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Interpolative Contraction and Discontinuity at Fixed Point on Partial Metric Spaces

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Abstract: *This paper proposes a novel technique for solving Rhodes' discontinuity problem by exploiting the features of a self-mapping that has a fixed point but is not continuous at that point within a partial metric space. Moreover, we investigate some geometric properties of F_T under interpolative-type contractions and establish a few results related to fixed-discs and fixed-circles.*

Keywords: *Fixed point, Partial metric, Interpolative contraction*

1. Introduction

The Banach contraction principle [4] is a classical result that ranks among the most commonly used and cited fixed point theorems. It states that if a self-mapping T on a complete metric space (X,d) satisfies the condition

$$d(Tx,Ty) \leq a\,d(x,y)$$

for all $x,y \in X$ with $0 \leq a < 1$, then T has a unique fixed point $x^* \in X$. It is well established that the Banach contraction mapping T is continuous over the entire domain X. Kannan [8] proved that there are contractive mappings with fixed points that may not be continuous across the entire domain.

Theorem 1.1. *[8] Let (X,d) be a metric space that is complete, and let $T : X \to X$ be a self-map. If T satisfies the Kannan contraction condition*

$$d(Tx,Ty) \leq b\left[d(x,Tx)+d(y,Ty)\right], \quad 0 \leq b < \tfrac{1}{2}$$

for all $x,y \in X$ then T admits a unique fixed point in X.

Kannan's contraction mapping is known to be continuous at its fixed point. In [17], Rhoades questioned whether a contractive condition could be formulated that guarantees the existence of a fixed point without assuming the continuity of the mapping at that point. This unresolved problem has motivated several attempts and contributions over time. Using the function $m(x,y) = \max\{d(x,Tx), d(y,Ty)\}$, Pant [16] found an initial solution in the metric space (X,d). Later, Bist and Pant [5] proposed another solution to this open problem.

Erdal Karapinar, a distinguished mathematician, introduced the notion of "interpolative contraction" in metric spaces in his work [9]. The interpolation approach has proven useful in exploring a wide range of classical and modern contraction types (see [3], [5], [7], [10], [14] for further details). More recently, Tas [19] addressed Rhoades' discontinuity problem by presenting a novel solution involving the existence of a fixed point for a self-map that is not continuous at that point. This was achieved by adapting the concepts of interpolative Boyd-Wong contractions and interpolative Matkowski-type contractions as follows:

Theorem 1.2. *[19] Let (X,d) be a complete metric space. Let $T : X \to X$ be a self-map such that for all $x,y \in X$ and $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq n(x,y) < \varepsilon + \delta(\varepsilon) \implies d(Tx,Ty) < \varepsilon.$$

If T is $k-$ continuous then T has a unique fixed point say z. Moreover, T is continuous at z if and only if

$$\lim_{x \to z} n(x,z) = 0,$$

where

$$n(x,y) = [d(x,y)]^{\beta}[d(x,Tx)]^{\alpha}[d(y,Ty)]^{\gamma}\left[\frac{d(x,Ty)+d(y,Tx)}{2}\right]^{1-\alpha-\beta-\gamma}$$

and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$.

Matthews [12] introduced partial metric spaces as a tool for investigating the denotational semantics of data flow networks and extended the classical Banach contraction principle to this more general setting of complete partial metric spaces. Subsequently, Karapinar, Alqahtani, and Aydi [11] explored a Hardy-Rogers type interpolative contraction and established a fixed point theorem within the framework of complete partial metric spaces.
In this work, we propose a revised form of the interpolative Boyd-Wong contraction and the Matkowski-type contraction, building upon the findings of and extending them within the setting of partial metric spaces.

2. Preliminaries

Consider a nonempty set X. The notations $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ stand for the set of real numbers, the set of positive real numbers, and the set of natural numbers, respectively. In [12], Matthews introduced the definition of a partial metric as follows:

Definition 2.1 ([12])**.** A partial metric on a nonempty set X is a function $p : X \times X \to [0,\infty)$ that satisfies the following conditions for all $x,y,z \in X$:

(P1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;

(P2) $0 \leq p(x,x) \leq p(x,y)$;

(P3) $p(x,y) = p(y,x)$;

(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. A partial metric becomes a metric if $p(x,x) = 0$ for every $x \in X$.

Definition 2.2 ([12])**.** Let (X,p) be a partial metric space, a point $x_0 \in X$ and $\varepsilon > 0$. The open ball for a partial metric p is set of the form

$$B_{\varepsilon}(x_0) = \{x \in X : p(x_0,x) < p(x_0,x_0) + \varepsilon\}.$$

In contrast to metric spaces, some open balls may be empty in partial metric spaces. For each partial metric p on X, a topology τ_p is induced on X, where the family of open p-balls

$$\{B_\varepsilon(x) : x \in X, \varepsilon > 0\}$$

forms a basis. These open p-balls are defined by

$$B_\varepsilon(x) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$. If p is a partial metric on X, then the function $d_p : X \times X \to [0,\infty)$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 2.3 ([12])**.** Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ converges to a point $x \in X$ if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

(ii) $\{x_n\}$ is called a Cauchy if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists.

(ii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that

$$p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x).$$

Lemma 2.4 ([12])**.** *Let (X,p) be partial metric space and $\{x_n\}$ be a sequence in X. Then*

(i) $\{x_n\}$ is a Cauchy in (X,p) if and only if it is Cauchy in (X,d_p).

(ii) (X,p) is complete if and only if (X,d_p) is complete.

(iii) $\lim_{n\to\infty} d_p(x_n,x) = 0$ if and only if $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x)$.

3. Main Results

In the following sequel, we denote

$$A(x,y) = [p(x,y)]^\beta \; [p(x,Tx)]^\alpha [p(y,Ty)]^\gamma \; \left[\frac{p(x,Ty)+p(y,Tx)}{2}\right]^{1-\alpha-\beta-\gamma}$$

where $\alpha,\beta,\gamma \in (0,1)$ with $\alpha+\beta+\gamma < 1$.

Theorem 3.1. *Let (X,p) be a complete partial metric space. Let $T : X \to X$ be a self-map. For a given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \le A(x,y) < \varepsilon + \delta(\varepsilon) \implies p(Tx,Ty) < \varepsilon$$

for all $x,y \in X$. Then the sequence $\{T^n x\}$ is a Cauchy sequence and $\lim_{n\to\infty} p(T^n x,z) = p(z,z)$ for some $z \in X$.

Proof. Suppose $A(x,y) > 0$. Then

$$p(Tx,Ty) \ < \varepsilon \ \leq \ A(x,y) \ \implies p(Tx,Ty) \ < \ A(x,y). \tag{1}$$

Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n = T^n x_0$$

and

$$q_n = p(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. Suppose $x_n \neq x_{n+1}$ for each n. Then by inequality (1), we have

$$\begin{aligned}
q_n = p(x_n,x_{n+1}) =& p(Tx_{n-1},Tx_n)\\
&< A(x_{n-1},x_n)\\
&=[p(x_{n-1},x_n)]^{\beta}[p(x_{n-1},Tx_{n-1})]^{\alpha}[p(x_n,Tx_n)]^{\gamma}\left[\frac{p(x_{n-1},Tx_n)+p(x_n,Tx_{n-1})}{2}\right]^{1-\alpha-\beta-\gamma}\\
&= [p(x_{n-1},x_n)]^{\beta}[p(x_{n-1},x_n)]^{\alpha}[p(x_n,x_{n+1})]^{\gamma}\left[\frac{p(x_{n-1},x_{n+1})+p(x_n,x_n)}{2}\right]^{1-\alpha-\beta-\gamma}\\
&\leq [p(x_{n-1},x_n)]^{\beta+\alpha}[p(x_n,x_{n+1})]^{\gamma}\left[\frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})}{2}\right]^{1-\alpha-\beta-\gamma}\\
&= q_{n-1}^{\alpha+\beta} q_n^{\gamma}\left[\frac{q_{n-1}+q_n}{2}\right]^{1-\alpha-\beta-\gamma}.
\end{aligned}$$

Therefore,

$$q_n < q_{n-1}^{\alpha+\beta} q_n^{\gamma} \ \left[\frac{q_{n-1}+q_n}{2}\right]^{1-\alpha-\beta-\gamma}. \tag{2}$$

Again, suppose $q_{n-1} < q_n$ for some $n \in \mathbb{N}$. Then we have

$$\frac{q_{n-1}+\ q_n}{2} < \ \ q_n.$$

From inequality (2) we get

$$\begin{aligned}
&q_n < [q_{n-1}]^{\alpha+\beta}[q_n]^{\gamma}[q_n]^{1-\alpha-\beta-\gamma}\\
\implies\ &q_n < [q_{n-1}]^{\alpha+\beta}[q_n]^{1-\alpha-\beta}\\
\implies\ &q_n^{\alpha+\beta} < q_{n-1}^{\alpha+\beta}\\
\implies\ &q_n < q_{n-1}
\end{aligned}$$

which is a contradiction of our assumption. Therefore, $q_n \leq q_{n-1}$ for all $n \in \mathbb{N}$. We come to the conclusion that the s $\{q_{n-1}\}$ is decreasing and contains real numbers that are not negative. Therefore, a non-negative constant q exists such that $\lim_{n\to\infty} q_{n-1} = q$. Since $q_n \leq q_{n-1}$, so

$$\frac{q_{n-1}+q_n}{2} \leq q_{n-1}$$

for all $n \geq 1$. Using the inequality (2) we have

$$\begin{aligned}
&q_n < \ [q_{n-1}]^{\alpha+\beta} \ \ [q_n]^{\gamma} \ \ [q_{n-1}]^{1-\alpha-\beta-\gamma}\\
\implies\ &q_n < \ \ \ [q_{n-1}]^{1-\gamma}[q_n]^{\gamma}\\
\implies\ &q_n < \ q_{n-1}\\
\therefore \lim_{n\to\infty} q_n = q <\ &\lim_{n\to\infty} q_{n-1} = q.
\end{aligned}$$

which is contradiction. Hence, $q = 0$.
i.e.,

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in (X, p). If possible suppose $\{x_n\}$ is not Cauchy sequence. As a result, one can find a positive constant $\varepsilon > 0$ and two sub-sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$m_k > n_k > k \quad \text{and} \quad p(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{3}$$

Choosing the smallest m_k satisfying (3). So, $p(x_{n_k}, x_{m_k-1}) < \varepsilon$. Consider any $k \in \mathbb{N}$. Then,

$$\begin{aligned}
\varepsilon \leq p(x_{m_k}, x_{n_k}) &\leq p(x_{m_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{n_k}) - p(x_{m_k-1,\, x_{m_k-1}}) \\
&\leq p(x_{m_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{n_k}) \\
&< \varepsilon + p(x_{m_k}, x_{m_k-1}).
\end{aligned}$$

Therefore,

$$\varepsilon \leq p(x_{m_k}, x_{n_k}) < \varepsilon + p(x_{m_k}, x_{m_k-1}). \tag{4}$$

Taking limit $k \to \infty$ in (4)

$$\begin{aligned}
\varepsilon \leq \lim_{k\to\infty} p(x_{m_k}, x_{n_k}) &< \varepsilon + \lim_{k\to\infty} p(x_{m_k}, x_{m_k-1}) \\
\Rightarrow \varepsilon \leq \lim_{k\to\infty} p(x_{m_k}, x_{n_k}) &< \varepsilon \\
\Rightarrow \lim_{k\to\infty} p(x_{m_k}, x_{n_k}) &= \varepsilon.
\end{aligned}$$

Using (P4) and above relation we have $\lim_{k\to\infty} p(x_{m_{k+1}}, x_{n_{k+1}}) = \varepsilon$. Again, we have

$$\begin{aligned}
p(x_{n_k+1}, x_{m_k+1}) &= p(Tx_{n_k}, Tx_{m_k}) \\
&< A(x_{n_k}, x_{m_k}) \\
&= [p(x_{n_k}, x_{m_k})]^{\beta} [p(x_{n_k}, Tx_{n_k})]^{\alpha} [p(x_{m_k}, Tx_{m_k})]^{\gamma} \left[\frac{p(x_{n_k}, Tx_{m_k}) + p(x_{m_k}, Tx_{n_k})}{2}\right]^{1-\alpha-\beta-\gamma} \\
&= [p(x_{n_k}, x_{m_k})]^{\beta} [p(x_{n_k}, x_{n_k+1})]^{\alpha} [p(x_{m_k}, x_{m_k+1})]^{\gamma} \left[\frac{p(x_{n_k}, x_{m_k+1}) + p(x_{m_k}, x_{n_k+1})}{2}\right]^{1-\alpha-\beta-\gamma}
\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} p(x_{n_k+1}, x_{m_k+1}) < 0$$

which contradicts the assumption. Therefore $\{x_n\}$ is a Cauchy sequence in the complete partial metric space (X, p). Hence, $\lim_{n\to\infty} p(T^n x, z) = p(z, z)$ for some $z \in X$. □

Definition 3.2. Let (X, p) be a partial metric space . A self-map $T : X \to X$ is called $k-$continuous, $k = 1, 2, 3, \cdots$, if $\lim_{n\to\infty} p(T^k x_n, x) = p(Tx, Tx)$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} p(T^{k-1} x_n, x) = p(x, x).$$

It was established in [15] that for $k > 1$, the notions of continuity of T^k and k-continuity of T are not dependent on each other within metric spaces. This conclusion is also applicable within the framework of partial metric spaces. Clearly, 1-continuity is just another way of stating continuity. Furthermore, there exists a one-way chain of implications:

continuity ⇒ 2-continuity ⇒ 3-continuity ⇒ $\cdots$,

though the reverse implications do not generally hold.

Theorem 3.3. *Let (X,p) be a complete partial metric space. Let $T: X \to X$ be a self-map such that for all $x,y \in X$ and $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq A(x,y) < \varepsilon + \delta(\varepsilon) \implies p(Tx,Ty) < \varepsilon. \tag{5}$$

If T is k-continuous, then it admits a fixed point z. Moreover, T is continuous at z if and only if

$$\lim_{x\to z} A(x,z) = p(z,z).$$

Proof. Let $x_0 \in X$, and construct a Picard sequence $\{x_n\}$ in X by setting

$$x_{n+1} = Tx_n = T^n x_0.$$

According to Theorem 3.1, the sequence $\{x_n\}$ is Cauchy. As (X,p) is a complete metric space, there exists a point $z \in X$ satisfying

$$\lim_{n\to\infty} p(x_n,z) = p(z,z).$$

i.e.,

$$\lim_{n\to\infty} p(T^n x,z) = p(z,z).$$

Since T is $k-$continuous then

$$\lim_{n\to\infty} p(T^{k-1}x_n,z) = p(z,z) \Rightarrow \lim_{n\to\infty} p(T^k x_n,Tz) = p(Tz,Tz).$$

Therefore,

$$p(z,z) = p(Tz,Tz).$$

So by (P1) $Tz = z$. Consequently, z is a point that remains fixed under T.
Assume that T is continuous at the fixed point z. and $\lim_{n\to\infty} p(x_n,z) = p(z,z)$. Then

$$\lim_{n\to\infty} p(Tx_n,Tz) = p(Tz,Tz) = p(z,z).$$

Hence

$$\begin{aligned}\lim_{x\to z} A(x,z) =& \lim_{x\to z}\left[[p(x,z)]^\beta [p(x,Tx)]^\alpha [p(z,Tz)]^\gamma \left(\frac{p(x,Tz)+p(z,Tx)}{2}\right)^{1-\alpha-\beta-\gamma}\right] \\ =& [p(z,z)]^\beta [p(z,Tz)]^\alpha [p(z,Tz)]^\gamma \left(\frac{p(z,Tz)+p(z,Tz)}{2}\right)^{1-\alpha-\beta-\gamma} \\ =& p(z,z).\end{aligned}$$

Conversely, suppose $\lim_{x\to z} A(x,z) = p(z,z)$. Let $\lim_{n\to\infty} p(x_n,z) = p(z,z)$. Then

$$\begin{aligned} &\text{“}\lim_{x\to z} A(x,z) = p(z,z) \\ \Rightarrow \lim_{x\to z}\left[[p(x,z)]^\beta [p(x,Tx)]^\alpha [p(z,Tz)]^\gamma \left(\frac{p(x,Tz)+p(z,Tx)}{2}\right)^{1-\alpha-\beta-\gamma}\right] &= p(z,z) \\ \Rightarrow [p(z,z)]^\beta \lim_{x\to z}[p(x,Tx)]^\alpha [p(z,z)]^\gamma \left(\frac{p(z,z)+p(z,z)}{2}\right)^{1-\alpha-\beta-\gamma} &= p(z,z) \\ \Rightarrow [p(z,z)]^{1-\alpha} \lim_{x\to z}[p(x,Tx)]^\alpha &= p(z,z) \\ \Rightarrow \lim_{x\to z}[p(x,Tx)]^\alpha &= [p(z,z)]^\alpha \\ \Rightarrow \lim_{x\to z}[p(x,Tx)] &= [p(z,z)] \\ \Rightarrow \lim_{x\to z}[p(Tx,Tz)] &= p(Tz,Tz). \end{aligned}$$

Hence T is continuous. □

Definition 3.4. [1] Let Ψ be the class of all functions $\phi : [0,\infty) \to [0,\infty)$ that satisfy the following requirements:

(i) ϕ is non-decreasing; that is, for any $\alpha_1 < \alpha_2$, we have $\phi(\alpha_1) \leq \phi(\alpha_2)$;

(ii) ϕ is continuous;

(iii) For every $\alpha > 0$, the series $\sum_{n=1}^{\infty} \phi^n(\alpha)$ converges.

Corollary 3.5. *Let (X,p) be a complete partial metric space, and let $T : X \to X$ be a self-map such that for all $x,y \in X$, the following conditions are satisfied:*

(i) $\forall\, \varepsilon > 0, \exists\, \delta(\varepsilon) > 0 : \varepsilon \leq A(x,y) < \varepsilon + \delta(\varepsilon) \Rightarrow p(Tx,Ty) < \varepsilon.$

(ii) $p(Tx,Ty) \leq \phi(A(x,y))$*, where* $\phi \in \Psi$.

If T is k-continuous, then T possesses a fixed point z. Furthermore, T is continuous at z iff

$$\lim_{x \to z} A(x,z) = p(z,z).$$

3.1. Fixed- Disc Results

Definition 3.6. Let (X,p) be a partial metric space, and let $T : X \to X$ be a self-mapping. The set

$$C_{x_0,r} = \{x \in X : p(x,x_0) = r + p(x_0,x_0)\}$$

is referred to as a circle centered at x_0 with radius r. If every point x in $C_{x_0,r}$ satisfies $Tx = x$, then $C_{x_0,r}$ is known as fixed circle of the mapping T.

Definition 3.7. Let (X,p) be a partial metric space, and let $T : X \to X$ be a self-mapping. The set

$$D_{x_0,r} = \{x \in X : p(x,x_0) \leq r + p(x_0,x_0)\}$$

is termed a disk centered at x_0 with radius r. If $Tx = x$ holds for every $x \in D_{x_0,r}$, then $D_{x_0,r}$ is called a fixed disk under the mapping T.

Theorem 3.8. *Let (X,p) be a complete partial metric space, and let $T : X \to X$ be a self-mapping. Define the number r by*

$$r = \inf\{p(x,Tx) : x \notin F_T\}. \tag{6}$$

Suppose there exists a point $x_0 \in X$ such that for every $x \in X \setminus F_T$,

$$p(x,Tx) < A(x,x_0) \tag{7}$$

and

$$0 < p(x_0,Tx) \leq r + p(x_0,x_0), \tag{8}$$

Then the following conclusions hold:

(i) x_0 is a fixed point of T.

(ii) The mapping T fixes the disc $D_{x_0,r}$.

(iii) The mapping T fixes the circle $C_{x_0,r}$.

Proof. (i) Let $x_0 \in X \setminus F_T$. Afterward,

$$\begin{aligned} p(x_0,Tx_0) < A(x_0,x_0) =& [p(x_0,x_0)]^{\beta}[p(x_0,Tx_0)]^{\alpha}[p(x_0,Tx_0)]^{\gamma}\left(\frac{p(x_0,Tx_0)+p(x_0,Tx_0)}{2}\right)^{1-\alpha-\beta-\gamma} \\ =& [p(x_0,x_0)]^{\beta}[p(x_0,Tx_0)]^{\alpha}[p(x_0,Tx_0)]^{\gamma}[p(x_0,Tx_0)]^{1-\alpha-\beta-\gamma} \\ =& [p(x_0,x_0)]^{\beta}[p(x_0,Tx_0)]^{1-\beta} \\ \implies [p(x_0,Tx_0)]^{\beta} <& [p(x_0,x_0)]^{\beta} \\ \implies p(x_0,Tx_0) <& p(x_0,x_0) \end{aligned}$$

which contradicts (P2). Hence $x_0 \in F_T$.

(ii) Suppose $x \in D_{x_0,r}$ and $x \in X \setminus F_T$. Then

$$p(x,x_0) \leq r + p(x_0,x_0).$$

Using part (i), we have

$$\begin{aligned} p(x,Tx) < A(x,x_0) =& [p(x,x_0)]^{\beta}[p(x,Tx)]^{\alpha}[p(x_0,Tx_0)]^{\gamma}\left(\frac{p(x,Tx_0)+p(x_0,Tx)}{2}\right)^{1-\alpha-\beta-\gamma} \\ =& [p(x,x_0)]^{\beta}[p(x,Tx)]^{\alpha}[p(x_0,x_0)]^{\gamma}\left(\frac{p(x,x_0)+p(x_0,Tx)}{2}\right)^{1-\alpha-\beta-\gamma} \\ \leq& [r+p(x_0,x_0)]^{\beta}[p(x,Tx)]^{\alpha}[r+p(x_0,x_0)]^{\gamma}[r+p(x_0,x_0)]^{1-\alpha-\beta-\gamma} \\ =& [r+p(x_0,x_0)]^{1-\alpha}[p(x,Tx)]^{\alpha} \\ \leq& [p(x,Tx)] \end{aligned}$$

which is contradiction. So $x = Tx$. Hence T fixes the disc.
(iii) Similar to (ii). □

Theorem 3.9. *Let (X,p) be a complete partial metric space, and let $T : X \to X$ be a self-mapping. Define the number r by*

$$r = \inf\{p(x,Tx) : x \notin F_T\}, \tag{9}$$

where F_T denotes the set of fixed points of T. Assume that there exists a point $x_0 \in X$ such that, for every $x \in X \setminus F_T$, the following conditions are met:

$$p(x,Tx) < \phi(A(x,x_0)), \tag{10}$$

$$0 < p(x_0,Tx) \leq r + p(x_0,x_0). \tag{11}$$

Under these conditions, the following conclusions can be drawn:

- *(i) x_0 is a fixed point of the mapping T;*
- *(ii) The disc $D_{x_0,r}$ is invariant under T;*
- *(iii) The circle $C_{x_0,r}$ is also invariant under T.*

Proof. Similar technique of Theorem 3.8. □

4. Conclusions

In conclusion, this work offers a new perspective on Rhoade's discontinuity problem by introducing a self-mapping with a fixed point that is discontinuous at the fixed point within a partial metric space. We have derived a few geometric properties of F_T under interpolative-type contraction, along with key findings related to fixed-disc and fixed-circle results. These contributions enhance our understanding of fixed point theory in spaces where discontinuities are present.

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