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Editorial

We are delighted to present Volume 5, Issue II of the *Nepal Journal of Mathematical Sciences (NJMS)* for the year 2024. This issue features five original research articles that span a range of topics within mathematics and mathematical sciences.

We would like to express our sincere gratitude to all the authors for their valuable contributions to this issue. Our heartfelt thanks also go to the reviewers and editors for their dedicated support and expert guidance in making this publication possible.

We would like to request research scientists, scholars and professors to submit their original research work for future issues of NJMS.

Thank you for your continued support.

March 16, 2025

Editor-in-Chief

CONTENTS

Nepal Journal of Mathematical Sciences (NJMS)
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Maximal Monotone Operators on Product Spaces

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Abstract: *Let X and Y be real Banach spaces with duals X^* and Y^*, respectively. Let $T: X \to 2^{X^*}$ and $S: Y \to 2^{Y^*}$ be maximal monotone operators. We discuss some known results on the maximality of the sum $T + S$ in the case $X = Y$ and an important characterization of maximal monotone operators in general Banach spaces. As our main result, we prove that the operator $H: X \times Y \to 2^{X^* \times Y^*}$ defined by $H(x, y) = \{(x^*, y^*): x^* \in Tx, y^* \in Sy\}, x \in X, y \in Y\}$, is maximal monotone.*

Keywords: Maximal monotone operators, Local boundedness, Upper semi-continuity

1. Introduction

In what follows, X denotes a real Banach space with norm $\|.\|$ and X^* denotes the dual space of X. The evaluation of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x^*, x\rangle$. Given a subset B of X, we denote its strong closure and strong interior by cl B and int B, respectively. The effective domain of a set-valued (multivalued) operator $T: X \to 2^{X^*}$ is defined as

$$D(T) = \{x \in X : Tx \neq \phi\}$$

and the graph of T is defined as $G(T) = \{(x, x^*) \in X \times X^*: x^* \in Tx\}$.

A multivalued operator $T: X \to 2^{X^*}$ is said to be monotone if for any $x, y \in D(T)$

$$\langle x^* - y^*, x - y\rangle \geq 0 \text{ for all } x^* \in Tx \text{ , } y^* \in Ty.$$

We say that T is maximal monotone if the graph of T has no proper monotone extension when $X \times X^*$ is partially ordered by the set inclusion. Equivalently, T is maximal monotone if for each $(x, x^*) \in X \times X^*$ such that

$$\langle x^* - y^*, x - y\rangle \geq 0 \text{ for all } (y, y^*) \in G(T), \text{we have } (x, x^*) \in G(T).$$

The theory of maximal monotone operators on real reflexive Banach spaces has been studied quite extensively. For example, sufficient conditions on the operators on reflexive Banach spaces have been studied for the sum of two maximal monotone to be maximal monotone (see, for example, [8]). We recall that the sum of two monotone operators is always monotone but this property may fail to hold for maximal monotonicity in general. The first result in this direction is given by Lescarret [6] for operators in Hilbert spaces. In reflexive Banach spaces, Browder [3, 4] established such results when at least one addendum is single-valued and maximal monotone. Specifically, if T_1 and T_2 are two monotone operators from X to X^* where T_1 is maximal, $D(T_2) = X$, and T_2 is bounded single-valued and hemi continuous, then $T_1 + T_2$ is maximal monotone. Rockafellar [9] generalized the results of [3, 4] by studying the following two conditions so that the sum $T_1 + T_2$ of two maximal monotone operators T_1 and T_2 from a reflexive real Banach space X to its dual X^* is maximal monotone:

$$D(T_1) \cap \text{int}\, D(T_2) = \phi, \text{or} \quad \text{there exists } x \in \text{cl}\, D(T_1) \cap \text{cl}\, D(T_2) \text{ and } T_2 \text{ is locally bounded at } x.$$

In the same context, Attouch [1] gave weaker condition in Hilbert spaces; namely, if $0 \in \text{int}\big(D(T_1) - D(T_2)\big)$, then $T_1 + T_2$ is a maximal monotone. This condition is weaker than the Rockafellar condition $D(T_1) \cap \text{int}\, D(T_2) \neq \phi$ because

$$\text{int}\, D(T_1) - D(T_2) \subset \text{int}\,(D(T_1) - D(T_2)).$$

Heisler's result holds for nonreflexive Banach spaces. Specifically, the sum $T_1 + T_2$ of two maximal monotone operators T_1 and T_2 defined from nonreflexive Banach space X to X^* with $D(T_1) = D(T_2) = X$ is maximal monotone [11]. Several weaker sufficient conditions for such results in both reflexive and general Banach spaces have been studied in recent years. The theory of maximal monotone operators on nonreflexive Banach spaces is still developing. For further details on these topics, the reader is referred to [1, 2, 11].

In this paper, we study the maximality of an operator on the product space of two (possibly nonreflexive) Banach spaces in terms of maximal monotone operators that are defined on the individual Banach spaces. Specifically, given monotone operators $T: X \to 2^{X^*}$ and $S: Y \to 2^{Y^*}$, we study the monotonicity and maximal monotonicity of $H: X \times Y \to 2^{X^* \times Y^*}$ defined by

$$H(x, y) = \{(x^*, y^*): x^* \in Tx, y^* \in Sx\}, \quad x \in D(T), y \in D(S).$$

2. Preliminary Results

For any normed space X, a canonical mapping $c: X \to X^{**}$ is defined by $c(x) = g_x \ \forall x \in X$. Here, for each $x \in X$, the linear functional $g_x: X^* \to \mathbb{R}$ is given by $g_x(f) = f(x)$ for all $f \in X^*$. The canonical mapping $c = g_x$ is a linear isometry, and therefore c is an isomorphism from X onto its range $R(C) \subseteq X^{**}$. The weak* topology on X^* is defined below.

Definition 1. The weak* topology is the smallest topology on X^* which makes all the canonical mappings $g_x: X^* \to \mathbb{R}$ continuous. A set $B \subset X^*$ is an open set in the weak* topology of X^* if and only if for every $g \in B$ there exist $\epsilon > 0$ and $x_1, x_2, \dots, x_n \in X$ such that

$$\{f \in X^* : |\langle f - g, x_i \rangle| < \epsilon\} \subset B.$$

We start with a classical compactness theorem (see [10]).

Theorem A (Banach-Alaoglu). The closed unit ball $B_{X^*} = \{x^* \in X^*: ||x^*|| \leq 1\}$ in X^* is compact in its weak* topology.

A net in X is a function defined on a directed set I with values in X. If $f : I \to X$ is a net, then for each α in I the α^{th} term $f(\alpha)$ of the net is denoted by x_α, and the entire net is often denoted by $(x_\alpha)_{\alpha \in I}$ or simply by (x_α).

Example 1. Every sequence is a net with the directed set being $\mathbb{N}$ in its natural order.

We next define the convergence of nets in topological spaces as in [7].

Definition 2. Let I be a directed set with an ordering denoted by $\preceq$. Let $(x_\alpha)_{\alpha\in I}$ be a net in a topological space X and let x be an element of X. Then we say that (x_α) converges to x, and x is called a limit point of (x_α) if for each neighborhood U of x, there is an α_U in I such that $x_\alpha \in U$ whenever $\alpha_U \preceq \alpha$. This convergence is expressed by writing $x_\alpha \to x$ or $lim_{\ \alpha} x_\alpha \to x$.

The following definition generalizes the notion of continuity in terms of net convergence in weak* topology.

Definition 3. Let X and Y be normed spaces and $\tau: X^* \to Y^*$ an operator. Then τ is said to be weak*-weak* continuous on X^* if $lim_\alpha \tau(x_\alpha^*) = \tau(x^*)$ for every net (x_α^*) in X^* that converges to $x^* \in X^*$.

Definition 4. An operator $T: X \to 2^{X^*}$ is said to be locally bounded at $x \in D(T)$ if there exists an open neighborhood V of x such that $T(V)$ is bounded. If T is locally bounded at each point of $U \subset D(T)$, then T is said to be locally bounded on.

The following theorem addresses the local boundedness of monotone operators (see [9]).

Theorem B. Let $T: X \to 2^{X^*}$be monotone. Then T is locally bounded at each $x \in \text{int}\, D(T)$.

For multivalued operators, the notion of continuity is generally replaced with that of the upper semicontinuity as defined below (e.g., see [5]).

Definition 5 (Upper Semi-continuity). Let X and Y be two linear topological spaces. A multivalued operator $T: X \to 2^Y$ is said to be upper semi-continuous if the set $\{x \in X: Tx \subset U\}$ is open in X whenever U is open in Y .

Note that $T : X \to \mathbb{R}$ is upper semicontinuous if and only if $\{x \in X: Tx \geq a\}$ is closed in $D(T)$ for every $a \in \mathbb{R}$.

We recall the following result from [5] that discusses sufficient conditions for a monotone operator to be maximal monotone.

Theorem C. Let X be real Banach space and $T : X \to 2^{X^*}$ a monotone operator.

(i) If T is maximal monotone, then for each $x \in D(T), Tx$ is convex and weak* closed, and T is norm-weak* upper semicontinuous.
(ii) If for each $x \in X, Tx$ is nonempty, convex, and weak* closed subset of X^* , and if T is norm - weak* upper semicontinuous, then T is maximal monotone.

We next discuss some basic results about maximal monotone operators (e.g., see [11]) that will be used in the proofs of our main results.

Lemma 1 (Simons, **[11]**). Let X be a Banach space and $T: X \to 2^{X^*}$ a maximal monotone operator. Then the following statements hold.

1. Tx is convex and weak* compact for any $x \in \text{int}\, D(T)$.
2. Assume that $\big((x_\alpha, x_\alpha^*)\big)$ is a bounded and norm-weak* converges to (x, x^*) in $G(T)$. Then $(x, x^*) \in G(T)$.
3. Let $y \in X$. Define $T_y: D(T) \to \mathbb{R} \cup \{\infty\}$ by

$$T_y(x) = \sup\{\langle y, T_x\rangle : x \in \text{int } D(T)\}.$$

Then T_y is real-valued and upper semicontinuous on int $D(T)$.

We have the following theorem as a characterization of the maximal monotone operators with full domain.

Theorem 2.1 (Simons, **[11]**). Let $D(T) = X$ and let $T: X \to 2^{X^*}$ be monotone. Then Tx is convex and weak* compact for all $x \in X$, and $T_y: X \to \mathbb{R} \cup \{\infty\}$ is upper semicontinuous for all $y \in X$ if and only if T is maximal monotone.

Proof.

Suppose T is maximal. Then from 1 and 3 of previous lemma, the if part is straightforward.
For the converse part, let $(z, z*) \in X \times X^*$ and

$$inf_{(s,s^*)\in G(T)}\langle s - z, s^* - z^*\rangle \geq 0. \qquad (1)$$

We have to show that $(z, z^*) \in G(T)$.
Let y be an arbitrary element of X. Let $\lambda > 0$. Since $D(T) = X$, there exists

$$(T_\lambda, T_\lambda^*) \in G(T): T_\lambda = z + \lambda y.$$

Then, $\langle \lambda y, T_\lambda^* - z^*\rangle = \langle T_\lambda - z, s_\lambda^* - z^*\rangle \geq 0$

and as a result, $\langle y, s_\lambda^* - z^*\rangle \geq 0$.

From the definition of T_λ, $T_\lambda(s\lambda) \geq \langle y, z^*\rangle$.

As $\lambda \to 0+$, $s\lambda \to z$ then as T_y is upper semicontinuous $T_y(z) \geq \langle y, z^*\rangle$.

This implies $\forall y \in X, \sup\langle y, T(z)\rangle \geq \langle y, z^*\rangle$.

As T_y is compact, $z^* \in Tz. i.e. (z, z^*) \in G(T)$.

Now we present the Heisler's result.

Theorem 2.2 (Simons, **[11]**). Assume, for two maximal operators

$T_1: X \to 2^{X^*}$ and $T_2: X \to 2^{X^*}$, $D(\mathrm{T}_1) = D(\mathrm{T}_2) = X$. Then $T_1 + T_2$ is maximal monotone.

Proof.

Denote $T = T_1 + T_2$. Then

$$D(T) = D(T_1) \cap D(T_2) = X.$$

As we know the sum of two weak* compact convex sets are also weak* compact and convex, for each $y \in X, T_y$ is weak* compact and convex. Also, the sum of two upper semicontinuous mappings $T_y = (T_1)_y + (T_2)_y$ is upper semi-continuous. Then $T = T_1 + T_2$ is maximal monotone from theorem 2.1.

3. Main Results

Let X and Y be two reals valued Banach spaces and X^* and Y^* be their duals. Let $T : X \to 2^{X^*}$ and $S : Y \to 2^{Y^*}$ be monotone operators. Define $H : X \times Y \to 2^{X^* \times Y^*}$ by

$$H(x,y) = \{(x^*, y^*) : x^* \in Tx, y^* \in Sx\}, (x,y) \in X \times Y.$$

The monotonicity of H is given in the following theorem.

Theorem 3.1. The mapping $H : X \times Y \to 2^{X^* \times Y^*}$ defined by

$$H(x,y) = \{(x^*, y^*) : x^* \in Tx, y^* \in Sx\} \text{ for all } (x,y) \in X \times Y \text{ is monotone.}$$

Proof.

Let $a, b \in X$. Since $T: X \to 2^{X^*}$ is monotone,

$$\langle a^* - b^*, a - b\rangle \geq 0, a^* \in Ta \text{ and } b^* \in Tb. \quad (2)$$

Similarly, from the monotonicity of $S: Y \to 2^{Y^*}$, for all $c, d \in Y$

$$\langle c^* - d^*, c - d\rangle \geq 0, c^* \in Sc \text{ and } d^* \in Sd. \quad (3)$$

Clearly $(a,c), (b,d) \in X \times Y$. Then from the definition of H

$$(a^*, c^*) \in H(a,c) \text{ and } (b^*, d^*) \in H(b,d).$$

We observe that

$$\langle (a^*, c^*) - (b^*, d^*), (a,c) - (b,d)\rangle = \langle (a^* - b^*, c^* - d^*), (a - b, c - d)\rangle.$$

Using the duality in $(X \times Y) \times (X^* \times Y^*)$ defined by

$$\langle (x^*, y^*), (x,y)\rangle = \langle x^*, x\rangle + \langle y^*, y\rangle, (x,y) \in X \times Y, (x^*, y^*) \in X^* \times Y^*,$$

we obtain $\quad \langle a^* - b^*, a - b\rangle + \langle c^* - d^*, c - d\rangle \geq 0,$

from the monotonicity of T and S. Thus H is monotone. □

The following result addresses the maximality of H whenever T and S are maximal monotone.

Theorem 3.2. Let X and Y be two real Banach spaces and $T : X \to 2^{X^*}$ and $S : Y \to 2^{Y^*}$ are maximal monotone operators. Then $H : (X \times Y) \to 2^{X^* \times Y^*}$ by

$$H(x,y) = \{(x^*, y^*) : x^* \in Tx, y^* \in Sx\} \text{ for } (x,y) \in X \times Y \text{ is maximal monotone.}$$

Proof.

By Theorem 3.1, H is monotone. To show that H is maximal monotone, we will show in view of Theorem C that

(i) for each $(x,y) \in X \times Y, H(x,y)$ is non-empty convex and weak*− closed, and

(ii) H is norm-to-weak* upper semicontinuous.

For each $(x,y) \in D(H)$, it follows from the maximal monotonicity of T and S that

$$\mathrm{H(x,y)} = \bigcap_{((x_1,y_1),(x_1^*,y_1^*) \in G(H)} \{(x^*, y^*) \in X^* \times Y^* : \langle (x^*, y^*) - (x_1^*, y_1^*), (x,y) - (x_1, y_1)\rangle \geq 0\},$$

Which implies

$$\mathrm{H(x,y)} = \bigcap_{((x_1,y_1),(x_1^*,y_1^*) \in G(H)} \{(x^*, y^*) \in X^* \times Y^* : \langle x^* - x_1^*, x - x_1\rangle + \langle y^* - y_1^*, y - y_1\rangle \geq 0\},$$

Since T and S are maximal monotone, it follows that Tx and Sy are nonempty, convex and weak* closed, and therefore, $H(x,y)$ also shares all these properties. This proves (i). Since T is maximal monotone, T is norm-to-weak* upper semicontinuous. Then for each $x \in X$, each weak* neighborhood U of Tx in $X*$, and each sequence $\{x_n\}$ in X such that $x_n^* \in Tx_n$ and $x_n \to x$, we have $x_n^* \in U$ for all sufficiently large $n \in \mathbb{N}$. Similarly, from the maximality of S, it follows that for each $y \in Y$, each weak* neighborhood V of Sx in Y^*, and each sequence $\{y_n\}$ in Y such that $y_n^* \in S_{y_n}$ and $y_n \to y$, we have $y_n^* \in V$ for all sufficiently large $n \in \mathbb{N}$ Suppose, on the contrary, that H is not weak* upper semicontinuous. Then there exists a point

$(x, y) \in X \times Y$, a weak* neighborhood W of $H(x, y)\ in\ X^* \times Y^*$, and a sequence $\{(x_n, y_n)\} \subseteq X \times Y$ such that $(x_n^*, y_n^*) \in H(x_n, y_n)$ and $(x_n, y_n) \to (x, y)$ and $(x_n^*, y_n^*) \notin W$ for infinitely many values of $n \in \mathbb{N}$. Consequently, there exist weak* neighborhoods U and V of x in X and y in Y, respectively, such that $(x_n^*, y_n^*) \notin U \times V$ for infinitely many values of n. This is a contradiction. Therefore, H is norm-to-weak* upper semicontinuous, which proves (ii). It follows from the part (ii) of Theorem C that $H : X \times Y \to 2^{X^* \times Y^*}$ is maximal monotone.

4. Conclusion

In this paper, we discussed some requirements for the sum $T + S$ of two maximal monotone mappings T and S to be again a maximal monotone in the case $X = \mathrm{Y}$ and an important characterization of maximal monotone operators in general Banach spaces. As our main result, we proved that the operator $\mathrm{H}: \mathrm{X} \times \mathrm{Y} \to 2^{\mathrm{X}^* \times \mathrm{Y}^*}$ defined by $\mathrm{H}(\mathrm{x}, \mathrm{y}) = \{(x^*, \mathrm{y}^*): \mathrm{x}^* \in \mathrm{T}x, \mathrm{y}^* \in \mathrm{Sy}\}, x \in \mathrm{X}, \mathrm{y} \in \mathrm{Y}$, is maximal monotone.

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Research Article
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On Two Closed Form Evaluations for the Generalized Hypergeometric Functions ${}_3F_2\left(\frac{1}{16}\right)$

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Abstract: *The main objective of this note is to provide two closed-form evaluations for the generalized hypergeometric functions with the argument* 1/16*. This is achieved by means of separating a generalized hypergeometric function* ${}_3F_2$ *into even and odd components together with the use of two known sums involving reciprocal of the certain binomial coefficients obtained very recently by Gencev.*

Keywords: Generalized hypergeometric functions, Central binomial coefficients, and Combinatorial sum.

1. Introduction

The concept of hypergeometric and generalized hypergeometric functions is essential in mathematics, engineering mathematics, and mathematical physics. Many of the frequently encountered functions in analysis are special or limiting cases of these two functions. Prof. John Wallis [2] was the first person to use the term "hypergeometric" in his work *Arithmetica Infinitorum* (1655), to refer to any series that is extended beyond the ordinary series. In fact, he explored the series...

$$1+a+a(a+1)+a(a+1)(a+2)+\ldots$$

Then Euler, Vandermode, Hidenberg, and other mathematicians studied similarly for the next hundred and fifty years.

In 1812, C.F. Gauss [3] used the symbol

$${}_2F_1\left[\begin{matrix} a, \quad b \\ c \end{matrix} ;z\right]$$

to define the infinite series given below:

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ; z\right] = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots \quad (1) $$

By using commonly used Pochhammer symbol for any $a \epsilon \mathbb{C}\ (\neq 0)$ and $n \in \mathbb{N}$ by

$$ (a)_n = \left\{\begin{matrix} a(a+1)(a+2)\ldots(a+n-1); & n \in \mathrm{N} \\ 1; & n = 0 \end{matrix}\right\} $$

$$ = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad \text{where } \Gamma(a) \text{ is the gamma function.} $$

The series (1) can be written as

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (2) $$

The quantities a, b and c expressed in (2) are called the parameters (real or complex) of the series provided that $c \neq 0, -1, -2, \ldots$. The term z is the variable of the series. By using the ratio tests, the series (2) is

(i) convergent for all values of z such that $|z| < 1$ and divergent for $|z| > 1$,

(ii) convergent for z =1 if $Re(c-a-b) > 0$ and divergent otherwise,

(iii) absolutely convergent for $z = -1$ if $Re(c-a-b) > 0$ and but

not absolutely convergent if -1< $Re(c-a-b) \leq 0$ and divergent for $Re(c-a-b) \leq -1$.

The observations for the Gauss's series (4) are given below:

(i) the series (4) reduces to the geometric series if a =1 and $b = c$ or b =1 and $a = c$. From this fact, this series (4) is called the "Hypergeometric Series".

(ii) the series becomes unity if a and b or both are zero.

(iii) the series becomes a polynomial (the series containing the finite number of terms and the issue of convergence does not arise) if a *or* b or both is a negative integer.

For the limiting case of (2), since $\frac{(b)_n}{b^n} z^n \to z^n$, and if we replace z by $\frac{z}{b}$ take the limit as $b \to \infty$, then we arrive at the infinite series denoted by the symbol ${}_1F_1\left[\begin{matrix} a, \\ c \end{matrix} ; z\right]$.

This is the Kummer's series or confluent hypergeometric function already defined in the literature viz.

$$ {}_1F_1\left[\begin{matrix} a, \\ c \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} \quad (3) $$

Gauss's hypergeometric function ${}_2F_1$ and its confluent hypergeometric function ${}_1F_1$ serve as foundational elements of special functions, encompassing many commonly used functions as special or limiting cases. These include the exponential function, trigonometric and inverse trigonometric functions, hyperbolic functions, Legendre's function, the incomplete Beta function, the complete elliptic functions of the first and second kinds, and most classical orthogonal polynomials; all of which are particular cases of the Gaussian hypergeometric function.

Similarly, the confluent hypergeometric function encompasses special cases such as Bessel's function, the parabolic cylinder function, and the Coulomb wave function. As previously noted, the Gaussian hypergeometric function is characterized by two numerator parameters a and b, along with one denominator parameter c. A natural generalization of this function is accomplished by introducing arbitrary number of p numerator and q number of denominator parameters. The resulting function is defined [2, 3, 9, 10, 12, 13, 16] as

$$ {}_pF_q\left[\begin{matrix}(a), \\ (c)\end{matrix} \quad ;z\right] = {}_pF_q\left[\begin{matrix}a_1,\dots,a_p \\ c_1,\dots,c_p\end{matrix} \quad ;z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(c_1)_n \dots (c_q)_n} \frac{z^n}{n!} \tag{4} $$

It is also assumed that the numerator parameters a_j $(j = 1,2, \dots, p)$, the denominator parameter c_j $(j = 1,2, \dots, q)$ and the variable z, can take real or complex values provided that $c_j \neq 0, -1, -2, \dots for\ j = 1,2, \dots, q$.

Also, the series (4) is

(i) convergent for $|z| < \infty$ if $p < q$,

(ii) convergent for $|z| < 1$ if $p = q + 1$ and divergent for all z, $z \neq 0$ if $p > q + 1$.

Further if we set $\omega = \left(\sum_{j=1}^{q} c_j - \sum_{j=1}^{q} a_j \right)$ then the series (4) with $p = q + 1$ is

(i) convergent absolutely for $|z| = 1$ if $Re(\omega) > 0$,

(ii) conditionally convergent for $|z| = 1, z \neq 1$ if $-1 < Re(\omega) \leq 0$ and divergent $|z| = 1$ if $Re(\omega) \leq -1$.

Hypergeometric functions ${}_2F_1$ and generalized hypergeometric functions ${}_pF_q$ have wide range of applications in the field of mathematics, engineering mathematics and mathematical physics. For details, see [1, 13, 16].

One the other hand, the binomial coefficients for non-negative integers $n\ and\ k$ are defined by

$$ \binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & ;n \geq k \\ 0 & ;n < k \end{cases} \tag{5} $$

The central binomial coefficients are defined by

$$ \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \quad (n = 0,1,2, \dots). \tag{6} $$

The binomial and the reciprocal of the binomial coefficients always plays an important role in many areas of mathematics (including number theory, probability and statistics). The sums containing the central binomial coefficients and the reciprocal of the central binomial coefficients are being studied since a long time ago.

The significant number of elegant results were published in the works of Kummer,et.al [7], Lehmer [8], Mansour [9], Pla [11], Sherman [15], Sprugnoli [17,18], Sury [19], Sury et.al [20], Trif [21], Wheelon [22], Zhng and J. [23] and Zhao and Wang [24]. Koshy [6] has mentioned the details on the central binomial coefficients and the reciprocal of the central binomial coefficients in his book. Gould [5] and Riordan [14] has collected numerous identities involving central binomial coefficients.

Recently, Gencev [4] studies the following interesting sums involving the reciprocal of the central binomial coefficients viz.

$$\sum_{k=1}^{\infty}\frac{1}{k^m\binom{2k}{k}} \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m\binom{2k}{k}}.$$

These sums are known as Aprey sums [3]. In particular, Uhl [9] obtained the following two interesting sums;

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)\binom{2k+2}{k+2}}=\frac{1}{2}-\frac{\sqrt{3}\pi}{18} \tag{7}$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)\binom{2k+4}{k+2}}=-\frac{1}{2}+\frac{3}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+1}{2}\right) \tag{8}$$

In terms of generalized hypergeometric functions, the results (7) and (8) can be written in the following manner;

$${}_3F_2\left[\begin{matrix}1, & 1, & 3 \\ 2, & \frac{5}{2} & \end{matrix};\frac{1}{4}\right]=3-\frac{\sqrt{3}\pi}{3} \tag{9}$$

and

$${}_3F_2\left[\begin{matrix}1, & 1, & 3 \\ 2, & \frac{5}{2} & \end{matrix};-\frac{1}{4}\right]=-3+\frac{18}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+1}{2}\right) \tag{10}$$

It is known that the new results can be obtained by resolving a generalized hypergeometric function ${}_pF_q\,(z)$ into odd and even components. We shall employ this procedure combined with the results (9) and (10) to obtain two new closed form evaluations of the series ${}_3F_2$ with argument $\frac{1}{16}$.

The same is given in the next section.

2.Two Closed Form Evaluations of ${}_3F_2\left(\frac{1}{16}\right)$

We shall establish the two new closed form evaluations for the generalized hypergeometric functions ${}_3F_2$ with the argument $\frac{1}{16}$ in this section via the theorem given below.

Theorem: The following two results hold true.

$${}_3F_2\left[\begin{matrix}\frac{1}{2}, & 1, & 2 \\ \frac{5}{4}, & \frac{7}{4} & \end{matrix};\frac{1}{16}\right]=\frac{9}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+1}{2}\right)-\frac{\sqrt{3}\pi}{6} \tag{11}$$

and

$$ {}_3F_2\left[\begin{matrix}1, & 1, & \frac{5}{2} \\ \frac{7}{4}, & \frac{9}{4} & \end{matrix}\;;\frac{1}{16}\right] = \frac{10}{3}\left\{6 - \frac{\sqrt{3}\pi}{3} - \frac{18}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+1}{2}\right)\right\} \tag{12} $$

Proof:

To prove the results (11) and (12), we shall use the following general results mentioned in [12

$$ {}_{q+1}F_q\left[\begin{matrix}a_1,\dots,a_{q+1} \\ c_1,\dots,c_q\end{matrix}\;;z\right] + {}_{q+1}F_q\left[\begin{matrix}a_1,\dots,a_{q+1} \\ c_1,\dots,c_q\end{matrix}\;;-z\right] = 2\; {}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{a_1}{2},\frac{a_1}{2}+\frac{1}{2},\dots,\frac{a_{q+1}}{2}+\frac{1}{2} \\ \frac{1}{2},\frac{c_1}{2},\frac{c_1}{2}+\frac{1}{2}\dots,\frac{c_q}{2}+\frac{1}{2}\end{matrix}\;;z^2\right] \tag{13} $$

and

$$ {}_{q+1}F_q\left[\begin{matrix}a_1,\dots,a_{q+1} \\ c_1,\dots,c_q\end{matrix}\;;z\right] - {}_{q+1}F_q\left[\begin{matrix}a_1,\dots,a_{q+1} \\ c_1,\dots,c_q\end{matrix}\;;-z\right] $$

$$ = \frac{2za_1a_2\dots a_{q+1}}{c_1c_2\dots c_q}\;{}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{a_1}{2}+\frac{1}{2},\frac{a_1}{2}+1,\dots,\frac{a_{q+1}}{2}+\frac{1}{2},\frac{a_{q+1}}{2}+1 \\ \frac{3}{2},\frac{c_1}{2}+\frac{1}{2},\frac{c_1}{2}+1,\dots,\frac{c_q}{2}+\frac{1}{2},\frac{c_q}{2}+1\end{matrix}\;;z^2\right] \tag{14} $$

These results (13) and (14) can be established by resolving a generalized hypergeometric function ${}_{q+1}F_q\left[\begin{matrix}a_1,\dots,a_{q+1} \\ c_1,\dots,c_q\end{matrix}\;;z\right]$ into even and odd components and making use of the following identities,

$$ (a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a}{2}+\frac{1}{2}\right)_n \quad \text{and} \quad (a)_{2n+1} = a2^{2n}\left(\frac{a}{2}+1\right)_n\left(\frac{a}{2}+\frac{1}{2}\right)_n $$

in (13) and (14), respectively.

Therefore, for the derivation of the results (11) and (12), we substitute in the results (9) and (10) by letting q = 2 and substituting $a_1 = a_2 = 1, a_3 = 3, b_1 = 2, b_2 = \frac{5}{2}$, $z = \frac{1}{4}$ in (3) and (4) respectively and we obtain the results (1) and (2) respectively after some simplification.

3. Conclusion

In this paper, two new and interesting closed-form evaluations of the generalized hypergeometric functions ${}_{q+1}F_q(z)$ for q = 2 with argument 1/16 have been established. This is done by separating the generalized hypergeometric function ${}_{q+1}F_q(z)$ into two components, even and odd, together with the use of two proven results by Gensev for the series involving reciprocals of the non-central binomial coefficients. We believe that the results established in this paper have not appeared in the literature before and represent a definite contribution in the area of generalized hypergeometric functions.

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On Difference Sequence Space $\ell_M(X, \alpha, P)$ Defined by Orlicz function

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Abstract: *The idea of difference sequence space is introduced by Kizmaz. Lindenstrauss and Tzafriri used the concept of Orlicz function M to construct the sequence space ℓ_M. In this paper, we introduce the class $\ell_M(X, \alpha, P)$ of the generalized form of difference sequence space and study some inclusion and linear properties.*

Keywords: Sequence space, Difference sequence space, Orlicz function, Banach spaces.

1. Introduction:

Sequence spaces have a major role in mathematics, especially in functional analysis and mathematical analysis. They give researchers a framework for examining infinite numerical sequences and provide information on their convergence, summability, and other characteristics. Mathematicians have extensively researched sequence spaces, which are vector spaces of sequences with elements in real or complex numbers ($\mathbb{R}$ or $\mathbb{C}$). Many mathematicians in classical analysis have investigated these spaces, which include convergence, bounded, null, and l_p spaces.

If ω denotes the set of all functions from the set of positive integers $\mathbb{N}$ to the field $\mathbb{K}$ of real or complex numbers then it becomes a vector space. Any Sequence space is defined as a set of all sequences $x = (x_n)$ linear subspace of ω over the field $\mathbb{C}$ with the usual operations defined as

$$(x_n) + (y_n) = (x_n + y_n) \text{ and } \lambda(x_n) = (\lambda x_n).$$

Several researchers, including Kamthan and Gupta [4] (1980), Maddox [7] (1981), Ruckle [15] (1981), and Malkowski and Rakocevic[8] (2004), have made substantial contributions to the theory of vector and scalar valued sequence spaces using Banach sequences. Similarly, Orlicz used the idea of Orlicz function to construct the space (L^M). Lindentrauss and Tzafriri [6] investigated Orlicz sequence spaces in more detail, and proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(1 \leq p < \infty)$.

Subsequently different classes of sequence spaces defined by Murasaleen et al.[9], Parashar and Choudhary [10], Subramanian [17] , Tripathy et al. [18] , and many others are studied.

A function $M: [0, \infty\ ) \longrightarrow [0, \infty\ )$ is called Orlicz function if it is continuous, non-decreasing, and convex with

$$M(0) = 0, M(t) > 0 \text{ for } t > 0 \text{ and } M(t) \to \infty \text{ } as\ t\ \to \infty.$$

If the convexity of Orlicz function M is replaced by $M(t+u) \leq M(t) + M(u)$ then this function is called modulus function.

In 1971, Lindenstrauss and Tzafriri[6] used the Orlicz function to construct the following class

$$\ell_M = \{\bar{x} = (x_k) \in \omega\colon \textstyle\sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

This class together with the norm defined by

$$\|\bar{x}\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}$$

forms a Banach space called an Orlicz sequence space and is related to the sequence space ℓ_p with

$$M(t) = t^p, 1 \leq p < \infty.$$

The concept of difference sequence spaces was first introduced by Kizmaz[5] in 1981 and defined as

i. $c_0(\Delta(\mathrm{X})) = \{\ \bar{x} = (x_k)\ \in X :\ \|\Delta x_k\|\ \to 0\ as\ k\ \to \infty\}$

ii. $c(\Delta(\mathrm{X})) = \{\ \bar{x} = (x_k)\ \in X :\ \exists\ l \in X\ s.t.\ \|\Delta x_k - l\|\ \to 0\ as\ k\ \to \infty\}$

iii. $\ell_\infty(\Delta(\mathrm{X})) = \{\ \bar{x} = (x_k)\ \in X :\ sup_k\|\Delta x_k\| < \infty\}$

iv. $\ell_p(\Delta(\mathrm{X})) = \{\ \bar{x} = (x_k)\ \in X :\ \sum_{k=1}^{\infty}\|\Delta x_k\|^p < \infty, 0 < p <\ \infty\}$

where $\Delta x_k = x_k - x_{k-1}$.

These spaces are Banach spaces with norm $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$.

Et[2], in 1993, generalized the concept of Kizmaz to study the Δ^2 sequence spaces of Banach space X-valued sequences defined as follows:

i. $c_0(\Delta^2(X)) = \{\ \bar{x} = (x_k)\ \in X :\ \|\Delta^2 x_k\|\ \to 0\ as\ k\ \to \infty\}$

ii. $c(\Delta^2(\mathrm{X})) = \{\ \bar{x} = (x_k)\ \in X :\ \exists\ l \in X\ s.t.\ \|\Delta^2 x_k - l\|\ \to 0\ as\ k\ \to \infty\}$

iii. $\ell_\infty(\Delta^2(\mathrm{X})) = \{\ \bar{x} = (x_k)\ \in X :\ sup_k\|\Delta^2\, x_k\| < \infty\}$

where

$$\Delta^2\, x_k\ = \Delta x_k - \Delta x_{k-1}\ = x_k -\ x_{k-1} - x_{k-1} -\ x_{k-2}\ =\ x_k -\ 2x_{k-1} + x_{k-2}.$$

These space are Banach spaces with the norm defined by

$$\|x\|_\Delta = |x_1| + |x_2| + \|\Delta^2 x\|_\infty .$$

Similarly, in 1995, Et and Colak [3] defined the following classes

i. $c_0(\Delta^m(X)) = \{\ \bar{x} = (x_k)\ \in X :\ \|\Delta^m x_k\|\ \to 0\ as\ k\ \to \infty\}$

ii. $c(\Delta^m(\mathrm{X})) = \{ \bar{x} = (x_k) \in X : \exists\, l \in X\ s.t.\ \|\Delta^m x_k - l\| \to 0\ as\ k \to \infty\}$

iii. $\ell_\infty(\Delta^m(\mathrm{X})) = \{ \bar{x} = (x_k) \in X : sup_k \|\Delta^m x_k\| < \infty\}$

where $m \in \mathbb{N}, \Delta^0 x = (x_k), \Delta x_k = x_k - x_{k-1}$,

and $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k-1} = \sum_{r=1}^{\infty}(-1)^r \binom{m}{r} x_{k+r}$.

They showed that these classes are Banach spaces with norm defined as

$$\|x\|_\Delta = \sum_{r=1}^{m}|x_r| + \|\Delta^m x\|_\infty .$$

In 2006, Tripathy and Esi [18] studied a new type of difference sequence spaces $c(\Delta_m), c_0(\Delta_m), c(\Delta_m)$ where $m \in \mathbb{N}$ as defined by

$$Z(\Delta_m) = \{ \bar{x} = (x_k) \in \omega\colon \Delta_m x \in Z\}, \quad \text{for } Z = \ell_\infty, c \text{ and } c_0$$

where, $\Delta_m x = (\Delta_m x_k) = (x_{k+m} - x_{k,})$for all $k \in \mathbb{N}$.

For m =1, $\ell_\infty(\Delta_m) = \ell_\infty(\Delta)$, $c(\Delta_m) = c(\Delta)$, $c_0(\Delta_m) = c_0(\Delta)$.

He proved that these spaces are Banach spaces with norm defined by

$$\|x\|_\Delta = \sum_{r=1}^{m}|x_r| + \sup_k|\Delta_m x_k|.$$

and the inclusion relations $c_0(\Delta_m) \subset c(\Delta_m) \subset \ell_\infty(\Delta_m)$

and $Z(\Delta) \subset Z(\Delta_m)$ for, $Z = \ell_\infty, c, c$ and c_0.

In 2012, Srivastava and Pahari [16] introduced the following class

$$\ell_M(X, \alpha, P) = \left\{x = (x_k) \in \omega(X) \colon \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\, x_k\|^{p_k}}{\rho}\right) < \infty \text{ for some } \rho > 0\right\},$$

and

$$\ell_M(X, \alpha, P, L) = \left\{x = (x_k) \in \omega(X) \colon \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\, x_k\|^{p_k/L}}{\rho}\right) < \infty \text{ for some } \rho > 0\right\},$$

where, $sup_k p_k = L < \infty$.

For $\alpha = (\alpha_k)$ for a sequence of nonzero complex numbers and $P = (p_k)$ for any sequence of strictly positive real numbers, they also investigated conditions for containment relation of $\ell_M(X, \alpha, P)$ and explore the linear topological structure of the class $\ell_M(X, \alpha, P, L)$.

Recently, in 2022 and 2023, Paudel, Pahari and et al. [11], [12] , [13] studied the various properties of sequence space using fuzzy concept. In 2023, Pokharel et al. [14] introduced a new class of double sequences in a normed space X to generalized the well-known sequence space ℓ by introducing and studying a new class $\ell^2 ((X, \|\,.\,\|), \bar{\gamma}, \bar{w})$ of double sequences with their terms in a normed space X focusing on exploring some of the preliminary results that characterize the linear topological structures.

Lemma [1]: Let (p_k) be a bounded sequence of strictly positive real numbers with $0 < p_k \le \sup p_k = L,\ D = \max\{1,\ 2^{L-1}\}$ then

i. $|x+y|^{p_k} \le D\{|x|^{p_k} + |y|^{p_k}\}$;

ii. $|\alpha|^{p_k} \le \max(1,\ [\alpha]^L)$.

Throughout the article, we shall denote for $\alpha = (\alpha_k),\ \beta = (\beta_k)$

$$\gamma_k = \frac{\alpha_k}{\beta_k} \text{ and } \delta_k = \left|\frac{\alpha_k}{\beta_k}\right|^{p_k}.$$

On the basis of the literature mentioned above, here we define the following class

$$\ell_M(X,\alpha,P) = \left\{x = (x_k) \in \omega(X) : \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ \Delta_m\ x_k\|^{p_k}}{\eta}\right) < \infty \text{ for some } \eta > 0\right\}$$

where, $\Delta_m x = (\Delta_m x_k) = (x_{k+m} - x_{k,})$ for all $k \in \mathbb{N}$.

2. Main Results

In this section, we shall study some of the containment relations on the class $\ell_M(X,\alpha,P)$ for different values of α and P and examine the linear structures of $\ell_M(X,\alpha,P)$.

Lemma 2.1. $\ell_M(X,\alpha,P) \subset \ell_M(X,\beta,P)$ if $\inf_k \delta_k > 0$.

Proof:

Let $\inf_k \delta_k > 0$ and let $x = (x_k) \in \ell_M(X,\alpha,P)$ then there exist $m_1 > 0, \eta > 0$ and positive integer K such that

$$M\,|\beta_k|^{p_k} < |\alpha_k|^{p_k}\ \forall\ k \ge K \text{ and } \sum_{k=K}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k}}{\eta}\right) < \infty.$$

Choose $\eta_1 > 0$ such that $\eta < m_1\,\eta_1$. Since M is non-decreasing, we have

$$\sum_{k=K}^{\infty} M\left(\frac{\|\beta_k \Delta_m x_k\|^{p_k}}{\eta_1}\right) = \sum_{k=K}^{\infty} M\left(\frac{|\beta_k|^{p_k}\|\Delta_m x_k\|^{p_k}}{\eta_1}\right)$$

$$\le \sum_{k=K}^{\infty} M\left(\frac{|\alpha_k|^{p_k}\|\Delta_m x_k\|^{p_k}}{m_1\,\eta_1}\right)$$

$$\le \sum_{k=K}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k}}{\eta}\right)$$

$$< \infty$$

which indicates that $x = (x_k) \in \ell_M(X,\beta,P)$. Thus

$$\ell_M(X,\alpha,P) \subset \ell_M(X,\beta,P).$$

□

Lemma 2.2. If $\ell_M(X,\alpha,P) \subset \ell_M(X,\beta,P)$, then $\inf_k \delta_k > 0$.

Proof:

Assume that $\ell_M(X,\alpha,P) \subset \ell_M(X,\beta,P)$ holds but $inf_k \delta_k = 0$ so that we can find a sequence ($k(n)$) of integers such that $k(n+1) > k(n) \geq 1$ for which

$$n^2 \left|\alpha_{k(n)}\right|^{p_{k(n)}} < \left|\beta_{k(n)}\right|^{p_{k(n)}} \quad \forall n \geq 1.$$

Corresponding to $z \in X$, with $\|z\| = 1$ we can define a sequence $x = (x_k)$ by

$$\Delta_m x_k = \begin{cases} \alpha_{k(n)}^{-1} n^{-2/p_{k(n)}} & \text{for } k = k(n), n \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

By using the convexity of M , we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k}}{\eta}\right) = \sum_{n=1}^{\infty} M\left(\frac{\left\|n^{-2/p_{k(n)}} z\right\|^{p_{k(n)}}}{\eta}\right) = \sum_{n=1}^{\infty} M\left(\frac{1}{n^2 \eta}\right)$$

$$\leq M\left(\frac{1}{\eta}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

$\Longrightarrow x \in \ell_M(X,\alpha,P)$.

But on the other hand for any $\eta > 0$ we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\beta_k \Delta_m x_k\|^{p_k}}{\eta}\right) = \sum_{n=1}^{\infty} M\left(\frac{\left\|\beta_{k(n)} n^{-2/p_{k(n)}} z\right\|^{p_{k(n)}}}{\alpha_{k(n)} \ \eta}\right)$$

$$= \sum_{n=1}^{\infty} M\left(\left|\frac{\beta_{k(n)}}{\alpha_{k(n)}}\right|^{p_{k(n)}} \frac{1}{n^2 \eta}\right)$$

$$\geq \sum_{n=1}^{\infty} M\left(\frac{1}{\eta}\right) = \infty,$$

showing that $x \notin \ell_M(X,\beta,P)$ which is a contradiction. So we have $inf_k \delta_k > 0$.

□

Theorem 2.3. $\ell_M(X,\alpha,P) \subset \ell_M(X,\beta,P)$ if and only if $\inf_k \delta_k > 0$.

After combining Lemmas 2.1 and 2.2, the result follows:

Theorem 2.4. $\ell_M(X,\beta,P) \subset \ell_M(X,\alpha,P)$ if only if $\lim sup_k \delta_k < \infty$.

Proof:

Let $\lim sup_k \delta_k < \infty$. Then we can find a positive integer L such that $L|\beta_k|^{p_k} > |\alpha_k|^{p_k}$ for sufficiently large k. Then by using lemma 1 the result follows.

For the necessity, suppose that $\ell_M(X,\beta,P) \subset \ell_M(X,\alpha,P)$ but $\lim sup_k \delta_k = \infty$. we can find a sequence (k(n)) of integers such that $k(n+1) > k(n) \geq 1$ for which

$$\left|\frac{\alpha_{k(n)}}{\beta_{k(n)}}\right|^{p_{k(n)}} > n^2, \forall n \geq 1.$$

Now corresponding to $z \in X$, with $\|z\| = 1$, we can define a sequence $x = (x_k)$ by

$$\Delta_m x_k = \begin{cases} \beta_{k(n)}^{-1} n^{-2/p_{k(n)}} & \text{for } k = k(n), n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

By using convexity of M , we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\beta_k \Delta_m x_k\|^{p_k}}{\eta}\right) = \sum_{n=1}^{\infty} M\left(\frac{\left\|n^{-2/p_{k(n)}} z\right\|^{p_{k(n)}}}{\eta}\right) = \sum_{n=1}^{\infty} M\left(\frac{1}{n^2 \eta}\right)$$

$$\leq M\left(\frac{1}{\eta}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

which shows that $x \in \ell_M(X, \beta, P)$.

On the other hand for any $\eta > 0$ we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k}}{\eta}\right) = \sum_{n=1}^{\infty} M\left(\frac{\left\|\alpha_{k(n)} n^{-2/p_{k(n)}} z\right\|^{p_{k(n)}}}{\beta_{k(n)}\ \eta}\right)$$

$$= \sum_{n=1}^{\infty} M\left(\left|\frac{\alpha_{k(n)}}{\beta_{k(n)}}\right|^{p_{k(n)}} \frac{1}{n^2 \eta}\right)$$

$$\geq \sum_{n=1}^{\infty} M\left(\frac{1}{\eta}\right) = \infty,$$

showing that $x \notin \ell_M(X, \alpha, P)$, a contradiction.

This completes the proof.

□

Lemma 2.5 : $\ell_M(X, \alpha, P)$ forms a linear space over $\mathbb{C}$ if $\lim sup_k p_k < \infty$.

Proof:

Let $sup_k p_k < \infty$. Let $x, y \in \ell_M(X, \alpha, P)$ and let $a, b \in \mathbb{C}$ then there exist scalars $\eta_1, \eta_2 > 0$ such that

$$\sum_{k=1}^{\infty} \alpha_k \Delta^m x_k < \infty \qquad \sum_{k=1}^{\infty} \alpha_k \Delta^m x_k < \infty.$$

Let us choose η_3 such that

$$2\mathrm{D}\eta_1 \max\{1, |a|\} \leq \eta_3 \text{ and } 2\mathrm{D}\eta_2 \max\{1, |b|\} \leq \eta_3.$$

For such $\leq \eta_3$, by using non-decreasing and convex properties of M ,we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k(a\Delta^m x_k + \mathrm{b}\Delta^m y_k)\|^{p_k}}{\eta_3}\right) \leq \sum_{k=1}^{\infty} M\left(\frac{D\|a\ \alpha_k \Delta^m x_k\|^{p_k}}{\eta_3} + \frac{D\|b\alpha_k \Delta^m y_k\|^{p_k}}{\eta_3}\right)$$

$$= \sum_{k=1}^{\infty} M\left(\frac{D|a|^{p_k}\|\alpha_k \Delta^m x_k\|^{p_k}}{\eta_3} + \frac{D|b|^{p_k}\|\alpha_k \Delta^m y_k\|^{p_k}}{\eta_3}\right)$$

$$\leq \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta^m x_k\|^{p_k}}{2\eta_1} + \frac{\|\alpha_k \Delta^m y_k\|^{p_k}}{2\eta_2}\right)$$

$$\leq \frac{1}{2} \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta^m x_k\|^{p_k}}{\eta_1}\right) + \frac{1}{2} \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta^m y_k\|^{p_k}}{\eta_2}\right) < \infty.$$

This implies that $\ell_M(X, \alpha, P)$ forms a linear space over $\mathbb{C}$. □

Lemma 2.6 : If $\ell_M(X, \alpha, P)$ forms a linear space over $\mathbb{C}$, then $\lim sup_k p_k < \infty$.

Proof:

Suppose that $\ell_M(X, \alpha, P)$ is a linear space over $\mathbb{C}$ but $\lim sup_k p_k = \infty$. we can find a sequence ($k(n)$) of integers such that $k(n+1) > k(n) \geq 1, n \geq 1$ for which $p_{k(n)} > n$ for which $n \geq 1$. Now corresponding to $z \in X$, with $\|z\| = 1$ we can define a sequence $x = (x_k)$ by

$$\Delta_m x_k = \begin{cases} \alpha_{k(n)}^{-1} n^{-2/p_{k(n)}} z & \text{for } k = k(n), n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then for $k = k(n)$, we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k a \Delta_m x_k\|^{p_k}}{\eta}\right) = \sum_{k=1}^{\infty} M\left(\frac{\left\| n^{-2/p_{k(n)}} z\right\|^{p_k}}{\eta}\right)$$

$$= \sum_{n=1}^{\infty} M\left(\frac{1}{n^2 \eta}\right) \leq M\left(\frac{1}{\eta}\right) \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$< \infty.$$

So, $x \in \ell_M(X, \alpha, P)$.

On the other hand since $p_{k(n)} > n$ for any $n \geq 1$ and scalar a = 4 we get

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k a \Delta_m x_k\|^{p_k}}{\eta}\right) = \sum_{k=1}^{\infty} M\left(\frac{\left\|4\, n^{-2/p_{k(n)}} z\right\|^{p_k}}{\eta}\right)$$

$$\geq \sum_{n=1}^{\infty} M\left(\frac{4^n}{n^2 \eta}\right)$$

$$\geq \sum_{n=1}^{\infty} M\left(\frac{1}{\eta}\right) = \infty \text{ as } \frac{4^n}{n^2} > 1, \text{ for each } n \geq 1.$$

This shows that $\alpha x \notin \ell_M(X, \alpha, P,)$ which gives a contradiction. This completes the proof.

□

On combining Lemmas 2.5 and 2.6, the theorem 2.7 follows.

Theorem 2.7: $\ell_M(X, \alpha, P)$ forms a linear space over $\mathbb{C}$ if and only if $\lim sup_k p_k < \infty$.

3. Conclusion

Sequence spaces are important in mathematics, particularly in mathematical and functional analysis. Here, using the concept of difference sequence spaces, we have discussed various fundamental topological properties of the generalized form of difference sequence space $\ell_M(X, \alpha, P)$.

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Sawi Transform of Hypergeometric Functions: A New Perspective on Special Functions

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Abstract: *In this article, we shall study the Sawi transform of the generalized Wright hypergeometric function. New results are derived involving the Sawi transform for generalized hypergeometric functions. Also, we extend our analysis to the Sawi transform of a product involving Mittag-Leffler function, generalized hypergeometric function, and several related results. These findings give us a new way to think about the Sawi Transform's role in the study of special functions.*

Mathematical Sciences Classification: 33C05, 44A05, 44A20.

Keywords: Integral transform, Sawi transform, Wright hypergeometric functions, hypergeometric function, Mittag-Leffler function.

1. Introduction

Special functions, especially Wright hypergeometric functions, are very important to many areas of mathematics, science, and engineering. The extended Wright hypergeometric function and Mittag-Leffler function are used in many areas, such as statistical physics, fluid dynamics, and quantum mechanics. Integral transforms are widely used to solve the differential equations.

A significant amount of research has been conducted on these functions using integral transforms like the Laplace transform, the Fourier transform, and the Mellin transform. As a new member of the family of integral transforms, the Sawi transform has been shown to facilitate the evaluation certain integrals and the solution of differential equations. The Sawi transform provides a more efficient and generalized approach compared to Laplace, Fourier, and Mellin transforms, especially for solving fractional differential equations and handling special function. Numerous researchers [3-6, 9-10] have studied in recent years. Awwad et al.

[1] introduced the double ARA-Sawi transform, extending its applicability to more complex integral equations. Additionally, Momani et al. [8] explored how combining the Laplace and Sawi transforms improves computational techniques in mathematical modeling. Saadeh et al. [11] investigated the use of Sawi transform in fractional differential equations, integrating it with iterative methods to enhance solution accuracy. The goal of this work is to bridge the gap by exploring the Sawi Transform of generalized Wright hypergeometric functions and demonstrating its applications.

The Sawi transform opens new research directions in integral transform, special functions, and fractional calculus, paving the way for future mathematical and applied studies.

2. Basic Definitions

2.1 The Sawi Transform

Mahgoub [7] came up with the Sawi transform of the function $f(t), t \geq 0$ in 2019 for a function in the set A marked by:

$$A = \left\{ f(t): \exists\, M, k_1, k_2 > 0;\ |f(t)| < M\, e^{\frac{|t|}{k_j}}, if\ t \in (-1)^j \times [0, \infty) \right\}$$

For a certain function in set A, M must be a real number, but k_1, k_2 can be a real number or an infinite number. Then $S[f(t)]$ stands for the Sawi transform of $f(t)$, which is defined as

$$S[f(t)] = R(\upsilon) = \frac{1}{\upsilon^2} \int_0^\infty e^{-\frac{t}{\upsilon}}\, f(t)\, dt\ , k_1 \leq \upsilon \leq k_2. \qquad (1)$$

2.2 The Sawi transform of some elementary functions [9, 12]

$f(t)$	1	t	$t^n, n \in N$	$t^\alpha, \alpha \in R^+$	e^{at}
$S[f(t)]$	$1/\upsilon$	1	$n!\,\upsilon^{n-1}$	$\Gamma(\alpha+1)\,\upsilon^{\alpha-1}$	$\frac{1}{\upsilon(1-a\upsilon)}$

2.3 The generalized Wright Hypergeometric function

The generalized Wright hypergeometric function [2, 14] for $z, a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in R$ $(\alpha_i, \beta_j \neq 0; i = 1,2,\dots,p; j = 1,2,\dots,q)$ is defined as:

$$p\Psi q\ [z] = p\Psi q \left[\begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} ; z \right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 n) \dots \Gamma(a_p+\alpha_p n)}{\Gamma(b_1+\beta_1 n) \dots \Gamma(b_q+\beta_q n)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q} \Gamma(b_j+\beta_j n)} \frac{z^n}{n!} \qquad (2)$$

2.4 The generalized Mittag-Leffler function

In 1971, Prabhakar [10] introduced the generalized Mittag-Leffler function

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \qquad (3)$$

where $\alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0.$

3. Main Results

Theorem 3.1. The Sawi Transform of generalized Wright hypergeometric function is given by

$$S\left[p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};z\right]\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\upsilon^{n-1}$$

Proof:

The generalized Wright hypergeometric function is defined by

$$p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}$$

Employing Sawi transform on both sides in above definition, determine

$$S\left[p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};z\right]\right]=S\left[\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}\right]$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\,n!}S[z^n]$$

After exercising Sawi transform for $f(z)=z^n$, yield the required result.

Corollary 3.2. When we substitute

$$\alpha_1=\alpha_2=\cdots=\alpha_n=\beta_1=\beta_2=\cdots=\beta_n=1$$

in Theorem (3.1), then we get new result as Sawi transform of generalized hypergeometric function [14] after a little simplification:

$$S\left[pFq\left[\begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix};z\right]\right]=S\left[\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\ \upsilon^{n-1}\right]$$

Corollary 3.3. Incorporating

$$p=q=1,a_1=\alpha_1=1,b_1=\beta,\qquad \beta_1=\alpha$$

in Theorem (3.1), we obtain the similar known result [12] in terms of Mittag-Leffler function:

$$S\left[1\Psi1\left[\begin{matrix}(1,1)\\(\alpha,\beta)\end{matrix};z\right]\right]=S\left[E_{\alpha,\beta}(z)\right]=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(\alpha n+\beta)}\upsilon^{n-1}$$

Theorem 3.4. The Sawi transform of a product comprising the generalized Wright hypergeomtric function is given by

$$S\left[z^m\, p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix};z\right]\right]=S\left[\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)\ (m+n)!}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\, n!}\ \upsilon^{m+n-1}\right]$$

Proof:

The generalized Wright hypergeomtric function is presented by

$$p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\, n!}z^n$$

Multiplying on both sides by z^m , yield

$$z^m\, p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\, n!}z^{m+n}$$

Taking Sawi transform on both sides of above equation, give

$$S\left[z^m\, p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix};z\right]\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\, n!}S[z^{m+n}].$$

The required results are obtained by using the result of Sawi transform for z^{m+n}.

Theorem 3.5. The Sawi transform of Mittag-Leffler function is given by

$$S\left[E_{\alpha,\beta}^{\gamma}(z)\right]=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\ \upsilon^{n-1}.$$

Proof:

The Mittag-Leffler function $E_{\alpha,\beta}^{\gamma}(z)$ is

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\ \frac{z^n}{n!}$$

Employing the Sawi transform on both sides, yield

$$S\left[E_{\alpha,\beta}^{\gamma}(z)\right]=S\left[\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\ \frac{z^n}{n!}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)\ n!}\ S[z^n]$$

With the aid of definition of Sawi transform of power function, we obtain the required result.

Theorem 3.6. For $\alpha > 0, a \in \boldsymbol{R}$ and we have the following formula for Sawi transform as:

$$S\left[E_{\alpha,\beta}^{\gamma}(-at^{\alpha})\right] = \upsilon^{-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\ \Gamma(\alpha n+1)\ (-a\upsilon^{\alpha})^n}{\Gamma(\alpha n+\beta)\ n!}$$

Proof:

The formula of Mittag-leffler function is

$$E_{\alpha,\beta}^{\gamma}(-at^{\alpha}) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\ \frac{(-at^{\alpha})^n}{n!}$$

Employing Sawi transform on both sides give

$$S\left[E_{\alpha,\beta}^{\gamma}(-at^{\alpha})\right] = S\left[\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\ \frac{(-at^{\alpha})^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_n\ (-a)^n}{\Gamma(\alpha n+\beta)n!}\ S[t^{\alpha n}]$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_n\ (-a)^n}{\Gamma(\alpha n+\beta)n!}\ \Gamma(\alpha n+1)\upsilon^{\alpha n-1}$$

$$= \upsilon^{-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\ \Gamma(\alpha n+1)\ (-a\upsilon^{\alpha})^n}{\Gamma(\alpha n+\beta)\ n!}$$

Corollary 3.7. Inserting $\beta = \gamma = 1$in Theorem (3.6), determine the known result as obtained by Wadi et.al. [12] in terms of Mittag-Leffler function or Wiman's function [13]:

$$S[E_{\alpha}\,(-at^{\alpha})]\ = \upsilon^{-1}\sum_{n=0}^{\infty}(-a\upsilon^{\alpha})^n = \frac{1}{\upsilon(1+a\upsilon^{\alpha})}$$

4. Conclusion

In this paper, the Sawi transform of the generalized Wright hypergeometric function was examined from a new perspective. We found the basic result that connects the transform to Mittag-Leffler functions and hypergeometric functions. The Sawi transform is a useful tool for more scientific research because it is easy to use and works well. These methods can be used in a lot of different areas of applied sciences (continuum mechanics and thermodynamics).

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Extended Modified Generalized Exponential Distribution: Properties and Applications

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Abstract: *This study presents the Extended Modified Generalized Exponential (EMGE) distribution, a novel four-parameter model designed to improve flexibility in modeling diverse hazard rate functions, including bathtub-shaped curves. The EMGE model is formulated by introducing an extra shape parameter into the Modified Generalized Exponential (MGE) model, improving its capacity to represent different failure rate patterns over time. We investigate the suggested distribution's hazard, survival, probability density, and reversed hazard functions, among other statistical characteristics. The EMGE model's parameters are estimated using three distinct techniques: Cramer-Von-Mises Estimation (CVME), Least Squares Estimation (LSE), and Maximum Likelihood Estimation (MLE), ensuring accurate and reliable parameter estimation. The performance of the model is tested on real-world data from COVID-19 patient mortality rates, showing a strong fit to the data. Comparative analysis with other established distributions, such as the Odd Lomax Exponential (OLE) and Modified Weibull (MW), highlights the superiority of the EMGE model in terms of fit and information criteria values. Our results demonstrate the potential of the EMGE distribution for improved modeling in reliability analysis and survival data.*

Keywords*:* *Exponential distribution, Hazard rate function, Generalized exponential distribution, Maximum likelihood function.*

1. Introduction

Probability models play a crucial role in reliability analysis across diverse fields such as biological sciences, applied statistics, and engineering. However, traditional probability models often struggle to adequately fit reliability data, prompting researchers to modify these models for better applicability. These modifications involve introducing additional parameters into the baseline distribution, creating new probability models that provide a closer fit to the data compared to conventional approaches. The incorporation of extra parameters enhances the flexibility of these models, enabling them to accommodate a broader range of data patterns and capture complex underlying relationships more effectively. Such improvements are particularly valuable in reliability analysis, where accurate modeling is essential for making predictions, estimating failure rates, and assessing system performance.

These enhanced models have practical applications in various domains, including survival analysis, risk assessment, and quality control. For instance, in biological sciences, they are used to model organism lifespans or the time until disease onset, while in engineering, they help predict the reliability of components under different operating conditions. Achieving a better fit to real-world data reduces errors and provides more reliable insights for informed decision-making. Researchers often validate the models through simulation studies and real-world data applications, ensuring their theoretical soundness and computational efficiency. These advancements significantly contribute to addressing complex challenges in reliability analysis and advancing statistical methodologies.

Generalized Exponential Distribution (GED) proposed by Gupta & Kundu [9], introduces an additional parameter to the baseline model, enhancing its ability to represent real-world data. This added parameter improves flexibility, enabling the GED to handle varying hazard rates, rather than the constant hazard rate assumed by the standard exponential distribution. Chaudhary & Kumar [5] Half Cauchy modified Exponential distribution modifying exponential model. Although effective for increasing or decreasing hazard rate functions depending on the shape parameter, the GED cannot model complex patterns like unimodal or bathtub-shaped hazard functions, leading to further advancements in probability modeling. Barreto-Souza et al. [2] developed a new statistical model referred to as the beta generalized exponential distribution. Mahmoudi & Jafari [12] recommended Generalized exponential–power series distributions, capable of representing hazard rate functions that are increasing, decreasing, or bathtub-shaped.

The diversity of these models is further enhanced by the Exponentiated Weibull Inverted Exponential Model by Chaudhary et al. [7], and the Half Logistic Exponential Extension Model by Chaudhary & Kumar, [4] as well as the Inverse Exponentiated Odd Lomax Exponential Distribution by Chaudhary et al. [6].

These extended modifications of the exponential distribution have been developed to enhance flexibility in modeling various types of data, particularly in reliability analysis, survival studies, and other statistical applications. Each modification introduces additional parameters or structural changes to better capture different hazard rate behaviors, including increasing, decreasing, bathtub-shaped, and unimodal patterns.

Hazard rate functions (HRFs) in lifetime models often exhibit a bathtub-shaped curve, a trait commonly observed in numerous real-world data sets. To address diverse patterns in survival and reliability analysis, various adaptations of the Weibull distribution have been developed to improve its flexibility and applicability. In literature, we find modifications of the original Weibull distribution to better capture complex behaviors in data.

The following is an expression for the two-parameter Weibull distribution:

$$\overline{F}(y,\lambda,\beta)=\exp[-(\lambda,y)]^{\beta} \tag{1}$$

The earlier mentioned model does not possess a failure rate function (HRF) with a bathtub shape. To overcome this drawback, it has been adapted into various versions that demonstrate a bathtub-shaped hazard rate. One such modification involves utilizing the exponentiated Weibull distribution, as proposed by Mudholkar & Srivastava [14].

Moreover, Lai et al. [10] demonstrate how adding certain constraints to the beta-integrated distribution facilitates the design of novel lifespan distributions. These constraints further enable the derivation of the following novel lifespan distributions.

$$\overline{F}(y) = \exp[ay^{b}.\exp(\lambda y)] \tag{2}$$

The Modified Generalized Exponential distribution recommended by Telee & Kumar [19] is an improved probabilistic model that builds upon the Generalized Exponential distribution. This enhancement was made by introducing an additional shape parameter, increasing the model's flexibility and applicability in statistical analysis. The original Generalized Exponential distribution was first developed by Gupta & Kundu [8].

The cumulative distribution function (CDF) for the Generalized Exponential distribution is given as follows:

$$F_{GED}(x,\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha}; x>0, \alpha>0, \lambda>0 \tag{3}$$

The Cumulative Distribution Function (or CDF) of the Modified Generalized Exponential (MGE) model given by Telee & Kumar [19], is as follows:

$$G(x;\alpha,\beta,\lambda) = \left[1-\exp\left(-\lambda x e^{\beta x}\right)\right]^{\alpha} \quad ; \ \alpha>0, \beta>0, \lambda>0, x>0 \tag{4}$$

Probability model introduced in this study, addresses the limitations of existing models by incorporating an additional shape parameter. This modification enhances flexibility in modeling diverse hazard rate functions, including bathtub-shaped curves. Unlike previous distributions such as the Weibull Extension and Modified Weibull, the EMGE model provides a better fit for real-world reliability and survival data, as demonstrated in our comparative analysis and empirical application to COVID-19 mortality rates

Study also introduces a novel probability distribution characterized by its cumulative distribution function (CDF) and probability density function (PDF), derived through a unique mathematical framework. The proposed model extends traditional distributions by introducing an innovative exponentiation mechanism that allows for greater flexibility in modeling skewed and heavy-tailed data.

This work builds upon existing probability models, particularly the Generalized Exponential and Modified Weibull distributions, by introducing a new shape parameter. The EMGE distribution unifies and extends existing distributions, making it applicable to a broader range of real-world datasets, including those with non-monotonic failure rates.

2. Extended Modified Generalized Exponential (EMGE)Distribution

The Extended Modified Generalized Exponential Distribution is created by adding an additional shape parameter to the Modified Generalized Exponential Distribution's cumulative distribution function (or CDF), as created by Telee & Kumar [19] in equation (4). The proposed Extended Modified Generalized Exponential (EMGE) model is characterized by its distribution and density functions, which are defined as follows:

$$F(x;\alpha,\beta,\lambda,\theta) = 1-\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha}; x,\alpha,\beta,\lambda,\theta>0 \tag{5}$$

$$f(x;\alpha,\beta,\lambda,\theta) = \alpha\lambda\left(\frac{e^{-\beta x}}{x^{\theta}}\right)\left(\beta+\theta x^{-1}\right)\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha-1}; x,\alpha,\beta,\lambda,\theta>0 \tag{6}$$

2.1 Survival Function

The reliability function associated with the proposed model is defined in equation (7).

$$S(x;\alpha,\beta,\lambda,\theta)=\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha}\ ;x,\alpha,\beta,\lambda,\theta>0 \tag{7}$$

2.2 Hazard Rate Function

Equation (8) provides a mathematical expression for the failure rate function, which represents the evolving failure probability over time.

$$h(x)=\alpha\lambda\left(\frac{e^{-\beta x}}{x^{\theta}}\right)\left(\beta+\theta x^{-1}\right)\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{-1} \tag{8}$$

2.3 The Reversed hazard function (RHR)

Equation (9) represents the reversed hazard rate function.

$$RHR(x)=\alpha\lambda\left(\frac{e^{-\beta x}}{x^{\theta}}\right)\left(\beta+\theta x^{-1}\right)\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha-1}\left[1-\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha}\right]^{-1} \tag{9}$$

2.4 Cumulative hazard function (CHF)

The CHF for the recommended model is shown in equation (10).

$$CHF(x)=-\log[S(x)]=-\alpha\log\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right] \tag{10}$$

2.5 The Quantile function

Equation (11) specifies the quantile function for the EMGE, based on the assumption that u is uniformly distributed over the interval [0,1].

$$Q(u)=-\frac{1}{\beta}\ln\left\{-\frac{x^{\theta}\ln[1-(1-u)^{1/\alpha}]}{\lambda}\right\} \tag{11}$$

2.6 Skewness and Kurtosis

The following formula can be used to get the quartile-based coefficient of skewness.

$$S_k=\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1} \tag{12}$$

Moors [13] states that the following formula may be used to define the kurtosis coefficient based on octiles:

$$K_M = \frac{Q(0.375)+Q(0.875)-Q(0.125)-Q(0.625)}{Q(0.75)-Q(0.25)} \tag{13}$$

Figure 1 displays the hazard rate function (HRF) and probability density function (PDF) of the suggested model at constant lambda = 3 & theta = 0.5. The PDF demonstrates a unimodal distribution with positive skewness, indicating that most values cluster near the center. In contrast, the HRF has an inverted bathtub shape and an increasing pattern, suggesting a variety of risk patterns over time.

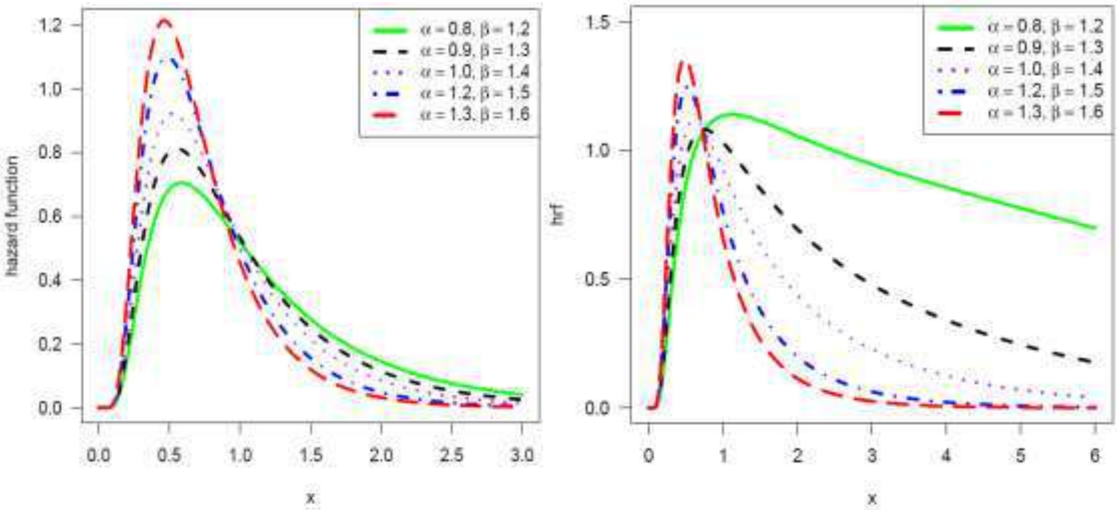

Figure 1: Density function (left) and hazard function (right) for lambda = 3 & theta = 0.5

3. Estimation Methods

MLE, LSE, and CVM methods are three well-known techniques used to estimate the parameters of the suggested EMGE model.

Maximum Likelihood Estimation (MLE)

A common statistical technique for estimating distribution parameters is Maximum Likelihood Estimation (MLE), which maximizes the likelihood function. In the case of the EMGE model, the likelihood function is formulated using its PDF, and the parameters are estimated by maximizing this function with respect to the unknown parameters.

Log likelihood function for EMGE is given as

$$l = n\log(\alpha\lambda) + \sum_{i=1}^{n} \log\left(\frac{e^{-\beta x_i}}{x_i^{\theta}}\right) + \sum_{i=1}^{n} \log\left(\beta + \theta x_i^{-1}\right) + \sum_{i=1}^{n} \left(\frac{-\lambda e^{-\beta x_{i=1}}}{x_{i=1}^{\theta}}\right) + (\alpha-1)\sum_{i=1}^{n} \log\left[1-\exp\left(\frac{-\lambda e^{-\beta x_{i=1}}}{x_{i=1}^{\theta}}\right)\right] \tag{14}$$

Partial derivatives can be obtained by differentiating equation (14) and solving derivatives, parameters can be estimated. Solution of these nonlinear equations are quite rigorous analytically, so the likelihood function is maximized using the Newton-Raphson algorithm, which aids in the development of the observed information matrix. Consequently, the variance-covariance matrix obtained is given by equation (15)

$$\left[-H(\Delta)_{|(\Delta=\hat{\Delta})}\right]^{-1}=\begin{pmatrix} V_{11}(\hat{\alpha}) & V_{12}(\hat{\alpha},\hat{\beta}) & V_{13}(\hat{\alpha},\hat{\lambda}) & V_{14}(\hat{\alpha},\hat{\theta}) \\ V_{21}(\hat{\beta},\hat{\alpha}) & V_{22}(\hat{\beta}) & V_{23}(\hat{\beta},\hat{\lambda}) & V_{24}(\hat{\beta},\hat{\theta}) \\ V_{31}(\hat{\lambda},\hat{\alpha}) & V_{32}(\hat{\lambda},\hat{\beta}) & V_{33}(\hat{\lambda}) & V_{34}(\hat{\lambda},\hat{\theta}) \\ V_{41}(\hat{\theta},\hat{\alpha}) & V_{42}(\hat{\theta},\hat{\beta}) & V_{43}(\hat{\theta},\hat{\lambda}) & V_{44}(\hat{\theta}) \end{pmatrix} \tag{15}$$

To construct approximately 100(1-δ) % confidence intervals for estimating α, β, λ and θ, the following method can be utilized, leveraging the asymptotic normality property of Maximum Likelihood Estimates (MLE).

$$\hat{\alpha}\pm Z_{\delta/2}\sqrt{V_{11}(\hat{\alpha})}\,,\ \hat{\beta}\pm Z_{\delta/2}\sqrt{V_{22}(\hat{\beta})}\,,\quad \hat{\lambda}\pm Z_{\delta/2}\sqrt{V_{33}(\hat{\lambda})}\ \text{and}\ \hat{\theta}\pm Z_{\delta/2}\sqrt{V_{44}(\hat{\theta})} \tag{16}$$

In this context, $Z_{\delta/2}$ refers to the upper critical value of the standard normal distribution.

Least Squares Estimation (LSE)

The total of the squared differences between the observed data points and the cumulative distribution values that the model predicts is minimized using the Least Squares Estimation (LSE) approach. This technique is particularly useful when dealing with data where the assumption of normality may not hold.

Cramér–von Mises (CVM) Criterion

The difference between the observed and theoretical distribution functions is reduced by applying the CVM approach. It offers an additional reliable technique for parameter estimation and is based on the weighted squared discrepancies between the theoretical and empirical cumulative distribution functions. These estimation techniques provide a comprehensive approach to fitting the EMGE model to data, ensuring that the parameters can be estimated with high precision and suitability for various applications in statistical modeling. By utilizing these methods, we aim to provide efficient and reliable estimates for the parameters of the EMGE distribution, enhancing its applicability in diverse fields such as reliability analysis, survival analysis, and other domains where flexible and accurate modeling of data is crucial.

4. Application to Real Dataset

According to Bantan et al. [1], the suggested model was tested on a real-world dataset that comprised the COVID-19 pandemic mortality rates in Mexico from March 4, 2020, to July 20, 2020, for 106 patients. The rates were divided by five to make them easier to analyze. The following is how the dataset is displayed:

1.7652, 1.2210, 1.8782, 2.9942, 2.0766, 1.4534, 2.6440, 3.2996, 2.3330, 1.2030, 2.1710, 1.2244, 1.3312, 0.6880, 1.1708, 2.1370, 2.0070, 1.0484, 0.8668, 1.0286, 1.5260, 2.9208, 1.5806, 1.2740, 0.7074, 1.2654, 0.9460, 0.6430, 1.8568, 2.5756, 1.7626, 2.0086, 1.4520, 1.1970, 1.2824, 0.6790, 0.8848, 1.9870, 1.5680, 1.9100, 0.6998, 0.7502, 1.3936, 0.6572, 2.0316, 1.6216, 1.3394, 1.4302, 1.3120, 0.4154, 0.7556, 0.5976, 0.6672, 1.3628, 1.5708, 1.6650, 1.7120, 0.6456, 1.4972, 1.3250, 1.2280, 0.9818, 0.9322, 1.0784, 2.4084, 1.7392, 0.3630, 0.6654,1.0812, 1.2364, 0.2082, 0.3600, 0.9898, 0.8178, 0.6718, 0.4140, 0.6596, 1.0634, 1.0884, 0.9114, 0.8584, 0.5000, 1.3070, 0.9296, 0.9394, 1.0918, 0.8240, 0.7884, 0.6438, 0.2804, 0.4876, 0.6514, 0.7264, 0.6466, 0.6054, 0.4704, 0.2410, 0.6436, 0.5852, 0.5202, 0.4130, 0.6058, 0.4116, 0.4652, 0.5052, 0.3846

The boxplot and TTT plot for the examined dataset are shown in Figure 2. A boxplot with a positive skew shows that the data is not distributed normally. Likewise, the concave shape of the TTT curve signifies an increasing failure rate.

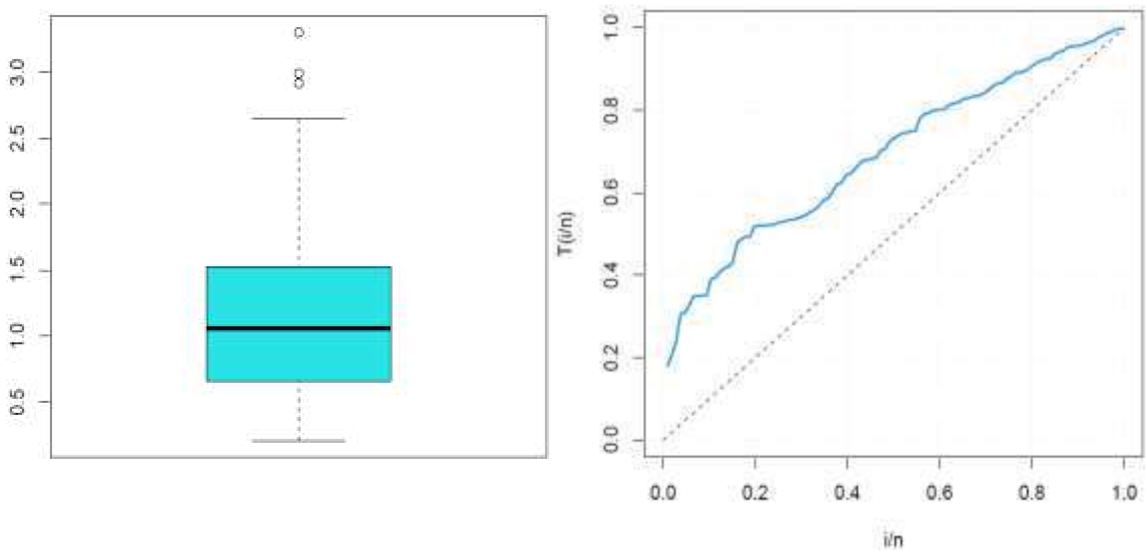

Figure2: Boxplot (Left panel) and TTT plot (Right panel) of the data

The dataset's descriptive statistics, shown in Table 1, show that it is positively skewed and deviates from normalcy.

Table 1: The data's descriptive statistics

Min	**Q_1**	**Md**	**$\bar{X}$**	**Q_3**	**Sd**	**Skewness**	**Kurtosis**	**Max**
0.2082	0.66	1.06	1.165	1.52	0.65	0.973	3.67	3.2996

The MLE, LSE, and CVME methods are used to estimate the model's parameters using the R software's optim() function (R Core Team[16]).The calculated parameters and the associated standard errors (SE) are shown in Table 2.

Table 2: Estimated Parameters using MLE, LSE and CVME along with respective S.E.

Methods	Alpha	Beta	Lambda	Theta
MLE	7.1294	0.2362	3.0860	0.5491
LSE	0.6660	3.3078	1.4800	1.9500
CVME	0.2182	0.0002	0.0002	4.9790

Furthermore, for each of the three estimating methods, the Log-Likelihood values and several information criteria, including BIC, AIC, CAIC, and HQIC, were computed in table 3.

Table 3: The BIC, HQIC, AIC, CAIC, and log likelihood (LL)

Model	LL	AIC	BIC	CAIC	HQIC
MLE	-90.70515	189.4103	200.0641	189.8063	193.7283
LSE	-94.75668	197.5134	208.1671	197.9094	201.8314
CVM	-96.37799	200.756	211.4097	201.1520	205.074

The goodness-of-fit test results are summarized in Table 4, which provides the statistics for Anderson-Darling (A^2), Cramér-von Mises (W), and Kolmogorov-Smirnov (KS), along with their respective p-values for different estimation approaches. Additionally, Table 4 compares the performance of various estimation techniques by evaluating how well they fit the observed data.

Table 4: Statistics for KS, W, and A^2 together with associated p-values

Methods	KS(p-value)	W(p-value)	A^2(p-value)
MLE	0.0817(0.4790)	0.1051(0.5613)	0.7129(0.5477)
LSE	0.0483(0.9652)	0.0347(0.9589)	0.2302(0.9798)
CVME	0.0486(0.9637)	0.0327(0.9669)	0.2049(0.9890)

Plots of PDF and CDF are frequently used to evaluate how well a given model fits data. Additionally, generating Q-Q and P-P plots provides deeper insights into the model's performance. The P-P plot highlights areas where the model may not fit well, while the Q-Q plot emphasizes the alignment in the tails of the distribution. Figure 3 demonstrates the EMGE model's strong fit to the data. The EMGE model's goodness of fit is assessed in a thorough manner by integrating Q-Q and P-P plots with PDF and CDF plots.

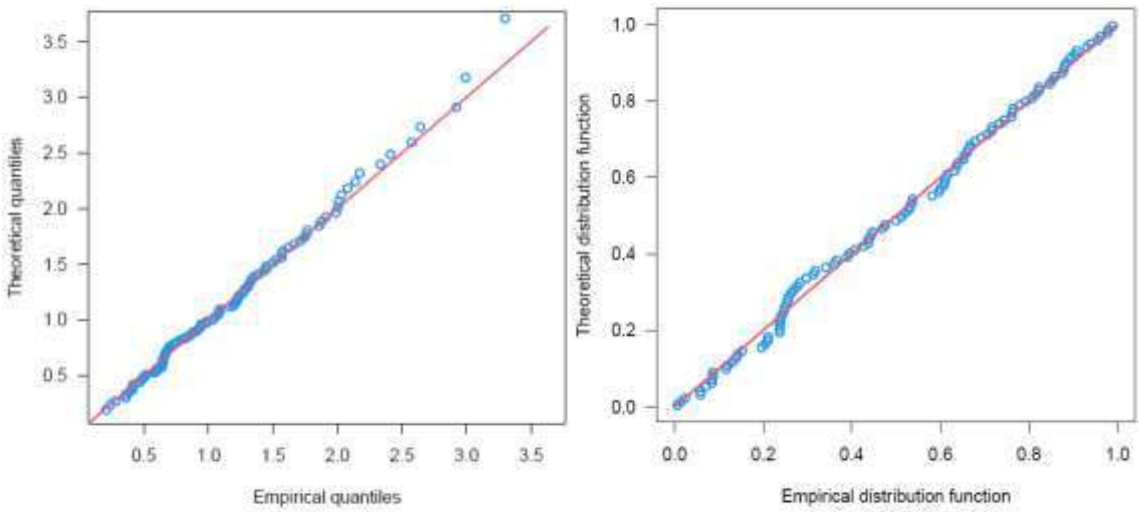

Figure 3: The EMGE model's P-P (left) and Q-Q (right) graphs.

The empirical cumulative distribution function (ECDF), density plot, and histogram are compared with the fitted CDF in Figure 4.

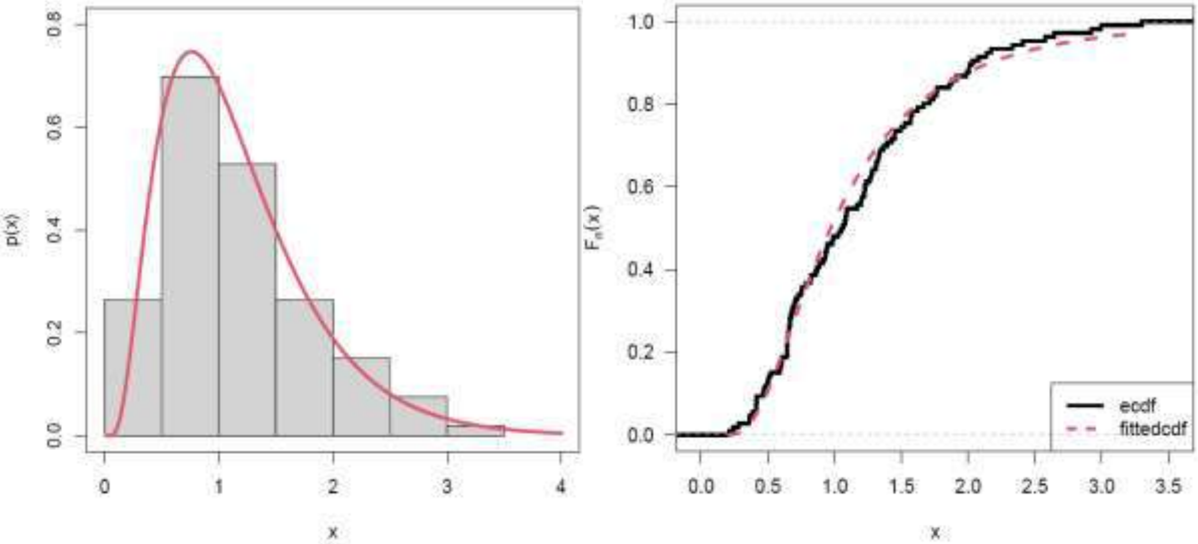

Figure 4: Ecdf against fitted cdf (right) and histogram versus pdf plot (left).

For model comparison, five previously published probability models were evaluated. Telee and Kumar [18] created the Lindley Generalized Inverted Exponential (LGIE) model, Lai et al. [11] developed the Modified Weibull (MW) model, Chaudhary and Kumar [3] developed the Logistic Inverse Exponential (LIE) distribution, Tang et al. [17] described the Weibull Extension (WE) model, and Ogunsanya et al. [15] introduced the Odd Lomax Exponential (OLE) distribution.

Table 5 provides the estimated parameter values and their corresponding standard errors for the proposed model in comparison with other models. These estimates are essential for evaluating and benchmarking the performance of the models. Furthermore, Table 6 showcases important statistical metrics, shedding light on the precision and reliability of these parameter estimates. Assessing the robustness and effectiveness of the proposed model in capturing the underlying relationships in the data in comparison to other models is made easy by this comprehensive analysis.

Table 5: “Values of estimated parameters for EMGE & their SE, along with competing models”

Model	Alpha	Beta	Theta	Lambda”
EMGE	7.1293	0.2361	0.5491	3.0860
OLE	0.1479	0.0119	-	0.1059
LGIE	7.7120	-	0.6487	1.4727
WE	20.2560	1.9589	-	10.3291
MW	0.5718	1.8937	-	0.0225
LIE	2.0429	-	-	0.6717

The effectiveness of the proposed model is assessed using various criteria, including the Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Corrected Akaike Information Criterion (CAIC), and Akaike Information Criterion (AIC).A summary of the results obtained is provided in Table 6.

Table 6: The HQIC, AIC, CAIC, BIC, and log likelihood (LL)

Model	LL	AIC	BIC	CAIC	HQIC
EMGE	-90.71	189.41	200.06	189.81	193.73
"OLE	-92.49	190.98	198.97	191.21	194.22
LGIE	-93.152	192.30	200.29	192.54	195.54
WE	-93.79	193.58	201.57	193.81	196.82
MW	-93.86	193.71	201.70	193.95	196.95
LIE	-96.39	196.78	202.10	196.89	198.94"

Simulation Study

Here, we have performed simulation analysis for the EMGE model. It helped in understanding its estimation properties, evaluating the performance of statistical methods, and determining its applicability to different types of data. It provides a foundation for making informed decisions about the model's effectiveness and its potential improvements for practical use in various domains. Below is a table summarizing the bias and mean squared error (MSE) for different combinations of sample size (n) and number of simulations (k) at $\alpha = 1.2$, $\beta = 1.0$, $\lambda = 2.0$ and $\theta = 1.2$:

Table 7: Bias and MSE for $\alpha = 1.2$, $\beta = 1.0$, $\lambda = 2.0$ and $\theta = 1.2$.

Sample Size (n)	Simulations (k)	Bias	MSE
50	500	0.0012	0.0044
50	1000	0.0061	0.0042
50	1500	0.0082	0.0042
100	500	0.0050	0.0023
100	1000	0.0071	0.0022
100	1500	0.0066	0.0023
150	500	0.0039	0.0014
150	1000	0.0052	0.0015
150	1500	0.0047	0.0015
200	500	0.0045	0.0011
200	1000	0.0064	0.0012
200	1500	0.0048	0.0011
250	500	0.0064	0.0009
250	1000	0.0064	0.0009
250	1500	0.0059	0.0009
300	500	0.0057	0.0007
300	1000	0.0072	0.0008
300	1500	0.0048	0.0008
350	500	0.0071	0.0007
350	1000	0.0066	0.0007
350	1500	0.0065	0.0007
400	500	0.0061	0.0006
400	1000	0.0061	0.0006
400	1500	0.0060	0.0006

The simulation study demonstrates that larger sample sizes (n) lead to more reliable and precise estimates, as reflected in the decreasing bias and MS. MSE consistently decreases with increasing n and k reinforcing the understanding that larger sample sizes and a higher number of simulations lead to better estimations of the true parameter values.

5. Conclusion

This research proposes the Extended Modified Generalized Exponential (EMGE) distribution as a valuable framework for reliability assessment and survival analysis. The EMGE model incorporates intricate patterns in hazard rates, such as those with bathtub-shaped curves, by adding a shape parameter to the Modified Generalized Exponential distribution. The comparative analysis with alternative models confirms the EMGE's superior performance in terms of fit and statistical criteria.

The derived CDF and PDF exhibit improved adaptability in representing diverse datasets, which is crucial for applications in reliability analysis, survival modeling, or econometrics. The mathematical derivation is rigorously validated, ensuring consistency with fundamental probability properties. Also, the model provides a better fit compared to existing distributions, as demonstrated through empirical or simulation-based comparisons. Furthermore, this model's ability to handle diverse failure rate functions makes it a valuable addition to the toolkit for reliability analysis, offering more accurate insights into system behavior and improving decision-making processes in various fields. Future work can explore further applications of the EMGE distribution in different domains, as well as potential extensions to address even more complex data patterns. Also, proposed distribution introduces additional parameters and a unique exponentiation mechanism, which provides more flexibility compared to traditional distributions.

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12. References

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[1] Faraji, H. & Nourouzi, K. (2017). Fixed and common fixed points for (ψ, ϕ)-weakly contractive mappings in b-metric spaces. *Sahand Communications in Mathematical Analysis*.**7(1):** 49-62.

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•Research Book Citation

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13. Manuscript Submission

At the time of submission, authors are requested to include a list of 3-5 panelists of experts in the related research area. All information, submissions, and correspondence of the articles should be addressed to

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