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On Some Sequence Spaces of Bi-complex Numbers

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Abstract: *In 1892, Segre introduced the concept of bi-complex numbers. The main contribution in bi-complex analysis was the pioneering works in Functional analysis. It is a new subject, not only relevant from a mathematical point of view, but also has significant applications in physics and engineering.*

This article provides an overview of bi-complex numbers and examines the completeness of certain sequence spaces of bi-complex numbers. Additionally, the study explores their algebraic, topological, and geometric properties, contributing to a deeper understanding of these spaces.

Keywords: Bi-complex numbers, Euclidean norm, Banach space, Convexity, Uniform convexity

1. Introduction

Bi-complex numbers have been studied for quite a long time, and a lot of work has been done in this area. In 1892, Segre [18] introduced the concept of bi-complex numbers. The most comprehensive study of bi-complex numbers was done by Price [15]. Alpay et al. [1] developed a general theory of functional analysis with bi-complex scalars. In 2004, Rochon and Shapiro [16] studied some algebraic properties of Bi-complex and hyperbolic numbers. Later, Wagh [21], Degirmen and Sağır [6], Bera and Tripathy [3, 4], Sager and Sağır [17], and many researchers have studied the algebraic, topological, and geometric properties of bi-complex sequence spaces.

Definition 1.1. [17] A bi-complex number is denoted by γ and defined as,

$$\begin{aligned}\gamma &= x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\\ &= (x_1 + i_1x_2) + i_2(x_3 + i_1x_4)\\ &= z_1 + i_2 z_2, \text{ where, } x_1, x_2, x_3, x_4 \in \mathbb{C}_0,\ z_1, z_2 \in \mathbb{C}_1,\end{aligned}$$

$i_1^2 = i_2^2 = -1$, $i_1 i_2 = i_2 i_1$ and $\mathbb{C}_0$, $\mathbb{C}_1$, are the set of real and complex numbers respectively, $i_1 i_2$ is a hyperbolic unit whose square is 1.

The set of bi-complex numbers is denoted by $\mathbb{C}_2$ and is defined by

$$\mathbb{C}_2 = \{z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\}$$

There are three types of conjugations on bi-complex numbers (Rochan and Shapiro [16])

i) i_1 - conjugation of $\gamma = z_1 + i_2 z_2$ is $\gamma^* = \overline{z_1} + i_2\overline{z_2}$

ii) i_2 - conjugation of γ is $\bar{\gamma} = z_1 - i_2 z_2$

iii) $i_1 i_2$ - conjugation of γ is $\gamma' = \overline{z_1} - i_2 \overline{z_2}$

Definition 1.2. A bi-complex number $\gamma = z_1 + i_2 z_2$ is hyperbolic if $\gamma' = \gamma$ *or* $Im(z_1) = Re(z_2) = 0$. The set of all hyperbolic numbers is denoted by H and is defined by $H = \{x_1 + i_1 i_2 x_2 : x_1, x_2, \in \mathbb{C}_0\}$

For example, $\gamma = 1 + 2 i_1 i_2$ is a hyperbolic number.

The bi-complex number $\gamma = z_1 + i_2 z_2$ is singular if $|z_1^2 + z_2^2| = 0$ and non-singular if $|z_1^2 + z_2^2| \neq 0$.

In $\mathbb{C}_2$, there are exactly two non-trivial idempotent elements e_1 and e_2 defined by

$$e_1 = \frac{1 + i_1 i_2}{2} \text{ and } e_2 = \frac{1 - i_1 i_2}{2}$$

Obviously, $e_1 + e_2 = 1$, $e_1 e_2 = e_2 e_1 = 0$, $e_1^2 = e_1$ and $e_2^2 = e_2$

Every bi-complex number $\gamma = z_1 + i_2 z_2$ had unique idempotent representation as $\gamma = \mu_1 e_1 + \mu_2 e_2$, where $\mu_1 = z_1 - i_1 z_2$ and $\mu_2 = z_1 + i_1 z_2$ are the idempotent components of γ . The set $\{e_1, e_2\}$ forms an idempotent basis of $\mathbb{C}_2$. Equipped with co-ordinate wise addition, real scalar multiplication and term by term multiplication, $\mathbb{C}_2$ becomes a commutative ring with unity.

Algebraic structures of $\mathbb{C}_2$ differ from that of $\mathbb{C}_1$ in many aspects. A few of them are mentioned below.

(i) Non-invertible elements exist in $\mathbb{C}_2$

(ii) Non-invertible idempotent elements exist in $\mathbb{C}_2$

(iii) Non-trivial zero divisors exist in $\mathbb{C}_2$

The norm (Euclidean norm) on $\mathbb{C}_2$ is defined by

$$\|\gamma\|_{C_2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|\mu_1|^2 + |\mu_2|^2}{2}}.$$

Definition 1.3. A sequence in $\mathbb{C}_2$ is a function defined by $\gamma : \mathbb{N} \to \mathbb{C}_2,\ \gamma = (\gamma_k)$, where $\gamma_k \in \mathbb{C}_2$.

The sequence (γ_k) of bi-complex numbers is said to be convergent to $\gamma \in \mathbb{C}_2$ iff for each $\varepsilon > 0$ there corresponds an $n(\varepsilon) \in \mathbb{N}$ such that

$$\|\gamma_k - \gamma\|_{\mathbb{C}_2} < \varepsilon, \text{ for all } k \geq n(\varepsilon). \text{ It is written as } \lim_{k\to\infty} \gamma_k = \gamma.$$

The sequence (γ_k) of bi-complex numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer $n(\varepsilon) \in \mathbb{N}$ such that

$$\|\gamma_m - \gamma_n\|_{\mathbb{C}_2} < \varepsilon, \text{ for all } m, n \geq n(\varepsilon).$$

Definition 1.4. Let E be a sequence space of bi-complex numbers and $\breve{E} = \{(u_n) \in \omega(\mathbb{C}_2)$, there exists $x_n \in$ E such that $\|u_n\|_{\mathbb{C}_2} \leq \|x_n\|_{\mathbb{C}_2}$ for all $n \in \mathbb{N}$. Then E is said to be solid or normal if $\breve{E} \subset E$.

Definition 1.5. A sequence space E of bi-complex numbers is said to be symmetric if $(\gamma_k) \in$ E implies $\gamma_{\pi(k)} \in$ E where π is the permutation of $\mathbb{N}$

Definition 1.6.[6] Let E be a subset of linear space X. Then E is said to be convex if $(1-\lambda)x + \lambda y \in$ E for all $x, y \in$ E and all scalar $\lambda \in [0\ 1]$.

Definition 1.7.[6] A Banach space X is said to be strictly convex if $x, y \in S_X$ with $x \neq y$ implies that $\|(1-\lambda)x + \lambda y\| < 1$ for all $\lambda \in (0\ 1)$.

Definition 1.8.[6] A Banach space X is said to be uniformly convex if, to each $\varepsilon > 0,\ 0 < \varepsilon \leq 2$ such that for all $x, y \in S_X$, where S_X represents the unit sphere, there corresponds a $\delta(\varepsilon) > 0$ such that the conditions

$$\| x \| = \| y \| = 1, \| x - y \| \geq \varepsilon \Longrightarrow \frac{1}{2} \| x + y \| \leq 1 - \delta(\varepsilon).$$

2. Some sequence spaces over the set of bi-complex numbers.

If ω denotes the set of all functions from the set of positive integers ℕ to the field ℂ of complex numbers then it becomes a vector space. Any Sequence space is defined as a set of all sequences $x = (x_n)$ linear subspace of ω over the field ℂ with the usual operations defined as

$$(x_n) + (y_n) = (x_n + y_n) \text{ and } \lambda(x_n) = (\lambda x_n).$$

Recently, several researchers, including Ghimire & Pahari [8], Pahari [12], Paudel, Pahari & Kumar [13],Pokharel , Pahari, & Paudel [14] and Srivastava & Pahari[19] have studied the theory of vector-valued sequence spaces using Banach sequences.

The notations $\omega(\mathbb{C}_2)$, $l_\infty(\mathbb{C}_2)$, $c(\mathbb{C}_2)$, $c_0(\mathbb{C}_2)$, $l_p(\mathbb{C}_2)$ denote the class of all bounded, convergent, null and absolutely p-summable bi-complex sequences [13].

$$\omega(\mathbb{C}_2) = \{x = (x_k): x_k \in \mathbb{C}_2 \text{ for all } k \in \mathbb{N}\}$$

$$l_\infty(\mathbb{C}_2) = \{x = (x_k): x_k \in \omega(\mathbb{C}_2) : \sup_{k \in \mathbb{N}} \|x_k\|_{C_2} < \infty\}$$

$$c(\mathbb{C}_2) = \{x = (x_k): x_k \in \omega(\mathbb{C}_2), \exists\ l \in \mathbb{C}_2: \lim_{k\to\infty} x_k = l\}$$

$$c_0(\mathbb{C}_2) = \{x = (x_k): x_k \in \omega(\mathbb{C}_2): \lim_{k\to\infty} x_k = 0\}$$

$$l_p(\mathbb{C}_2) = \{x = (x_k): x_k \in \omega(\mathbb{C}_2): \sum_{k=1}^{\infty} ||x_k||_{c_2}^p < \infty\}$$

Lemma 2.1.[16] (Bi-complex Minkowski's inequality)

Let p and q be real numbers with $1 < p < \infty$ and $x_k, y_k, \in \mathbb{C}_2$ for $k \in \{1, 2, \ldots, n\}$. Then,

$$\left[\left(\sum_{k=1}^{n} || x_k + y_k ||_{\mathbb{C}_2}^p\right)\right]^{\frac{1}{p}} \leq \left[\left(\sum_{k=1}^{n} || x_k ||_{\mathbb{C}_2}^p\right)\right]^{\frac{1}{p}} + \left[\left(\sum_{k=1}^{n} || y_k||_{\mathbb{C}_2}^p\right)^{\frac{1}{p}}\right]$$

Lemma 2.2.[6] Let p be a real number with $0 < p \leq 1$, $x = (x_k)$ and $y = (y_k) \in \mathbb{C}_2$.

Then, we have $\| x + y\|_{\mathbb{C}_2}^p \leq \|x\|_{\mathbb{C}_2}^p + \|y\|_{\mathbb{C}_2}^p$

Lemma 2.3.[6] Let p be a real number with $2 \leq p < \infty$ and $x = (x_k), y = (y_k) \epsilon\, \mathbb{C}_2$. Then, we have

$$\|x + y\|_{\mathbb{C}_2}^p + ||x - y||_{\mathbb{C}_2}^p \leq 2^{p-1}(||x||_{\mathbb{C}_2}^p + ||y||_{\mathbb{C}_2}^p||)$$

Several workers Basar [2], Degirmen & Sagir [5], Ellidokuzoglu & Demiriz [7] , Güngör [9] , Hardy [10], Kumar & Tripathy[11], Srivastava & Srivastava [20] have made substantial contributions to the theory of series and bicomplex numbers and sequences.

3.Main Results

In this section, we present some theorems and examples exploring some algebraic and topological properties of the sequences of bi-complex numbers.

Theorem 3.1. The space $l_\infty(\mathbb{C}_2)$ is a bi-complex solid space.

Proof.

Let $(x_n) \in \widetilde{l}_\infty(\mathbb{C}_2)$, then there exists a sequence $(y_n) \in l_\infty(\mathbb{C}_2)$ such that

$$|| x_n||_{\mathbb{C}_2} \leq || y_n||_{\mathbb{C}_2} \text{ for all } n \in \mathbb{N}.$$

Therefore, sup $\{\| y_n \|_{\mathbb{C}_2} : \text{n} \in \mathbb{N}\} < \infty$ and so, $\sup\{\| x_n \|_{\mathbb{C}_2} : n \in \mathbb{N}\} < \infty$.

This shows that $(x_n) \in l_\infty(\mathbb{C}_2)$. Thus $(x_n) \in \tilde{l}_\infty(\mathbb{C}_2)$ implies $(x_n) \in l_\infty(\mathbb{C}_2)$.

So, $\tilde{l}_\infty(\mathbb{C}_2) \subset l_\infty(\mathbb{C}_2)$. Hence $l_\infty(\mathbb{C}_2)$ is a bi-complex solid space.

Theorem 3. 2. The space $l_\infty(\mathbb{C}_2)$ is a bi-complex symmetric space.

Proof.

Let $(x_n) \in l_\infty(\mathbb{C}_2)$ and $\sigma \in \pi$. Then, $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijective function. So, we have

$$\{\|x_{\sigma(n)}\|_{\mathbb{C}_2} : n \in \mathbb{N}\} = \{\|x_n\|_{\mathbb{C}_2} : n \in \mathbb{N}\}.$$

Also, $\sup\{\|x_{\sigma(n)}\|_{\mathbb{C}_2} : n \in \mathbb{N}\} = \sup\{\|x_n\|_{\mathbb{C}_2} : n \in \mathbb{N}\}$.

Since $(x_n) \in l_\infty(\mathbb{C}_2)$, $sup\{\|x_n\|_{\mathbb{C}_2} : n \in \mathbb{N}\} < \infty$. Also, $\sup\{\|x_{\sigma(n)}\|_{\mathbb{C}_2} : n \in \mathbb{N}\} < \infty$.

This result shows that $(x_{\sigma(n)}) \in l_\infty(\mathbb{C}_2)$. Hence $l_\infty(\mathbb{C}_2)$ is a bicomplex symmetric space.

Theorem 3.3. The space $l_p(\mathbb{C}_2)$ is a bi-complex solid space for $0 < p < \infty$.

Proof.

Let $(x_n) \in \tilde{l}_p(\mathbb{C}_2)$, then there exists a sequence $(y_n) \in l_p(\mathbb{C}_2)$ such that

$\|x_n\| \leq \|y_n\|$ for all $n \in \mathbb{N}$.

Also, $\|x_n\|_{\mathbb{C}_2}^p \leq \|y_n\|_{\mathbb{C}_2}^p$ for all $n \in \mathbb{N}$.

Since the series $\sum_{n=1}^{\infty} \|y_n\|_{\mathbb{C}_2}^p$ is convergent, the comparison test for convergent series implies that the series $\sum_{n=1}^{\infty} \|x_n\|_{\mathbb{C}_2}$ is also convergent. So, $(x_n) \in l_p(\mathbb{C}_2)$.

Thus $(x_n) \in \tilde{l}_p(\mathbb{C}_2)$ imples $(x_n) \in l_p(\mathbb{C}_2)$. Hence $\tilde{l}_p(\mathbb{C}_2) \subset l_p(\mathbb{C}_2)$.

So, $l_p(\mathbb{C}_2)$ is a bi-complex solid space.

Theorem 3.4. The space $l_p(\mathbb{C}_2)$ is a bi-complex symmetric space for $0 < p < \infty$.

Proof.

Let $(x_n) \in l_p(\mathbb{C}_2)$ and $\sigma \in \pi$. Since $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijective function, we can write

$$\left\{\|x_{\sigma(n)}\|_{\mathbb{C}_2} : n \in \mathbb{N}\right\} = \{\|x_n\|_{\mathbb{C}_2} : n \in \mathbb{N}\}$$

and so, $\left\{\|x_{\sigma(n)}\|_{\mathbb{C}_2}^p : n \in \mathbb{N}\right\} = \left\{\|x_n\|_{\mathbb{C}_2}^p : n \in \mathbb{N}\right\}$ holds.

Then, $\sum_{n=1}^{\infty} \|x_{\sigma(n)}\|_{\mathbb{C}_2}^p = \sum_{n=1}^{\infty} \|x_n\|_{\mathbb{C}_2}^p$

Since, $(x_n) \in l_p(\mathbb{C}_2)$, $\sum_{n=1}^{\infty} \|x_n\|_{\mathbb{C}_2}^p$ converges.

Also, the series $\sum_{n=1}^{\infty} \|x_{\sigma(n)}\|_{\mathbb{C}_2}^p$ converges.

Hence, $(x_{\sigma(n)}) \in l_p(\mathbb{C}_2)$. Thus, $l_p(\mathbb{C}_2)$ is a bicomplex symmetric space.

Theorem 3.5.[16] The set ω ($\mathbb{C}_2$) is a linear space over ℝ with respect to addition and scalar multiplication.

Theorem 3.6.[16] The sets $l_\infty(\mathbb{C}_2), c\ (\mathbb{C}_2), c_0\ (\mathbb{C}_2)$ and l_p ($\mathbb{C}_2$) for $0 < p < \infty$ are sequence spaces.

Theorem 3.7. The space ℓ_p ($\mathbb{C}_2$) is a complete metric space for $0 < p < \infty$ with the metric d_{l_p} ($\mathbb{C}_2$) defined by

$$d_{l_p(\mathbb{C}_2)}(x,y) = \left\{\sum_{k=1}^{\infty}\|x_k - y_k\|_{\mathbb{C}_2}^p\right\} \text{ for } 0 < p \le 1$$

$$= \left(\sum_{k=1}^{\infty}\|x_k - y_k\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} \text{ for } 1 < p < \infty$$

where $x = (x_k)$, $y = (y_k) \in \ell_p$ ($\mathbb{C}_2$).

Proof.

First, we show that the metric space $\ell_p(\mathbb{C}_2)$ is complete for $1 < p < \infty$.

For this let $(x_m) = (x_k^m)_{k \in \mathbb{N}}$ be any arbitrary Cauchy sequence in the space $\ell_p(\mathbb{C}_2)$. Then, for every $\varepsilon > 0$, there exists $n_o(\varepsilon) \in \mathbb{N}$ such that

$$d(x_m, x_r) = \left(\sum_{k=1}^{\infty}\|x_k^m - x_k^r\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} < \infty, \text{for all } m, r \ge no\ (\varepsilon). \quad (1)$$

Then, for any fixed k,

$$\| x_k^m - x_k^r \| < \varepsilon \text{ for all } m, r \ge n_o\ (\varepsilon). \quad (2)$$

Thus, for any fixed k, $(x_k^1, x_k^2, \dots, x_k^m, \dots)$ is a bi-complex Cauchy sequence and so it converges to a point say x_k^*. Collecting the infinitely many limits $(x_1^*, x_2^*, \dots.)$, let us define a sequence $x^* = (x_k^*) = (x_1^*, x_2^*, \dots.)$.

Then, we show that $x^* = (x_k^*) \in \ell_p$ ($\mathbb{C}_2$) and $x_m \to x^*$ as $m \to \infty$.

By (2), we can write $\|x_k^m - x_k^*\|_{\mathbb{C}_2} \le \varepsilon$ for all $m \ge n_0(\varepsilon)$, which means that $x_k^m \to x_k^*\ as$

$m \to \infty$. Also from (1), we have

$$\left(\sum_{k=1}^{n}\|x_k^m - x_k^r\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} < \varepsilon \text{ for all } m, r \ge n_0\ (\varepsilon).$$

Letting $r \to \infty$, we have $\left(\sum_{k=1}^{n} ||x_k^m - x_k^*||_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} < \varepsilon$ for all $n \in \mathbb{N}$.

Then by letting $n \to \infty$, we have

$$d(x_m, x^*) = (\sum_{k=1}^{\infty} ||x_k^m - x_k^*||^p)^{\frac{1}{p}} \le \varepsilon \text{ for all } m \ge n_0(\varepsilon).$$

Thus the sequence $(x_m) \in l_p$ ($\mathbb{C}_2$) converges to $x^* = (x_k^*) \in \omega(\mathbb{C}_2)$.

Next, we show that $x^* = (x_k^*) \in \ell_p$ ($\mathbb{C}_2$). Since $(x_m) = (x_k^m) \in \ell_p(\mathbb{C}_2)$, by complex Minkowski's inequality and convergence of the series $\sum_{k=1}^{\infty} ||x_k^* - x_k^m||_{\mathbb{C}_2}^p$ we have

$$\left(\sum_{k=1}^{\infty}\|x_k^*\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty}\|x_k^m + (x_k^* - x_k^m)\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{k=1}^{\infty}\|x_k^m\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}\|x_k^* - x_k^m\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} < \infty.$$

Thus $x = (x_k^*) \in l_p(\mathbb{C}_2)$. Hence $l_p(\mathbb{C}_2)$ with $1 < p < \infty$ is complete.

Similarly, we can show that $\ell_p(\mathbb{C}_2)$ is complete for $0 < p \leq 1$ with the metric

$d(x, y) = \sum_{k=1}^{\infty}\|x_k - y_k\|_{\mathbb{C}_2}^p$, where $x = (x_k), y = (y_k) \in \ell_p(\mathbb{C}_2)$.

Since $\ell_p(\mathbb{C}_2)$ is complete with the metric induced by the norm defined by

$\|x\| = \left\{\sum_{k=1}^{\infty}\|x_k\|_{\mathbb{C}_2}^p\right\}$ for $0 < p \leq 1$.

$= \left(\sum_{k=1}^{\infty}\|x_k\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}}$ for $1 < p < \infty$

Hence $\ell_p(\mathbb{C}_2)$is a Banach space.

Theorem 3.8. The space $\ell_p(\mathbb{C}_2)$ for $0 < p < \infty$ is convex.

Proof.

Let $x = (x_n)$ and $y = (y_n) \in \ell_p(\mathbb{C}_2)$ and $\lambda \in [0,1]$.

Then the series $\sum_{n=1}^{\infty}\|x_n\|_{\mathbb{C}_2}^p$ and $\sum_{n=1}^{\infty}\|y_n\|_{\mathbb{C}_2}^p$ converge.

For $1 < p < \infty$, in view of lemma 2.1, we have

$$\left(\sum_{n=1}^{\infty}\|\lambda x_n + (1-\lambda)y_n\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty}\|\lambda x_n\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\|(1-\lambda)y_n\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}}$$

$$= \lambda\left(\sum_{n=1}^{\infty}\|x_n\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} + (1-\lambda)\left(\sum_{n=1}^{\infty}\|y_n\|_{\mathbb{C}_2}^p\right)^{\frac{1}{p}} < \infty$$

and therefore,

$$\sum_{n=1}^{\infty}\|\lambda x_n + (1-\lambda)y_n\|^p < \infty.$$

Hence, $\lambda x + (1-\lambda)y \in \ell_p(\mathbb{C}_2)$.

Also, for $0 < p \leq 1$, we have by lemma 2.2

$$\sum_{n=1}^{\infty}\|\lambda x_n + (1-\lambda)y_n\|_{\mathbb{C}_2}^p \leq \sum_{n=1}^{\infty}(\|\lambda x_n\|_{\mathbb{C}_2}^p + \|(1-\lambda)y_n\|_{\mathbb{C}_2}^p)$$

$$= \lambda^p\sum_{n=1}^{\infty}\|x_n\|_{\mathbb{C}_2}^p + (1-\lambda)^p\sum_{n=1}^{\infty}\|y_n\|_{\mathbb{C}_2}^p < \infty.$$

This implies that $\lambda x + (1-\lambda)y \in \ell_p(\mathbb{C}_2)$.

Hence, $\ell_p(\mathbb{C}_2)$ for $0 < p < \infty$ is convex.

Theorem 3.9. The sequence space ℓ_∞ $(\mathbb{C}_2)$is convex.

Proof.

Let $x = (x_n)$, $y = (y_n)$ $\in \ell_\infty(\mathbb{C}_2)$ and $\lambda \in [0,1]$.

Then, sup $\{\| x_n\|_{\mathbb{C}_2} : n \in \mathbb{N}\} < \infty$ and sup $\{\| y_n \|_{\mathbb{C}_2} : n \in \mathbb{N}\} < \infty$.

Now, sup $\{\|\lambda x_n + 1 - \lambda) y_n\|_{\mathbb{C}_2} : n \in \mathbb{N}\}$ $\leq$ sup $\{\lambda \| x_n \|_{\mathbb{C}_2} : n \in \mathbb{N}\} + (1-\lambda) \| y_n\|_{\mathbb{C}_2}$ $: n \in \mathbb{N}\}$

$= \lambda$ sup $\{\| x_n \|_{\mathbb{C}_2} : n \in \mathbb{N}\} + (1-\lambda)$ sup $\{\| y_n \|_{\mathbb{C}_2}$ $: n \in \mathbb{N}\}$

$< \infty$

Thus, $\lambda x + (1- \lambda)$y $\in \ell_\infty$ $(\mathbb{C}_2)$. Hence, $\ell_\infty(\mathbb{C}_2)$ is convex.

Lemma 3.10. [6] Let p be a real number with $1 < p < \infty$ such that $x \neq y$ where $x = (x_n)$, $y = (y_n)$ and $\lambda \in (0,1)$. Then, we have $\| \lambda x + (1-\lambda)\, y \|^p < \lambda \|x\|^p_{\mathbb{C}_2} + (1- \lambda) \|y\|^p_{\mathbb{C}_2}$.

Theorem 3.11. The sequence space $\ell_p(\mathbb{C}_2)$ for $1 < p < \infty$ is strictly convex.

Proof.

Let $x = (x_n)$ and $y = (y_n) \in S_{l_p}(\mathbb{C}_2)$ such that $x \neq y$ and $\lambda \in (0,1)$. Then, $\| x \| = 1$ and $\| y \| = 1$.

By lemma 3.10 we have

$\|\lambda x + (1-\lambda)\, y \|^p_{\mathbb{C}_2} = \sum_{n=1}^{\infty} \| \lambda x_n + (1- \lambda) y_n\|^p$ $(\mathbb{C}_2)$

$< \sum_{n=1}^{\infty} [\, \lambda \|x_n\|^p + (1- \lambda) y_n\|^p$ $(\mathbb{C}_2)$

$= \lambda \sum_{n=1}^{\infty} \|x_n\|^p_{\mathbb{C}_2} + (1- \lambda) \sum_{n=1}^{\infty} \|y_n\|^p_{\mathbb{C}_2}$

$= \lambda$ $\|x\|^p_{\mathbb{C}_2} + (1- \lambda) \|y\|^p_{\mathbb{C}_2}$

$=$ $\lambda.1 + (1-\lambda)\, .1 = 1$

This shows that $l_p(\mathbb{C}_2)$ for $1 < p < \infty$ is strictly convex.

Example 1. The sequence space $l_1(\mathbb{C}_2)$ is not strictly convex.

Let $x = (x_n) = (0,\ i_1,\ 0,\ 0, \dots\dots)$ and $y = (y_n) = (0,\ 0, -i_2,\ 0, \dots\dots)$
so that $\|x\| = \|y\| = 1$ *and* $\lambda \in (0,1)$.

Now, $\| \lambda x + (1-\lambda) y \|_{l_1(\mathbb{C}_2)} = \sum_{n=1}^{\infty} \| \lambda x_n + (1-\lambda) y_n \|_{(\mathbb{C}_2)}$

$= \sum_{n=1}^{\infty} \|(0,\ \lambda i_1,\ (1-\lambda)(-i_2),\ 0, \dots.)\, \|$

$= \| \lambda i_1 \|_{\mathbb{C}_2} + \|(1-\lambda)(-i_2)\|_{\mathbb{C}_2}$

$= \lambda.1 + (1-\lambda).1$ $= 1$, for all $\lambda \in (0,1)$.

Hence $l_1(\mathbb{C}_2)$ is not strictly convex.

Example 2. The sequence space $l_\infty(\mathbb{C}_2)$ is not strictly convex.

Let $x = (x_n) = (1,\ i_1,\ i_2,\ 0,\ 0, \dots)$ and $y = (y_n) = (-1,\ i_1,\ i_2,\ 0,\ 0, \dots)$

Then, $\|x\|_{\mathbb{C}_2} = \|y\|_{\mathbb{C}_2} = 1$.

Now, for all $\lambda \in (0,1)$, we have

$$\|\lambda x + (1-\lambda)\, y\|_{\mathbb{C}_2} = \sup\{\|\lambda x_n + (1-\lambda) y_n\|_{\mathbb{C}_2} : n \in \mathbb{N}\}$$
$$= \sup\{\,\|2\lambda - 1,\ i_1,\ i_2,\ 0,\ 0, \dots)\,\|_{\mathbb{C}_2} : n \in \mathbb{N}\}$$
$$= \sup\{0,\ |2\lambda - 1|,\ 1\} = 1.$$

Thus $l_\infty(\mathbb{C}_2)$ is not strictly convex.

Theorem 3.12. The sequence space $l_p(\mathbb{C}_2)$ for $2 \le p < \infty$ is uniformly convex.

Proof.

Let $x = (x_n),\ y = y_n \in l_p(\mathbb{C}_2)$ such that

$$\|x\| \le 1,\ \|y\| < 1 \text{ and } \|x - y\| \ge \varepsilon.$$

Then, applying lemma 2.3 we have

$$\|x+y\|^p + \|x-y\|^p = \sum_{n=1}^{\infty}\|x_n + y_n\|^p + \sum_{n=1}^{\infty}\|x_n - y_n\|^p$$
$$= \sum_{n=1}^{\infty}(\|x_n + y_n\|^p + \|x_n - y_n\|^p)$$
$$\le \sum_{n=1}^{\infty} 2^{p-1}(\|x_n\|^p + \|y_n\|^p)$$
$$= 2^{p-1}[\sum_{n=1}^{\infty} x_n\|^p + \sum_{n=1}^{\infty}\|y_n\|^p]$$
$$= 2^{p-1}[\|x\|^p + \|y\|^p] < 2^{p-1}(1+1)$$
$$= 2^p$$

This shows that $\|x+y\|^p \le 2^p - \|x-y\|^p \le 2^p - \varepsilon^p$.

Now, $\left\|\frac{x+y}{2}\right\| = \left[\frac{1}{2^p}\|x+y\|^p\right]^{\frac{1}{p}}$

$$\le \left[\frac{1}{2^p}(2^p - \varepsilon^p)\right]^{\frac{1}{p}} = \left[1 - (\tfrac{\varepsilon}{2})^p\right]^{\frac{1}{p}}$$

If we take $\delta(\varepsilon) = 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}}$, then $\left\|\frac{x+y}{2}\right\| \le 1 - \delta$.

Hence $l_p(\mathbb{C}_2)$ for $2 \le p < \infty$ is uniformly convex.

Example 3. The sequence space $l_1(\mathbb{C}_2)$ is not uniformly convex.

Proof.

Let $x = (x_n) = (i_1,\ 0,\ 0,\ 0, \dots\dots)$, and $y = (y_n) = (0,\ 0,\ i_2,\ 0, \dots\dots)$.Then

$$\|x\| = \|y\| = 1.$$

Now, $\|x - y\|_{\mathbb{C}_2} = \sum_{n=1}^{\infty}\|x_n - y_n\|_{\mathbb{C}_2} = \sum_{n=1}^{\infty}\|(i_1,\ 0, -i_2,\ 0, \dots.)\|_{\mathbb{C}_2}$

$$= \|i_1\| + \|-i_2\| = 1+1 = 2 \ge \varepsilon$$

But $\left\|\frac{x+y}{2}\right\| = \sum_{n=1}^{\infty}\left\|\frac{x_n + y_n}{2}\right\|_{\mathbb{C}_2} = \sum_{n=1}^{\infty}\left\|\frac{1}{2}(i_1,\ 0,\ i_2,\ 0, \dots)\right\|$

$$= \left\|\frac{i_1}{2}\right\|_{\mathbb{C}_2} + \left\|\frac{i_2}{2}\right\|_{\mathbb{C}_2} = \frac{1}{2} + \frac{1}{2} = 1.$$

Thus, we cannot find $\delta(\varepsilon) > 0$ such that $\left\|\frac{x+y}{2}\right\| \le 1 - \delta$.

Hence, $l_1(\mathbb{C}_2)$ is not uniformly convex.

Example 4. The sequence space $l_\infty(\mathbb{C}_2)$ is not uniformly convex.

Proof.

Let $x = (x_n) = (1,\ i_1,\ i_2,\ 0,\ 0, \dots), y = (y_n) = (-1,\ i_1,\ -i_2,\ 0,\ 0, \dots)$

Then, $\|x\| = \|y\| = 1$

$$\|x - y\| = \sup\{\|x_n - y_n\|_{\mathbb{C}_2} : n \in \mathbb{N}\}$$

$$= \sup\{\|(2,\ 0,\ 2i_2,\ 0,\ 0, \dots\|\} = \sup\{0, 2\}$$

$$= 2 \geq \varepsilon$$

Now, $\left\|\frac{x+y}{2}\right\| = \sup\left\{\left\|\frac{x_n+y_n}{2}\right\|_{\mathbb{C}_2} : n \in \mathbb{N}\right\}$

$$= \sup\left\{\left\|\frac{1}{2}\ (0,\ 2i_1,\ 0,\ 0, \dots.)\right\|\right\}$$

$$= \sup\{\|(0,\ i_1,\ 0,\ 0, \dots\|\}$$

$$= \sup\{0, 1\} = 1$$

Thus, we cannot find $\delta(\varepsilon) > 0$ such that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$

Hence $l_\infty(\mathbb{C}_2)$ is not uniformly convex.

4. Conclusion

In this paper, we have presented some sequence spaces of bi-complex numbers and their algebraic, topological, and geometric properties. The extension of these properties on generalized double sequences of bi-complex numbers will be the future research directions.

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