A(2) & \ddots \\ 0 & A(2) & B(3) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

is called a block Jacobi matrix. Some studies about the block Jacobi matrix can be found in the paper [7]. Equation (1.1) is a particular case of Jacobi equation with $A(n) \equiv 1$.

2. Some basic theory

In this section, we will give an operator theoretic analysis of the equation (1.1). The Equation (1.1) induces an operator J on $l^2(I, \mathbb{C}^d)$ as

$$Jy(n) = y(n+1) + y(n-1) + B(n)y(n).$$

If $I = \mathbb{N}$, then we need to slightly modify the definition of J as

$$Jy(1) = y(2) + B(1)y(1).$$

The matrix $B(n)$ is called the potential. We assume that

$$B(n)^* = B(n), \text{ and } \|B(n)\| \leq C$$

.

Proposition 2.1. If $B(n) \in \mathbb{C}^{d\times d}$, $B(n)^* = B(n)$, then J is a self-adjoint operator on $l^2(\mathbb{N}, \mathbb{C}^d)$.

Proof. It is clear by definition of J and Cauchy Schwartz inequality that J is a bounded linear operator on $l^2(\mathbb{N}, \mathbb{C}^d)$. In order to see self adjointness, suppose $u, v \in l^2(\mathbb{N}, \mathbb{C}^d)$ then

$$\begin{aligned}
\langle u, Jv\rangle =& \sum_{n=1}^{\infty} u(n)^* Jv(n) = u(1)^*(J(v)(1)) + \sum_{n=2}^{\infty} u(n)^*(v(n+1) + v(n-1) + B(n)v(n)) \\
=& u(1)^*(v(2) + B(1)v(1)) + \sum_{n=2}^{\infty} u(n)^* v(n+1) + \sum_{n=2}^{\infty} u(n)^* v(n-1) + \sum_{n=2}^{\infty} u(n)^* B(n)v(n)) \\
=& (u(2) + B(1)u(1))^* v(1) + \sum_{n=2}^{\infty} (u(n+1) + u(n-1) + B(n)u(n))^* v(n)) \\
=& \sum_{n=1}^{\infty} (Ju(n))^* v(n) \\
=& \langle Ju, v\rangle
\end{aligned}$$

□

Since J is a self adjoint operator, the spectrum of such operator is a set of real numbers: $\sigma(J) \subset \mathbb{R}$.

To get a solution of the equation (1.1), we may fix any two vectors $c_1, c_2 \in C^d$ at two consecutive sites, that is, we fix the values $u_k = c_1, u_{k+1} = c_2$ and evolve according to (1.1). Suppose τ is the difference expression in the left side of (1.1), then we have the following remark.

Remark 2.2. Let c_1, c_2 be any two vectors in $\mathbb{C}^d$ and $k \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\})$. For any arbitrary sequence $f(n)$ there exists a unique solution $u(n)$ of $(\tau - z)u(n) = f(n)$ with $u(k) = c_1$ and $u(k+1) = c_2$.

Consequently the following preposition holds.

Proposition 2.3. The set of solutions u to $(\tau - z)u(n) = 0$ is a $2d$-dimensional vector space.

Proof. By above remark, for each $c_1, c_2 \in \mathbb{C}^d$ and $k \in \mathbb{N}_0$ there exists a unique solution u such that $u(k) = c_1$ and $u(k+1) = c_2$. Since $\mathbb{C}^d$ is a d dimensional space, the solution space is $2d$ dimensional vector space. □

In the following theorem we show that the number of linearly independent $l^2(\mathbb{N}, \mathbb{C}^d)$ solutions of $d-$ dimensional Schrödinger equations (1.1) is d.

Theorem 2.4. *Let $z \in \mathbb{C} - \mathbb{R}$. Then $(\tau - z)u(n) = 0$ has exactly d linearly independent solutions in $l^2(\mathbb{N}, \mathbb{C}^d)$.*

Proof. Since J is self adjoint the spectrum $\sigma(J) \subset \mathbb{R}$ and therefore for each $z \in \mathbb{C}^+,\ (J-z)$ is invertible in $B(l^2)$. Let

$$\delta_k = (e(k), 0, 0, , \dots)$$

where for each $k, e(k)^t = (0, 0, \dots, 1, \dots, 0, 0)$ is a vector in C^d with 1 in the kth component and 0 otherwise. Let $u_k = (J-z)^{-1}\delta_k$ for each $k = 1, 2, ...d$. Clearly, u_k are linearly independent (being the images of linearly independent vector under bounded linear operator). Moreover,

$$(J - z)u_k = \delta_k.$$

For each $k, k = 1, 2, ..., d,\ (\tau - z)u_k(n) = 0$ for $n \geq 2$. By suitably defining at $n = 0$ we can also achieve that $(\tau - z)u_k(1) = 0$. So this extended u_k is an l^2 solution. Thus there are d linearly independent solutions.

Suppose there is another solution v linearly independent to $u_1, u_2, ...u_d$. Then $v(0), u_1(0), ..., u_d(0)$ being $d+1$ vectors in $\mathbb{C}^d$, are linearly dependent. So there exists constants $\alpha_1, \alpha_2, ...\alpha_d$ not all zero such that

$$v(0) = \alpha_1 u_1(0) + ... + \alpha_d u_d(0).$$

Define

$$f(n, z) = v(n, z) - (\alpha_1 u_1(n, z) + ... + \alpha_d u_d(n, z)).$$

Clearly $f(n, z)$ satisfies the difference equation and $f(n, z) \in l^2(\mathbb{N}, \mathbb{C}^d)$. Moreover, $f(0, z) = 0$. So $f(n, z)$ is an eigenvector for J, which contradicts that $\sigma(J) \subset \mathbb{R}$. So there are exactly d linearly independent l^2 solutions. $\square$

As we know that the Wronskian of the solutions of a differential equations have close connection with the linearly independent solutions. It is also important in the difference equations as well. One of the applications of Wronskian can be found in [6]. We define the Wronskian of any two vector valued sequences in $l^2(\mathbb{N}, \mathbb{C}^d)$.

Definition 2.5. The Wronskian of any two sequences $f(n, z), g(n, z) \in l^2(\mathbb{N}, \mathbb{C}^d)$ is defined by

$$W_n(f, g) = [f^*(n + 1, \bar{z})g(n, z) - f^*(n, \bar{z})g(n + 1, z)].$$

This definition incorporate with the definition in one dimensional space and in the continuous case. Analogous to a result in ordinary differential equations, we have the following lemma.

Lemma 2.6. *If $f(n, z), g(n, z) \in l^2(\mathbb{N}, \mathbb{C}^d)$ are any two solutions of (1.1) then $W_n(f, g)$ is independent of n.*

Proof. Since $f(n,z), g(n,z)$ are solutions of (1.1) we have,

$$f(n+1,z)+f(n-1,z)+B(n)f(n,z)=zf(n,z)$$

and

$$g(n+1,z)+g(n-1,z)+B(n)g(n,z)=zg(n,z).$$

Multiply the complex conjugate of first equation by $g^*(n,z)$ from left and take the conjugate transpose we obtain,

$$f^*(n+1,\bar{z})g(n,z)+f^*(n-1,\bar{z})g(n,z)+f^*(n,\bar{z})B(n)g(n,z)=f^*(n,\bar{z})zg(n,z)$$

Similarly, multiplying second equation from left by $f^*(n,\bar{z})$ we get,

$$f^*(n,\bar{z})g(n+1,z)+f^*(n,\bar{z})g(n-1,z)+f^*(n,\bar{z})B(n)g(n,z)=f^*(n,\bar{z})zg(n,z).$$

On subtraction we get

$$f^*(n+1,\bar{z})g(n,z)-f^*(n,\bar{z})g(n+1,z)=f^*(n,\bar{z})g(n-1,z)-f^*(n-1,\bar{z})g(n,z)$$

so that $W_n(f,g)=W_{n-1}(f,g)$. ☐

Next we establish the Green's identity corresponding to equation (1.1).

Lemma 2.7 (Green's Identity)**.** *Let* $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. *For* $f(n,z), g(n,z)\in l^2(\mathbb{N}_0,\mathbb{C}^d)$

$$\sum_{j=0}^{n}\Big(f^*(\tau g)-(\tau f)^*g\Big)(j)=W_0(\bar{f},g)-W_n(\bar{f},g).$$

Proof.

$$\begin{aligned}
&\sum_{j=0}^{n}\Big(f^*(\tau g)-(\tau f)^*g\Big)(j)\\
&=\sum_{j=0}^{n}\Big[f^*(j)\Big(g(j+1)+g(j-1)+B(n)g(j)\Big)-\Big(f(j+1)+f(j-1)+B(n)f(j)\Big)^*g(j)\Big]\\
&=\sum_{j=0}^{n}\Big[\Big(f^*(j)g(j-1)-f^*(j-1)g(j)\Big)-\Big(f^*(j+1)g(j)-f^*(j)g(j+1)\Big)\Big]\\
&=\sum_{j=0}^{n}W_j(\bar{f},g)-W_{j-1}(\bar{f},g)\\
&=W_0(\bar{f},g)-W_n(\bar{f},g)
\end{aligned}$$

☐

It is now convenient to fix a basis of the solution space. An easier way to choose a basis of the solution space of (1.1) is to prescribe a pair of initial conditions. For $z\in\mathbb{C}$, let

$$\begin{aligned}
U(n,z)&=[u_1(n),u_2(n),\dots,u_d(n)],\\
u_i(n)&=[u_{1,i}(n)\ u_{2,i}(n)\ \dots u_{d,i}(n)]^t\\
V(n,z)&=[v_1(n),v_2(n),\dots,v_d(n)]\\
v_i(n)&=[v_{1,i}(n)\ v_{2,i}(n)\ \dots v_{d,i}(n)]^t
\end{aligned}\tag{2.1}$$

be the set of solutions. Both of the sets $U(n,z)$ and $V(n,z)$ consists of d linearly independent solutions of $(\tau - z)u(n) = 0$. For a convenience, we may consider these sets as matrices in $M^{d\times d}(\mathbb{C})$. We further suppose that these solutions satisfy the following initial conditions

$$U(0,z) = -I, \qquad V(0,z) = O, \qquad U(1,z) = O, \qquad V(1,z) = I. \tag{2.2}$$

By iterating the difference equation, we see that for fixed $n \in \mathbb{N}, U(n,z), V(n,z)$ are polynomial of degree $n-2$ over $M^{d\times d}(\mathbb{C})$. So $\overline{U(n,z)} = U(n,\bar{z})$ and $\overline{V(n,z)} = V(n,\bar{z})$.

We extend the definition of Wronskian from above for the sets $U(n,z), V(n,z)$, each contains d linearly independent solutions of (1.1).

$$W_n(U,V) = \det[U^*(n+1,\bar{z})V(n,z) - U^*(n,\bar{z})V(n+1,z)].$$

We now extend the lemma 2.6 in the following proposition.

Proposition 2.8. $W_n(U,V)$ is independent of $n \in \mathbb{N}$

Proof. Since $U(n,z), V(n,z)$ are solutions of (1.1),

$$U(n+1,z) + U(n-1,z) + B(n)U(n,z) = zU(n,z)$$

and

$$V(n+1,z) + V(n-1,z) + B(n)V(n,z) = zV(n,z)$$

By multiplying the complex conjugate of first equation by $V^*(n,z)$ and taking the conjugate transpose we obtain,

$$U^*(n+1,\bar{z})V(n,z) + U^*(n-1,\bar{z})V(n,z) + U^*(n,\bar{z})B(n)V(n,z) = U^*(n,\bar{z})zV(n,z)$$

Similarly, multiplying second equation by $U^*(n,\bar{z})$ we get,

$$U^*(n,\bar{z})V(n+1,z) + U^*(n,\bar{z})V(n-1,z) + U^*(n,\bar{z})B(n)V(n,z) = U^*(n,\bar{z})zV(n,z).$$

On subtraction we get

$$U^*(n+1,\bar{z})V(n,z) - U^*(n,\bar{z})V(n+1,z) = U^*(n,\bar{z})V(n-1,z) - U^*(n-1,\bar{z})V(n,z)$$

so that

$$\det[U^*(n+1,\bar{z})V(n,z) - U^*(n,\bar{z})V(n+1,z)] = \det[U^*(n,\bar{z})V(n-1,z) - U^*(n-1,\bar{z})V(n,z)].$$

It follows that $W_n(U,V) = W_{n-1}(U,V)$. Continuing we get,

$$W_n(U,V) = W_{n-1}(U,V) = \ldots = W_0(U,V) = \det I = 1.$$

$\square$

3. Titchmarsh-Weyl m function

Titchmarsh-Weyl m functions associated with the Schrödinger equations are very important objects in the direct and inverse spectral theory of the corresponding operator. These functions provides asymptotic behavior of the solutions of these equations. The history of these functions goes back to 1910 when H. Weyl introduce these functions in [13] for Sturn-Liouville differential equations. It was further studied by E. C. Titchmarsh in [12] and establish the connection between the analyticity of the solution and the spectrum of the operator of Sturn-Liouville differential equations. There has been tremendous work about the Weyl theory in one dimension which can be found in many literatures, please see [1, 5, 8, 10, 11] as a few references.

We introduce the Titchmarsh-Weyl m function for the vector-valued discrete Schrödinger operators and express in terms of resolvent operator.

Definition 3.1. Let $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The Titchmarsh-Weyl m function is defined as the unique $M(z) \in \mathbb{C}^{d\times d}$ for which

$$F(n,z) = U(n,z) + M(z)V(n,z) \tag{3.1}$$

where $U(n,z), V(n,z)$ are the sets of d linearly independent solutions with initial values (2.2) and $F(n,z)$ is a set of d linearly independent solutions of (1.1) in $l^2(\mathbb{N}, \mathbb{C}^d)$.

This definition, is in fact well defined. As we mentioned above that there are only d linearly independent solutions in $l^2(\mathbb{N}_0, \mathbb{C}^d)$, if there is another $M(z)$ satisfying the above conditions then the solutions from both sets $U(n,z)$ and $V(n,z)$ will be in $l^2(\mathbb{N}_0, \mathbb{C}^d)$. The solution set $V(n,z)$ is such that $V(0,z) = 0$ which implies that $V(n,z)$ is the set of eigenfunctions for the self adjoint operator J. This contradicts that the spectrum of J is a set of real numbers.

The following is the main theorem in this section.

Theorem 3.2. *$z \in \mathbb{C}^+$. If $(\tau - z)F = 0$ and F consists of d solutions in $l^2(\mathbb{N}, \mathbb{C}^d)$. Then*

$$M(z) = -F(1,z)F(0,z)^{-1}. \tag{3.2}$$

Moreover,

$$M(z) = (m_{ij}(z))_{d\times d} \in \mathbb{C}^{d\times d}, \ m_{ij}(z) = \langle \delta_j, (J-z)^{-1}\delta_i \rangle. \tag{3.3}$$

Proof. If F is specifically the set of l^2 solutions from (3.1). Then $F(0,z) = -I$ and $F(1,z) = M(z)$. So (3.2) holds. A set of arbitrary d solutions $G(n,z)$ is a constant (matrix) multiple of the solution set $F(n,z)$ from (3.1) because (3.1) is a set of d linearly independent solutions. That is,

$$G(n,z) = F(n,z)C$$
$$F(n,z) = G(n,z)C^{-1}$$

so that

$$\begin{aligned}-G(1,z)G(0,z)^{-1} &= -\,F(1,z)CC^{-1}F(0,z)^{-1}\\ &= -\,F(1,z)F(0,z)^{-1}\\ &= M(z).\end{aligned}$$

Let $F(n,z)$ as in (3.2) and let

$$g_i = (J-z)^{-1}\delta_i.$$

Then $(J-z)g_i = \delta_i$. So $(\tau - z)g_i(n) = 0$ for $n \geq 2$. Moreover $g_i \in l^2$ for all $i = 1, 2, \ldots\ldots., d$. Let

$$G(n,z) = [g_1, g_2, \ldots\ldots., g_d].$$

Then $G(n,z) = F(n,z)C,\ C \in \mathbb{C}^{d\times d}$. By comparing values at

$$n = 1,\ G(1,z) = [g_1(1), g_2(1), \ldots\ldots\ldots, g_d(1)].$$

Here

$$g_1(1) = (J-z)^{-1}\delta_1(1)$$

and

$$g_1 = [g_{11}, g_{21}, \ldots, \ldots, \ldots, g_{d1}]^t,\ \ g_{i1} = \langle \delta_i, g_1\rangle, i = 1, 2, \ldots., d.$$

Then $M(z) = G(1,z)C^{-1}$ and

$$\begin{aligned}M(z) &= (m_{ij}(z))\\ &= (\langle \delta_j, (J-z)^{-1}\delta_i\rangle)C^{-1}.\end{aligned}$$

To find the value of C, we compare values at $n = 2$.

First $(J-z)G(1,z) = (\delta_1, \delta_2, \ldots\ldots, \delta_d)$ so

$$(J-z)G(1,z) = \begin{pmatrix} 1 & 0\ldots & 0\\ 0 & 1\ldots & 0\\ \vdots & \vdots & \vdots\\ 0 & 0\ldots & 1\end{pmatrix} = I$$

It follows that

$$G(2,z) + B(1)G(1,z) - zG(1,z) = I$$

$$G(2,z) = (z - B(1))G(1,z) + I\ldots\ldots\ldots\ldots(i)$$

Also,

$$F(2,z) = (z - B(1))F(1,z) - F(0,z)C$$

$$G(2,z) = (z - B(1))G(1,z) - G(0,z)\ldots\ldots\ldots\ldots(ii)$$

Comparing (i) and (ii), we get $-F(0,z)C = I$ and so $I.C = I \implies C = I$. Hence (3.3) holds. That is

$$\begin{aligned}M(z) &= (m_{ij}(z))\\ &= (\langle \delta_j, (J-z)^{-1}\delta_i\rangle).\end{aligned} \tag{3.4}$$

This result allows us to connect the m function with a matrix valued Borel measure using functional calculus for these resolvent operators $\langle \delta_j, (J-z)^{-1}\delta_i \rangle$. We aim to extend the Weyl theory and discuss the spectrum of vector-valued discrete Schrödinger operators in our next studies.

Acknowledgement: The author would like to thank the Department of Mathematics and the office of sponsored research, Embry-Riddle Aeronautical University for support.

□

References

[1] J.Behrndt, J. Rohleder, Titchmarsh-Weyl Theory for Schrödinger operators on unbounded domain, arXiv: 12085224v2.

[2] H. L. Cycon, R.G. Froese, W. Kirsch, B. Simon, Schrödinger Operators: With Applications to Quantum Mechanics and Global Geometry, Springer, 2008.

[3] D. Damanik, A. Pushnitski, B. Simon, The analytic theory of matrix orthogonal polynomials. *Surv. Approx. Theory*, 4: 1-85, 2008.

[4] J. S. Geronimo, Scattering theory and matrix orthogonal polynomials on the real line, *Circuits Systems Signal Process.*, 1(3-4): 472-495, 1982.

[5] F. Gesztesy, E. Rsekanovskii, On matrix-valued Herglotz functions. *Math. Machr.*, 218: 61-138, 2000.

[6] F. Gesztesy, B. Simon, G. Teschl, Zeros of the Wronskian and renormalized oscillation theory, *Amer. J. Math.* 118 (1996), 571 - 594.

[7] R. Kozhan, Equivalence classes of block Jacobi matrices. *Proc. Amer. Math. Soc.*, (139) 799-805, 2011.

[8] C. Remling, The absolutely continuous spectrum of Jacobi Matrices, *Annals of Math.*, **174**, 125-171, 2011.

[9] C. Remling, The absolutely continuous spectrum of one-dimensional Schrödinger operators, *Math. Phys. Anal. Geom.*, 10(4), 359–373, 2007.

[10] B. Simon, m-functions and the absolutely continuous spcetrum of one-dimensional almost periodic Schrödinger operators, *Differential equation (Birmingham, Ala., 1983)*, 519, North-Holland Math. Stud. 92, North-Holland, Amsterdam, 1984.

[11] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Latices, Mathematical Monographs and Surveys, Vol.72, American Mathematical Society, Providence, 2000.

[12] E.C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations*, Part I, Second Edition, Clarendon Press, Oxford, 1962.

[13] H. Weyl, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann., 68 (1910), no. 2, 220-269.