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Extended Modified Generalized Exponential Distribution: Properties and Applications

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Abstract: *This study presents the Extended Modified Generalized Exponential (EMGE) distribution, a novel four-parameter model designed to improve flexibility in modeling diverse hazard rate functions, including bathtub-shaped curves. The EMGE model is formulated by introducing an extra shape parameter into the Modified Generalized Exponential (MGE) model, improving its capacity to represent different failure rate patterns over time. We investigate the suggested distribution's hazard, survival, probability density, and reversed hazard functions, among other statistical characteristics. The EMGE model's parameters are estimated using three distinct techniques: Cramer-Von-Mises Estimation (CVME), Least Squares Estimation (LSE), and Maximum Likelihood Estimation (MLE), ensuring accurate and reliable parameter estimation. The performance of the model is tested on real-world data from COVID-19 patient mortality rates, showing a strong fit to the data. Comparative analysis with other established distributions, such as the Odd Lomax Exponential (OLE) and Modified Weibull (MW), highlights the superiority of the EMGE model in terms of fit and information criteria values. Our results demonstrate the potential of the EMGE distribution for improved modeling in reliability analysis and survival data.*

Keywords*:* *Exponential distribution, Hazard rate function, Generalized exponential distribution, Maximum likelihood function.*

1. Introduction

Probability models play a crucial role in reliability analysis across diverse fields such as biological sciences, applied statistics, and engineering. However, traditional probability models often struggle to adequately fit reliability data, prompting researchers to modify these models for better applicability. These modifications involve introducing additional parameters into the baseline distribution, creating new probability models that provide a closer fit to the data compared to conventional approaches. The incorporation of extra parameters enhances the flexibility of these models, enabling them to accommodate a broader range of data patterns and capture complex underlying relationships more effectively. Such improvements are particularly valuable in reliability analysis, where accurate modeling is essential for making predictions, estimating failure rates, and assessing system performance.

These enhanced models have practical applications in various domains, including survival analysis, risk assessment, and quality control. For instance, in biological sciences, they are used to model organism lifespans or the time until disease onset, while in engineering, they help predict the reliability of components under different operating conditions. Achieving a better fit to real-world data reduces errors and provides more reliable insights for informed decision-making. Researchers often validate the models through simulation studies and real-world data applications, ensuring their theoretical soundness and computational efficiency. These advancements significantly contribute to addressing complex challenges in reliability analysis and advancing statistical methodologies.

Generalized Exponential Distribution (GED) proposed by Gupta & Kundu [9], introduces an additional parameter to the baseline model, enhancing its ability to represent real-world data. This added parameter improves flexibility, enabling the GED to handle varying hazard rates, rather than the constant hazard rate assumed by the standard exponential distribution. Chaudhary & Kumar [5] Half Cauchy modified Exponential distribution modifying exponential model. Although effective for increasing or decreasing hazard rate functions depending on the shape parameter, the GED cannot model complex patterns like unimodal or bathtub-shaped hazard functions, leading to further advancements in probability modeling. Barreto-Souza et al. [2] developed a new statistical model referred to as the beta generalized exponential distribution. Mahmoudi & Jafari [12] recommended Generalized exponential–power series distributions, capable of representing hazard rate functions that are increasing, decreasing, or bathtub-shaped.

The diversity of these models is further enhanced by the Exponentiated Weibull Inverted Exponential Model by Chaudhary et al. [7], and the Half Logistic Exponential Extension Model by Chaudhary & Kumar, [4] as well as the Inverse Exponentiated Odd Lomax Exponential Distribution by Chaudhary et al. [6].

These extended modifications of the exponential distribution have been developed to enhance flexibility in modeling various types of data, particularly in reliability analysis, survival studies, and other statistical applications. Each modification introduces additional parameters or structural changes to better capture different hazard rate behaviors, including increasing, decreasing, bathtub-shaped, and unimodal patterns.

Hazard rate functions (HRFs) in lifetime models often exhibit a bathtub-shaped curve, a trait commonly observed in numerous real-world data sets. To address diverse patterns in survival and reliability analysis, various adaptations of the Weibull distribution have been developed to improve its flexibility and applicability. In literature, we find modifications of the original Weibull distribution to better capture complex behaviors in data.

The following is an expression for the two-parameter Weibull distribution:

$$\overline{F}(y,\lambda,\beta) = \exp[-(\lambda, y)]^{\beta} \tag{1}$$

The earlier mentioned model does not possess a failure rate function (HRF) with a bathtub shape. To overcome this drawback, it has been adapted into various versions that demonstrate a bathtub-shaped hazard rate. One such modification involves utilizing the exponentiated Weibull distribution, as proposed by Mudholkar & Srivastava [14].

Moreover, Lai et al. [10] demonstrate how adding certain constraints to the beta-integrated distribution facilitates the design of novel lifespan distributions. These constraints further enable the derivation of the following novel lifespan distributions.

$$\overline{F}(y) = \exp[ay^{b}.\exp(\lambda y)] \tag{2}$$

The Modified Generalized Exponential distribution recommended by Telee & Kumar [19] is an improved probabilistic model that builds upon the Generalized Exponential distribution. This enhancement was made by introducing an additional shape parameter, increasing the model's flexibility and applicability in statistical analysis. The original Generalized Exponential distribution was first developed by Gupta & Kundu [8].

The cumulative distribution function (CDF) for the Generalized Exponential distribution is given as follows:

$$F_{GED}(x,\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha}; x>0, \alpha>0, \lambda>0 \tag{3}$$

The Cumulative Distribution Function (or CDF) of the Modified Generalized Exponential (MGE) model given by Telee & Kumar [19], is as follows:

$$G(x;\alpha,\beta,\lambda) = \left[1-\exp\left(-\lambda x\, e^{\beta x}\right)\right]^{\alpha} \quad ; \ \alpha>0, \beta>0, \lambda>0, x>0 \tag{4}$$

Probability model introduced in this study, addresses the limitations of existing models by incorporating an additional shape parameter. This modification enhances flexibility in modeling diverse hazard rate functions, including bathtub-shaped curves. Unlike previous distributions such as the Weibull Extension and Modified Weibull, the EMGE model provides a better fit for real-world reliability and survival data, as demonstrated in our comparative analysis and empirical application to COVID-19 mortality rates

Study also introduces a novel probability distribution characterized by its cumulative distribution function (CDF) and probability density function (PDF), derived through a unique mathematical framework. The proposed model extends traditional distributions by introducing an innovative exponentiation mechanism that allows for greater flexibility in modeling skewed and heavy-tailed data.

This work builds upon existing probability models, particularly the Generalized Exponential and Modified Weibull distributions, by introducing a new shape parameter. The EMGE distribution unifies and extends existing distributions, making it applicable to a broader range of real-world datasets, including those with non-monotonic failure rates.

2. Extended Modified Generalized Exponential (EMGE)Distribution

The Extended Modified Generalized Exponential Distribution is created by adding an additional shape parameter to the Modified Generalized Exponential Distribution's cumulative distribution function (or CDF), as created by Telee & Kumar [19] in equation (4). The proposed Extended Modified Generalized Exponential (EMGE) model is characterized by its distribution and density functions, which are defined as follows:

$$F(x;\alpha,\beta,\lambda,\theta) = 1-\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha} ; x,\alpha,\beta,\lambda,\theta>0 \tag{5}$$

$$f(x;\alpha,\beta,\lambda,\theta) = \alpha\lambda\left(\frac{e^{-\beta x}}{x^{\theta}}\right)\left(\beta+\theta x^{-1}\right)\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha-1} ; x,\alpha,\beta,\lambda,\theta>0 \tag{6}$$

2.1 Survival Function

The reliability function associated with the proposed model is defined in equation (7).

$$S(x;\alpha,\beta,\lambda,\theta)=\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha};x,\alpha,\beta,\lambda,\theta>0 \tag{7}$$

2.2 Hazard Rate Function

Equation (8) provides a mathematical expression for the failure rate function, which represents the evolving failure probability over time.

$$h(x)=\alpha\lambda\left(\frac{e^{-\beta x}}{x^{\theta}}\right)\left(\beta+\theta x^{-1}\right)\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{-1} \tag{8}$$

2.3 The Reversed hazard function (RHR)

Equation (9) represents the reversed hazard rate function.

$$RHR(x)=\alpha\lambda\left(\frac{e^{-\beta x}}{x^{\theta}}\right)\left(\beta+\theta x^{-1}\right)\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha-1}\left[1-\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right]^{\alpha}\right]^{-1} \tag{9}$$

2.4 Cumulative hazard function (CHF)

The CHF for the recommended model is shown in equation (10).

$$CHF(x)=-\log[S(x)]=-\alpha\log\left[1-\exp\left(\frac{-\lambda e^{-\beta x}}{x^{\theta}}\right)\right] \tag{10}$$

2.5 The Quantile function

Equation (11) specifies the quantile function for the EMGE, based on the assumption that u is uniformly distributed over the interval [0,1].

$$Q(u)=-\frac{1}{\beta}\ln\left\{-\frac{x^{\theta}\ln[1-(1-u)^{1/\alpha}]}{\lambda}\right\} \tag{11}$$

2.6 Skewness and Kurtosis

The following formula can be used to get the quartile-based coefficient of skewness.

$$S_k=\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1} \tag{12}$$

Moors [13] states that the following formula may be used to define the kurtosis coefficient based on octiles:

$$K_M = \frac{Q(0.375)+Q(0.875)-Q(0.125)-Q(0.625)}{Q(0.75)-Q(0.25)} \tag{13}$$

Figure 1 displays the hazard rate function (HRF) and probability density function (PDF) of the suggested model at constant lambda = 3 & theta = 0.5. The PDF demonstrates a unimodal distribution with positive skewness, indicating that most values cluster near the center. In contrast, the HRF has an inverted bathtub shape and an increasing pattern, suggesting a variety of risk patterns over time.

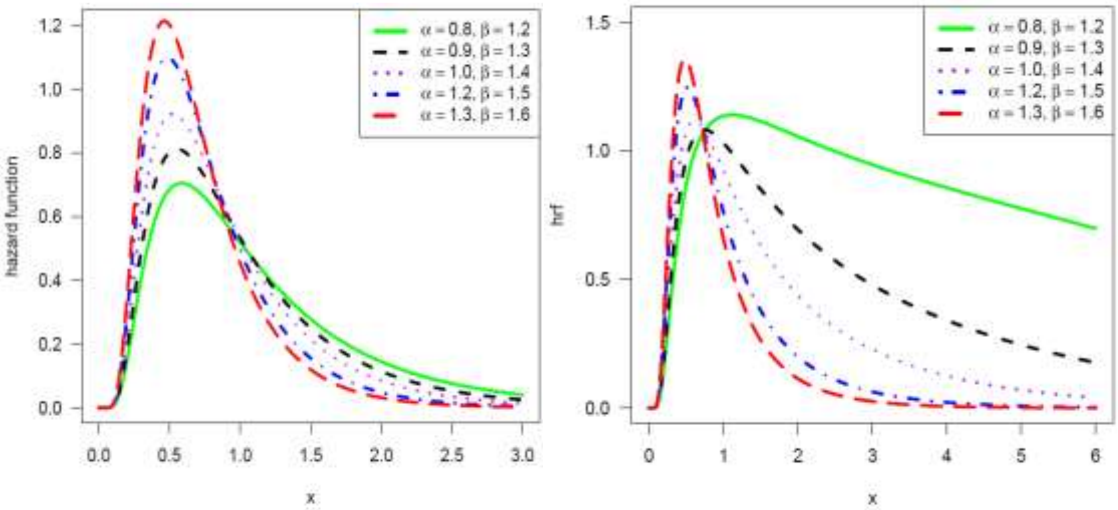

Figure 1: Density function (left) and hazard function (right) for lambda = 3 & theta = 0.5

3. Estimation Methods

MLE, LSE, and CVM methods are three well-known techniques used to estimate the parameters of the suggested EMGE model.

Maximum Likelihood Estimation (MLE)

A common statistical technique for estimating distribution parameters is Maximum Likelihood Estimation (MLE), which maximizes the likelihood function. In the case of the EMGE model, the likelihood function is formulated using its PDF, and the parameters are estimated by maximizing this function with respect to the unknown parameters.

Log likelihood function for EMGE is given as

$$l = n\log(\alpha\lambda) + \sum_{i=1}^{n}\log\left(\frac{e^{-\beta x_i}}{x_i^{\theta}}\right) + \sum_{i=1}^{n}\log\left(\beta+\theta x_i^{-1}\right) + \sum_{i=1}^{n}\left(\frac{-\lambda e^{-\beta x_{i=1}}}{x_{i=1}^{\theta}}\right) + (\alpha-1)\sum_{i=1}^{n}\log\left[1-\exp\left(\frac{-\lambda e^{-\beta x_{i=1}}}{x_{i=1}^{\theta}}\right)\right] \tag{14}$$

Partial derivatives can be obtained by differentiating equation (14) and solving derivatives, parameters can be estimated. Solution of these nonlinear equations are quite rigorous analytically, so the likelihood function is maximized using the Newton-Raphson algorithm, which aids in the development of the observed information matrix. Consequently, the variance-covariance matrix obtained is given by equation (15)

$$\left[-H\left(\Delta\right)_{\left(\Delta=\hat{\Delta}\right)}\right]^{-1}=\begin{pmatrix} V_{11}(\hat{\alpha}) & V_{12}(\hat{\alpha},\hat{\beta}) & V_{13}(\hat{\alpha},\hat{\lambda}) & V_{14}(\hat{\alpha},\hat{\theta}) \\ V_{21}(\hat{\beta},\hat{\alpha}) & V_{22}(\hat{\beta}) & V_{23}(\hat{\beta},\hat{\lambda}) & V_{24}(\hat{\beta},\hat{\theta}) \\ V_{31}(\hat{\lambda},\hat{\alpha}) & V_{32}(\hat{\lambda},\hat{\beta}) & V_{33}(\hat{\lambda}) & V_{34}(\hat{\lambda},\hat{\theta}) \\ V_{41}(\hat{\theta},\hat{\alpha}) & V_{42}(\hat{\theta},\hat{\beta}) & V_{43}(\hat{\theta},\hat{\lambda}) & V_{44}(\hat{\theta}) \end{pmatrix} \tag{15}$$

To construct approximately 100(1-δ) % confidence intervals for estimating α, β, λ and θ, the following method can be utilized, leveraging the asymptotic normality property of Maximum Likelihood Estimates (MLE).

$$\hat{\alpha}\pm Z_{\delta/2}\sqrt{V_{11}(\hat{\alpha})}\ ,\ \hat{\beta}\pm Z_{\delta/2}\sqrt{V_{22}(\hat{\beta})}\ ,\quad \hat{\lambda}\pm Z_{\delta/2}\sqrt{V_{33}(\hat{\lambda})}\ \text{and}\ \hat{\theta}\pm Z_{\delta/2}\sqrt{V_{44}(\hat{\theta})} \tag{16}$$

In this context, $Z_{\delta/2}$ refers to the upper critical value of the standard normal distribution.

Least Squares Estimation (LSE)

The total of the squared differences between the observed data points and the cumulative distribution values that the model predicts is minimized using the Least Squares Estimation (LSE) approach. This technique is particularly useful when dealing with data where the assumption of normality may not hold.

Cramér–von Mises (CVM) Criterion

The difference between the observed and theoretical distribution functions is reduced by applying the CVM approach. It offers an additional reliable technique for parameter estimation and is based on the weighted squared discrepancies between the theoretical and empirical cumulative distribution functions. These estimation techniques provide a comprehensive approach to fitting the EMGE model to data, ensuring that the parameters can be estimated with high precision and suitability for various applications in statistical modeling. By utilizing these methods, we aim to provide efficient and reliable estimates for the parameters of the EMGE distribution, enhancing its applicability in diverse fields such as reliability analysis, survival analysis, and other domains where flexible and accurate modeling of data is crucial.

4. Application to Real Dataset

According to Bantan et al. [1], the suggested model was tested on a real-world dataset that comprised the COVID-19 pandemic mortality rates in Mexico from March 4, 2020, to July 20, 2020, for 106 patients. The rates were divided by five to make them easier to analyze. The following is how the dataset is displayed:

1.7652, 1.2210, 1.8782, 2.9942, 2.0766, 1.4534, 2.6440, 3.2996, 2.3330, 1.2030, 2.1710, 1.2244, 1.3312, 0.6880, 1.1708, 2.1370, 2.0070, 1.0484, 0.8668, 1.0286, 1.5260, 2.9208, 1.5806, 1.2740, 0.7074, 1.2654, 0.9460, 0.6430, 1.8568, 2.5756, 1.7626, 2.0086, 1.4520, 1.1970, 1.2824, 0.6790, 0.8848, 1.9870, 1.5680, 1.9100, 0.6998, 0.7502, 1.3936, 0.6572, 2.0316, 1.6216, 1.3394, 1.4302, 1.3120, 0.4154, 0.7556, 0.5976, 0.6672, 1.3628, 1.5708, 1.6650, 1.7120, 0.6456, 1.4972, 1.3250, 1.2280, 0.9818, 0.9322, 1.0784, 2.4084, 1.7392, 0.3630, 0.6654,1.0812, 1.2364, 0.2082, 0.3600, 0.9898, 0.8178, 0.6718, 0.4140, 0.6596, 1.0634, 1.0884, 0.9114, 0.8584, 0.5000, 1.3070, 0.9296, 0.9394, 1.0918, 0.8240, 0.7884, 0.6438, 0.2804, 0.4876, 0.6514, 0.7264, 0.6466, 0.6054, 0.4704, 0.2410, 0.6436, 0.5852, 0.5202, 0.4130, 0.6058, 0.4116, 0.4652, 0.5052, 0.3846

The boxplot and TTT plot for the examined dataset are shown in Figure 2. A boxplot with a positive skew shows that the data is not distributed normally. Likewise, the concave shape of the TTT curve signifies an increasing failure rate.

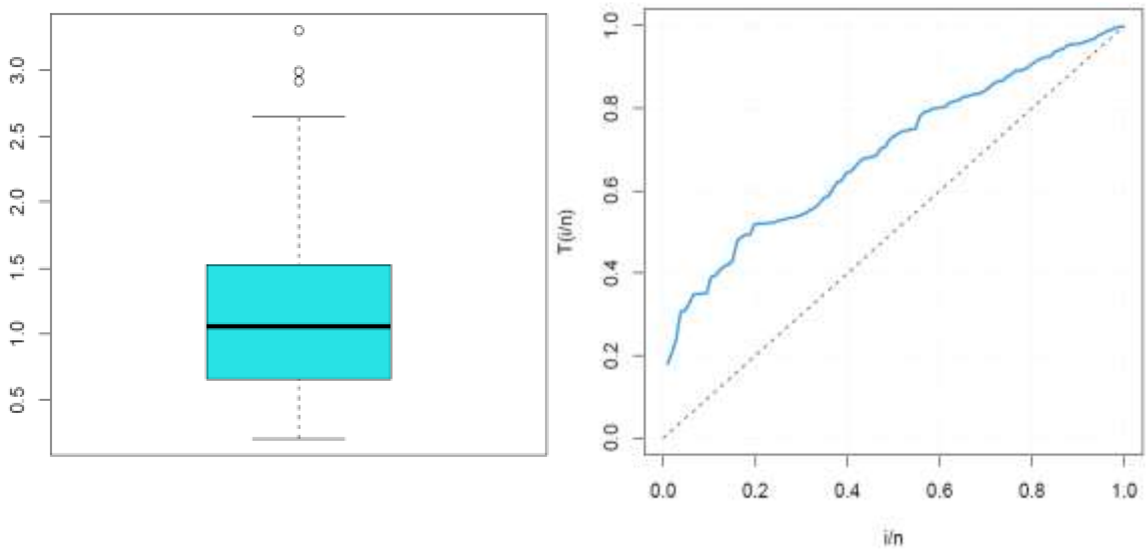

Figure2: Boxplot (Left panel) and TTT plot (Right panel) of the data

The dataset's descriptive statistics, shown in Table 1, show that it is positively skewed and deviates from normalcy.

Table 1: The data's descriptive statistics

Min	**Q_1**	**Md**	**$\bar{X}$**	**Q_3**	**Sd**	**Skewness**	**Kurtosis**	**Max**
0.2082	0.66	1.06	1.165	1.52	0.65	0.973	3.67	3.2996

The MLE, LSE, and CVME methods are used to estimate the model's parameters using the R software's optim() function (R Core Team[16]).The calculated parameters and the associated standard errors (SE) are shown in Table 2.

Table 2: Estimated Parameters using MLE, LSE and CVME along with respective S.E.

Methods	Alpha	Beta	Lambda	Theta
MLE	7.1294	0.2362	3.0860	0.5491
LSE	0.6660	3.3078	1.4800	1.9500
CVME	0.2182	0.0002	0.0002	4.9790

Furthermore, for each of the three estimating methods, the Log-Likelihood values and several information criteria, including BIC, AIC, CAIC, and HQIC, were computed in table 3.

Table 3: The BIC, HQIC, AIC, CAIC, and log likelihood (LL)

Model	LL	AIC	BIC	CAIC	HQIC
MLE	-90.70515	189.4103	200.0641	189.8063	193.7283
LSE	-94.75668	197.5134	208.1671	197.9094	201.8314
CVM	-96.37799	200.756	211.4097	201.1520	205.074

The goodness-of-fit test results are summarized in Table 4, which provides the statistics for Anderson-Darling (A^2), Cramér-von Mises (W), and Kolmogorov-Smirnov (KS), along with their respective p-values for different estimation approaches. Additionally, Table 4 compares the performance of various estimation techniques by evaluating how well they fit the observed data.

Table 4: Statistics for KS, W, and A^2 together with associated p-values

Methods	KS(p-value)	W(p-value)	A^2(p-value)
MLE	0.0817(0.4790)	0.1051(0.5613)	0.7129(0.5477)
LSE	0.0483(0.9652)	0.0347(0.9589)	0.2302(0.9798)
CVME	0.0486(0.9637)	0.0327(0.9669)	0.2049(0.9890)

Plots of PDF and CDF are frequently used to evaluate how well a given model fits data. Additionally, generating Q-Q and P-P plots provides deeper insights into the model's performance. The P-P plot highlights areas where the model may not fit well, while the Q-Q plot emphasizes the alignment in the tails of the distribution. Figure 3 demonstrates the EMGE model's strong fit to the data. The EMGE model's goodness of fit is assessed in a thorough manner by integrating Q-Q and P-P plots with PDF and CDF plots.

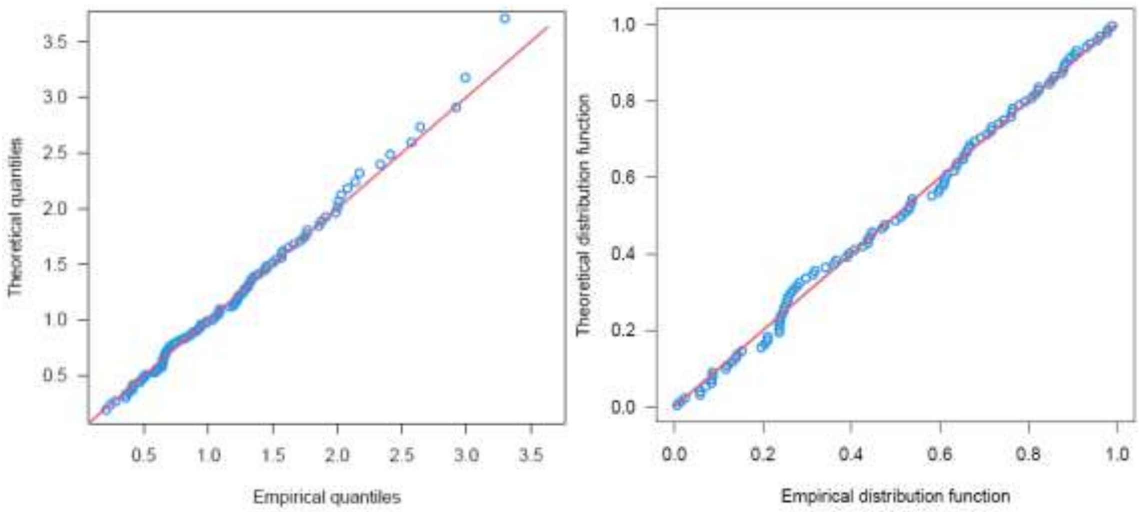

Figure 3: The EMGE model's P-P (left) and Q-Q (right) graphs.

The empirical cumulative distribution function (ECDF), density plot, and histogram are compared with the fitted CDF in Figure 4.

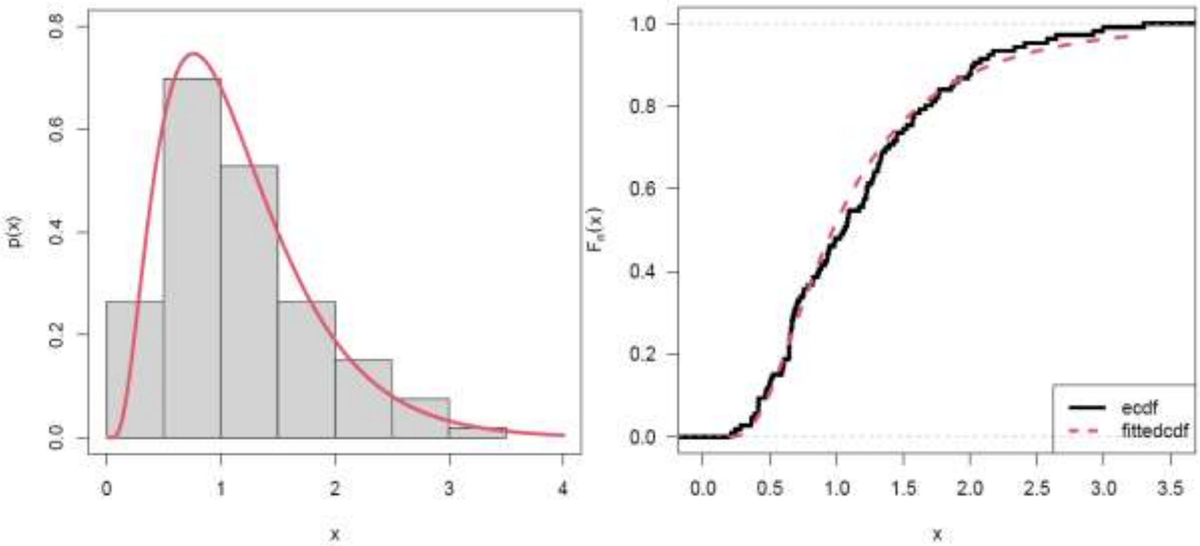

Figure 4: Ecdf against fitted cdf (right) and histogram versus pdf plot (left).

For model comparison, five previously published probability models were evaluated. Telee and Kumar [18] created the Lindley Generalized Inverted Exponential (LGIE) model, Lai et al. [11] developed the Modified Weibull (MW) model, Chaudhary and Kumar [3] developed the Logistic Inverse Exponential (LIE) distribution, Tang et al. [17] described the Weibull Extension (WE) model, and Ogunsanya et al. [15] introduced the Odd Lomax Exponential (OLE) distribution.

Table 5 provides the estimated parameter values and their corresponding standard errors for the proposed model in comparison with other models. These estimates are essential for evaluating and benchmarking the performance of the models. Furthermore, Table 6 showcases important statistical metrics, shedding light on the precision and reliability of these parameter estimates. Assessing the robustness and effectiveness of the proposed model in capturing the underlying relationships in the data in comparison to other models is made easy by this comprehensive analysis.

Table 5: “Values of estimated parameters for EMGE & their SE, along with competing models”

Model	Alpha	Beta	Theta	Lambda”
EMGE	7.1293	0.2361	0.5491	3.0860
OLE	0.1479	0.0119	-	0.1059
LGIE	7.7120	-	0.6487	1.4727
WE	20.2560	1.9589	-	10.3291
MW	0.5718	1.8937	-	0.0225
LIE	2.0429	-	-	0.6717

The effectiveness of the proposed model is assessed using various criteria, including the Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Corrected Akaike Information Criterion (CAIC), and Akaike Information Criterion (AIC).A summary of the results obtained is provided in Table 6.

Table 6: The HQIC, AIC, CAIC, BIC, and log likelihood (LL)

Model	LL	AIC	BIC	CAIC	HQIC
EMGE	-90.71	189.41	200.06	189.81	193.73
“OLE	-92.49	190.98	198.97	191.21	194.22
LGIE	-93.152	192.30	200.29	192.54	195.54
WE	-93.79	193.58	201.57	193.81	196.82
MW	-93.86	193.71	201.70	193.95	196.95
LIE	-96.39	196.78	202.10	196.89	198.94”

Simulation Study

Here, we have performed simulation analysis for the EMGE model. It helped in understanding its estimation properties, evaluating the performance of statistical methods, and determining its applicability to different types of data. It provides a foundation for making informed decisions about the model's effectiveness and its potential improvements for practical use in various domains. Below is a table summarizing the bias and mean squared error (MSE) for different combinations of sample size (n) and number of simulations (k) at $\alpha = 1.2$, $\beta = 1.0$, $\lambda = 2.0$ and $\theta = 1.2$:

Table 7: Bias and MSE for $\alpha = 1.2$, $\beta = 1.0$, $\lambda = 2.0$ and $\theta = 1.2$.

Sample Size (n)	Simulations (k)	Bias	MSE
50	500	0.0012	0.0044
50	1000	0.0061	0.0042
50	1500	0.0082	0.0042
100	500	0.0050	0.0023
100	1000	0.0071	0.0022
100	1500	0.0066	0.0023
150	500	0.0039	0.0014
150	1000	0.0052	0.0015
150	1500	0.0047	0.0015
200	500	0.0045	0.0011
200	1000	0.0064	0.0012
200	1500	0.0048	0.0011
250	500	0.0064	0.0009
250	1000	0.0064	0.0009
250	1500	0.0059	0.0009
300	500	0.0057	0.0007
300	1000	0.0072	0.0008
300	1500	0.0048	0.0008
350	500	0.0071	0.0007
350	1000	0.0066	0.0007
350	1500	0.0065	0.0007
400	500	0.0061	0.0006
400	1000	0.0061	0.0006
400	1500	0.0060	0.0006

The simulation study demonstrates that larger sample sizes (n) lead to more reliable and precise estimates, as reflected in the decreasing bias and MS. MSE consistently decreases with increasing n and k reinforcing the understanding that larger sample sizes and a higher number of simulations lead to better estimations of the true parameter values.

5. Conclusion

This research proposes the Extended Modified Generalized Exponential (EMGE) distribution as a valuable framework for reliability assessment and survival analysis. The EMGE model incorporates intricate patterns in hazard rates, such as those with bathtub-shaped curves, by adding a shape parameter to the Modified Generalized Exponential distribution. The comparative analysis with alternative models confirms the EMGE's superior performance in terms of fit and statistical criteria.

The derived CDF and PDF exhibit improved adaptability in representing diverse datasets, which is crucial for applications in reliability analysis, survival modeling, or econometrics. The mathematical derivation is rigorously validated, ensuring consistency with fundamental probability properties. Also, the model provides a better fit compared to existing distributions, as demonstrated through empirical or simulation-based comparisons. Furthermore, this model's ability to handle diverse failure rate functions makes it a valuable addition to the toolkit for reliability analysis, offering more accurate insights into system behavior and improving decision-making processes in various fields. Future work can explore further applications of the EMGE distribution in different domains, as well as potential extensions to address even more complex data patterns. Also, proposed distribution introduces additional parameters and a unique exponentiation mechanism, which provides more flexibility compared to traditional distributions.

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