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On ϑ – Curvature Tensor of Finslerian Hypersurfaces Given by Generalised Kropina Type Metric

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Abstract: *The purpose of the present paper is to find angular metric tensor ,carton torsion tensor, v-curvature tensor in a generalized Kropina space and the relation between v -curvatures with respect to Cartan connection $C\Gamma$ of a Finsler space $F^n = (M^n, L)$ and a Finsler space $F^{*n} = (M^n, L^*)$ whose metric L^* is derived from the metric L of F^n by $L^* (x, y) = \mu^{1/2} (x, y)\, \beta^{1/2} (x, y)$, where $\mu^{1/2} (x, y) = (L^{1/2} + \beta^{1/2}) (x, y)$ and $\beta = b_i(x)\, y^i$. The Finsler space F^{*n} is called a generalized Kropina space under certain conditions.*

Keywords: *Finsler metric, Kropina space, Cartan connection, h- vectors ϑ -curvature tensor.*

Mathematics subject classification: 2000: 53B 20, 53B 28, 53B 40, 53B 18.

1. Introduction

The study of Finsler spaces, with special metrics, has attracted considerable attention over the years. Among these, the Kropina metric, introduced by V.K. Kropina, has been a focal point due to its unique properties and applications. The Kropina metric is a special case of the (α, β)-metric, where the metric function is given by

$$L(x,y) = \frac{\alpha^2}{\beta},$$

with α being a Riemannian metric and β a 1-form.

This metric has been extensively studied in the context of Finslerian hypersurfaces, where the interplay between the intrinsic and induced geometries provides deep insights into the structure of the space.

A significant milestone in the development of Finslerian hypersurfaces was the introduction of the Kropina metric by Kropina himself. This class of Finsler metrics, given by

$$L(x,y) = \alpha^m \beta^n \text{ (where } m \neq 0, -1)$$

was extensively studied by Shibata [8], who investigated its geometrical properties and provided foundational results on its structure. Later, Shibata et al. [9] extended this work by introducing the transformation of Finsler metrics using

$$L^*(x, y) = f(L, \beta), \tag{1}$$

where f is a positively homogeneous function of degree one in L. This transformation played a crucial role in understanding the induced and intrinsic theories of hypersurfaces in Kropina spaces.

Hashiguchi et al. [1] studied the properties of Landsberg spaces with (α, β) metrics, providing insights into two-dimensional Finslerian structures. Their work was further expanded by Kitayama [2], who explored metric transformations and their impact on hypersurfaces in Finsler spaces. Additionally, Matsumoto [3] developed the induced and intrinsic connections of Finslerian hypersurfaces, contributing significantly to the study of projective geometry in Finsler spaces.

Prasad [4] and Prasad & Tripathi [5] examined torsion tensors and hypersurface structures in Finsler spaces with Kropina changes, establishing important results on the interactions between different types of metric transformations. Rastogi [6] extended these ideas by analyzing the properties of (α, β) metrics, further refining our understanding of Finslerian geometry.

Recent studies have continued to build on these foundational works. Shanker et al. [7] investigated curvature properties in homogeneous Finsler spaces, revealing new relationships between curvature tensors and metric deformations. Singh et al. [10] and Singh & Srivastava [11] explored h-transformations and Kropina-type modifications in special Finsler spaces, shedding light on the structural modifications induced by such transformations.

Izumi introduced the concept of h-vectors b_i that are ϑ-covariantly constant with respect to the Cartan connection, leading to new insights into the interplay between conformal transformations and directional dependencies in Finsler spaces. Srivastava and Pandey [12] , [13], extended this idea by examining generalized Kropina-type metrics under β-change and their implications for Finslerian hypersurfaces. Their work provided key relations between the original and transformed hypersurfaces, establishing conditions under which these transformations preserve geometric properties.

In this paper, we consider a generalized Kropina-type metric given by:

$$L^*(x, y) = \mu^{1/2}(x, y)\beta^{1/2}(x, y), \tag{2}$$

where

$$\mu^{1/2}(x, y) = \left(L^{1/2} + \beta^{1/2}\right)(x, y) \text{ and } \beta = b_i y^i,$$

with the vector b_i as a function of positional coordinates x^i only. When $L(x, y)$ corresponds to a Riemannian space, $L^*(x, y)$ reduces to the Kropina metric function.

Izumi while studying a conformal transformation of a Finsler space, introduced the h – vector b_i which is ϑ covariantly constant with respect to Cartan connection CΓ and

$$LC_{h\,i\,j}\, b^h = \rho\, h_{ij}.$$

The h – vector b_i is not only a function of positional coordinates x^i but also a function of directional arguments y^i. In fact

$$L(\partial b_i / \partial y^j) = \rho\, h_{ij}.$$

Here $b_i(x, y)$ is an h – vector in (M^n, L).

Let b_i is an h-vector in the Finsler space (M^n, L) and (M^n, L^*) be another Finsler space. The fundamental metric function $L^*(x ,y)$ is defined by

$$L^*(x, y) = (L^{1/2} + \beta^{1/2})(x, y)\, \beta^{1/2}(x, y). \tag{3}$$

Let us call the Finsler space $F^{n*} = (M^n, L^*)$ as generalazed Kropina space. To distinguish the geometrical objects of F^{n*} from those of the Finsler space F^n, we shall put *a* * sign on the corresponding objects of F^n.

This metric plays a crucial role in analyzing the curvature properties of Finslerian hypersurfaces, particularly through the ϑ-curvature tensor, which provides insights into the geometric deformations induced by the generalized Kropina-type metric.

Building upon the existing body of research, this study aims to further examine the properties of Finslerian hypersurfaces defined by the generalized Kropina-type metric and analyze the implications of the ϑ-curvature tensor in this context.

For an *h*-vector b_i we have the following lemmas [1] :

Lemma 1. If b_i is an *h*-vector then the function ρ and $l^*_i = b_i - \rho l_i$ are independent of *y*.

Lemma 2. The magnitude of an h-vector b_i is independent of *y*.

2. Preliminaries

Let b_i is a vector field in the Finsler space (M^n, L). If b_i satisfy the conditions [4]

$$b_{i\,|\,j} = 0$$

and
$$LC^h_{ij}\, b_h = \rho\, h_{ij}, \tag{4}$$

then the vector field b_i is called an *h*-vector. Here $|_j$ denotes the covariant differentiation with respect to Cartan's connection CΓ, C_{hij} is the Cartan's C-tensor, h_{ij} is the angular metric tensor, and ρ is a function described by

$$\rho = (n\text{-}1)^{-1} L\, C^i\, b_i\,, \tag{5}$$

We have $\frac{\partial\beta}{\partial y^i} = b_i$ by using the indicatory property.

Differentiating of (3) with respect to y^i yields.

$$l_i^* = \left(\frac{1}{2}\frac{\beta^{\frac{1}{2}}}{L^{\frac{1}{2}}}\right) l_i + \left(1 + \frac{1}{2}\frac{L^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\right) b_i. \tag{6}$$

We know that

$$\frac{\partial l_i}{\partial y^j} = L^{-1} h_{ij}.$$

From the above relation

$$h_{ij}^* = \left(\frac{\beta}{2L}\right) A_0 h_{ij} + \frac{1}{4} A_0\left(l_i b_j + l_j b_i\right) - \left(\frac{\beta}{4L}\right) A_0 l_i l_j - \left(\frac{L}{4\beta}\right) A_0 b_i b_j. \tag{7}$$

where

$$A_0 = 1 + \frac{\beta^{\frac{1}{2}}}{l^{\frac{1}{2}}}.$$

Theorem 2.1 (Angular Metric Tensor Transformation)

Under the transformation (3)

$$L^*(x, y) = (L^{1/2} + \beta^{1/2})(x, y)\,\beta^{1/2}(x, y).$$

the angular metric tensor (h^*_{ij}) of F^{*n} is given by (7) as follow:

$$h^*_{ij} = \left(\frac{\beta}{2L}\right) A_0 h_{ij} + \frac{1}{4} A_0 \left(l_i b_j + l_j b_i\right) - \left(\frac{\beta}{4L}\right) A_0 l_i l_j - \left(\frac{L}{4\beta}\right) A_0 b_i b_j$$

where $\qquad A_0 = 1 + \frac{\beta^{\frac{1}{2}}}{l^{\frac{1}{2}}}.$

Further from equation (7) and $g_{ij} = h_{ij} + l_i l_j$ one gets

$$g^*_{ij} = \left(\frac{\beta}{2L}\right) A_0 g_{ij} + \frac{1}{2} U_0 \left(l_i b_j + l_j b_i\right) - \left(\frac{\beta}{2L}\right) U_0 l_i l_j + U_1 b_i b_j \tag{8}$$

where $\qquad U_0 = 1 + \frac{3\beta^{\frac{1}{2}}}{2l^{\frac{1}{2}}}$

and $\qquad U_1 = 1 + \frac{3L^{\frac{1}{2}}}{2\beta^{\frac{1}{2}}}.$

Theorem 2.2. Under the transformation (3), the metric tensor of F^{*n} (g^*_{ij}) is described by (8).

From equation (8) and $C_{ijk} = \left(\frac{1}{2}\right)\frac{\partial g_{ij}}{\partial y^k}$,we get

$$C^*_{ijk} = \left(\frac{\beta}{2L}\right) A_0 C_{ijk} + \frac{1}{2} U_0 \left(h_{ij} m_k + h_{jk} m_i + h_{ki} m_J\right) - \left(\frac{3L^{\frac{1}{2}}}{8\beta^{\frac{3}{2}}}\right)\left(m_I + m_J + m_K\right). \tag{9}$$

where $\quad m_i = b_i - \left(\frac{\beta}{L}\right) l_i.$

Theorem 2.3. If the angular metric tensor h_{ij} of F^n vanishes, the torsion tensor of F^{*n} (C^*_{ijk}) also vanishes.

With the help of lemma (1) and relation

$$\sigma = \left(1 + \frac{\beta\rho}{L}\right),$$

we get

$$\frac{\partial\sigma}{\partial y^i} = \frac{\rho}{L} m_i. \tag{10}$$

From the definition of m_i, we get the following identities:

(i) $\quad m_i l^i = 0,$

(ii) $\quad m_i b^i = m_i m^i = b^2 - (\beta^2/L^2),$

(ii) $\quad h_{ij} m^i = h_{ij} b^i = m_j \quad$ and

(iv) $\quad C^h_{ij} m_h = L^{-1} \rho\, h_{ij}. \qquad (11)$

3. ϑ – Curvature Tensor

Definition 3.1. The v – curvature tensor S_{hijk} of $F^n = (M^n, L)$ with respect to Cartan's connection $C\Gamma$ is defined in [4] by

$$S_{hijk} = C_{hkm}C^m_{ij} - C_{hjm}C^m_{ik} \tag{12}$$

From (4), (8) and (11), we get

$$C^{*h}_{ij} = C^h_{ij} + R_1\left(h_{ij}m^k + h^k_j m_i + h^k_i m_j\right) - R_2\left(h_{ij}l^k R_3 + m_i m_j l^k\right) + R_4\left(h_{ij}b^h R_3 + m_i m_j b^h\right). \tag{13}$$

where $R_1 = \left(\frac{3\beta}{2L}\right)\frac{\rho}{\sigma}$,

$R_2 = \left(\frac{3(1-\sigma)\beta\rho}{L^2}\right)R_0$,

$R_3 = \frac{\beta^{\frac{1}{2}}}{4l^{\frac{1}{2}}}\left(b^2 - \frac{\beta^2}{L^2}\right) + \sigma$,

$R_4 = \left(\frac{\beta}{4L}\right)\rho R_0$

and $R_0 = \frac{1}{\frac{\sigma\{(1-\sigma)\beta^2 - L^2\}}{L^2} - b^2}$.

From equation (9) and (13) ,we get

$$C^*_{hkm}C^*_{ij} = C_{hkm}C^h_{ij} + \mu_1 h_{ij}h_k + C_1 h_{hk}m_i m_j + C_2 h_{ij}m_h m_k + C_0\left(C_{ijk}m_h + C_{ijh}m_h + C_{ihk}m_j + C_{jhk}m_i\right)$$
$$+C_0^2 A_3\left(h_{jk}m_i m_h + h_{ih}m_j m_k + h_{jh}m_i m_k + h_{ik}m_j m_k\right) \tag{14}$$

where

$\mu_1 = \left(b^2 - \frac{3\beta^2}{4L^2}\right)\left(\frac{\sigma\rho^2}{L^2}R_0 + \frac{\rho^2}{4L^2} + \frac{\rho^2}{4L^2}R_0\right) + \frac{\rho^2\sigma^2}{L^2}R_0,$

$C_0 = \frac{3\beta}{2L}\rho,$

$C_1 = \frac{\beta^2\rho^2}{4L^2}R_0(\sigma+1) + \frac{\beta^2\rho^2}{\sigma L^2}$

and $C_2 = \frac{3\beta^2\rho^2}{4\sigma L^2} + \frac{\rho^2}{2L^2}R_0\left\{\sigma + \frac{1}{2}\left(b^2 - \frac{\beta^2}{L^2}\right)\right\}.$

Thus from (12) and (14) , we obtain the following

Theorem 3.1. (ϑ - Curvature Tensor Transformation)

Under the transformation (3) the v-curvature tensor (S^*_{hijk}) of F^{*n} is written in the form

$$S^*_{hijk} = \sigma\left(\frac{\beta^{\frac{1}{2}}}{L^{\frac{1}{2}}}\right)s_{hijk} + \frac{3}{4\beta}h_{ij}e_{hk} + \frac{3L}{4}h_{hk}\left(e_{ij}+1\right) - \frac{L^{\frac{1}{2}}}{4\beta^{\frac{1}{2}}}h_{ik}e_{hj} - \frac{\beta^{\frac{1}{2}}}{4L^{\frac{1}{2}}}h_{hj}e_{ik}. \tag{15}$$

where $e_{ij} = \frac{1}{4}\mu_1 h_{ij} + \frac{3}{4}\mu_2 m_i m_j$

and $\mu_2 = \frac{3\beta^2\rho^2}{4L^2}R_0\left\{\sigma + \left(b^2 - \frac{\beta^2}{L^2}\right)\right\}.$

Theorem 3.2. The v - curvature tensor (S^*_{hijk}) of the transformed Finsler space F^{*n} vanishes if the angular metric tensor (h_{ij}) of F^n also vanishes, i. e. $S^*_{hijk} = 0$.

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