Nepal Journal of Mathematical Sciences (NJMS)
ISSN: **2738-9928 (*online*), 2738-9812 (*print*)**
Vol. 5, No. 2, 2024 (August): 21-26
DOI: https://doi.org/10.3126/njmathsci.v5i2.76630

Research Article
Received Date: December 7, 2024
Accepted Date: February 27, 2025
Published Date: March 16, 2025

Sawi Transform of Hypergeometric Functions: A New Perspective on Special Functions

Meena Kumari Gurjar[1*], Rajneesh Kumar[2], Preeti Chhattry[3] & Anil Kumar Vishnoi[4]

[1,4]*Department of Mathematics and Statistics, J.N.V. University, Jodhpur-342001, Rajasthan, India.*
[2]*Department of Mathematics, Kurukshetra University, Kurukshetra-136119, India.*
[3]*Govt. Girls High School Pathariya, Mungeli, Chattisgarh-495113, India.*

Corresponding author: [*]meenanetj@gmail.com

Abstract: *In this article, we shall study the Sawi transform of the generalized Wright hypergeometric function. New results are derived involving the Sawi transform for generalized hypergeometric functions. Also, we extend our analysis to the Sawi transform of a product involving Mittag-Leffler function, generalized hypergeometric function, and several related results. These findings give us a new way to think about the Sawi Transform's role in the study of special functions.*

Mathematical Sciences Classification: 33C05, 44A05, 44A20.

Keywords: Integral transform, Sawi transform, Wright hypergeometric functions, hypergeometric function, Mittag-Leffler function.

1. Introduction

Special functions, especially Wright hypergeometric functions, are very important to many areas of mathematics, science, and engineering. The extended Wright hypergeometric function and Mittag-Leffler function are used in many areas, such as statistical physics, fluid dynamics, and quantum mechanics. Integral transforms are widely used to solve the differential equations.

A significant amount of research has been conducted on these functions using integral transforms like the Laplace transform, the Fourier transform, and the Mellin transform. As a new member of the family of integral transforms, the Sawi transform has been shown to facilitate the evaluation certain integrals and the solution of differential equations. The Sawi transform provides a more efficient and generalized approach compared to Laplace, Fourier, and Mellin transforms, especially for solving fractional differential equations and handling special function. Numerous researchers [3-6, 9-10] have studied in recent years. Awwad et al.

[1] introduced the double ARA-Sawi transform, extending its applicability to more complex integral equations. Additionally, Momani et al. [8] explored how combining the Laplace and Sawi transforms improves computational techniques in mathematical modeling. Saadeh et al. [11] investigated the use of Sawi transform in fractional differential equations, integrating it with iterative methods to enhance solution accuracy. The goal of this work is to bridge the gap by exploring the Sawi Transform of generalized Wright hypergeometric functions and demonstrating its applications.

The Sawi transform opens new research directions in integral transform, special functions, and fractional calculus, paving the way for future mathematical and applied studies.

2. Basic Definitions

2.1 The Sawi Transform

Mahgoub [7] came up with the Sawi transform of the function $f(t), t \geq 0$ in 2019 for a function in the set A marked by:

$$A = \left\{ f(t) : \exists\, M, k_1, k_2 > 0;\ |f(t)| < M\, e^{\frac{|t|}{k_j}}, if\ t \in (-1)^j \times [0, \infty) \right\}$$

For a certain function in set A, M must be a real number, but k_1, k_2 can be a real number or an infinite number. Then $S[f(t)]$ stands for the Sawi transform of $f(t)$, which is defined as

$$S[f(t)] = R(\upsilon) = \frac{1}{\upsilon^2} \int_0^\infty e^{-\frac{t}{\upsilon}} f(t)\, dt\ , k_1 \leq \upsilon \leq k_2. \qquad (1)$$

2.2 The Sawi transform of some elementary functions [9, 12]

$f(t)$	1	t	$t^n, n \in N$	$t^\alpha, \alpha \in R^+$	e^{at}
$S[f(t)]$	$1/\upsilon$	1	$n!\,\upsilon^{n-1}$	$\Gamma(\alpha+1)$ $\upsilon^{\alpha-1}$	$\frac{1}{\upsilon(1-a\upsilon)}$

2.3 The generalized Wright Hypergeometric function

The generalized Wright hypergeometric function [2, 14] for $z, a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in R$ $(\alpha_i, \beta_j \neq 0; i = 1,2, \ldots, p; j = 1,2, \ldots, q)$ is defined as:

$$p\Psi q\,[z] = p\Psi q \left[\begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} ; z \right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 n) \ldots \Gamma(a_p+\alpha_p n)}{\Gamma(b_1+\beta_1 n) \ldots \Gamma(b_q+\beta_q n)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q} \Gamma(b_j+\beta_j n)} \frac{z^n}{n!} \qquad (2)$$

2.4 The generalized Mittag-Leffler function

In 1971, Prabhakar [10] introduced the generalized Mittag-Leffler function

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \qquad (3)$$

where $\alpha, \beta, \gamma \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0.$

3. Main Results

Theorem 3.1. The Sawi Transform of generalized Wright hypergeometric function is given by

$$S\left[p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};z\right]\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\upsilon^{n-1}$$

Proof:

The generalized Wright hypergeometric function is defined by

$$p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}$$

Employing Sawi transform on both sides in above definition, determine

$$S\left[p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix};z\right]\right]=S\left[\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}\right]$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\,n!}S[z^n]$$

After exercising Sawi transform for $f(z)=z^n$, yield the required result.

Corollary 3.2. When we substitute

$$\alpha_1=\alpha_2=\cdots=\alpha_n=\beta_1=\beta_2=\cdots=\beta_n=1$$

in Theorem (3.1), then we get new result as Sawi transform of generalized hypergeometric function [14] after a little simplification:

$$S\left[pFq\left[\begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix};z\right]\right]=S\left[\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\upsilon^{n-1}\right]$$

Corollary 3.3. Incorporating

$$p=q=1,a_1=\alpha_1=1,b_1=\beta,\qquad\beta_1=\alpha$$

in Theorem (3.1), we obtain the similar known result [12] in terms of Mittag-Leffler function:

$$S\left[1\Psi 1\left[\begin{matrix}(1,1)\\(\alpha,\beta)\end{matrix};z\right]\right]=S\left[E_{\alpha,\beta}(z)\right]=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(\alpha n+\beta)}\upsilon^{n-1}$$

Theorem 3.4. The Sawi transform of a product comprising the generalized Wright hypergeomtric function is given by

$$S\left[z^m\, p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix};z\right]\right]=S\left[\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)\ (m+n)!}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\,n!}\ \upsilon^{m+n-1}\right]$$

Proof:

The generalized Wright hypergeomtric function is presented by

$$p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\,n!}z^n$$

Multiplying on both sides by z^m , yield

$$z^m\, p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\,n!}z^{m+n}$$

Taking Sawi transform on both sides of above equation, give

$$S\left[z^m\, p\Psi q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix};z\right]\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)\,n!}S[z^{m+n}].$$

The required results are obtained by using the result of Sawi transform for z^{m+n}.

Theorem 3.5. The Sawi transform of Mittag-Leffler function is given by

$$S\left[E_{\alpha,\beta}^{\gamma}(z)\right]=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\ \upsilon^{n-1}.$$

Proof:

The Mittag-Leffler function $E_{\alpha,\beta}^{\gamma}(z)$ is

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\,\frac{z^n}{n!}$$

Employing the Sawi transform on both sides, yield

$$S\left[E_{\alpha,\beta}^{\gamma}(z)\right]=S\left[\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\,\frac{z^n}{n!}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)\ n!}\ S[z^n]$$

With the aid of definition of Sawi transform of power function, we obtain the required result.

Theorem 3.6. For $\alpha > 0, a \in \boldsymbol{R}$ and we have the following formula for Sawi transform as:

$$S\left[E^{\gamma}_{\alpha,\beta}(-at^{\alpha})\right] = \upsilon^{-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\ \Gamma(\alpha n+1)\ (-a\upsilon^{\alpha})^n}{\Gamma(\alpha n+\beta)\ n!}$$

Proof:

The formula of Mittag-leffler function is

$$E^{\gamma}_{\alpha,\beta}(-at^{\alpha}) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\ \frac{(-at^{\alpha})^n}{n!}$$

Employing Sawi transform on both sides give

$$S\left[E^{\gamma}_{\alpha,\beta}(-at^{\alpha})\right] = S\left[\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\ \frac{(-at^{\alpha})^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_n\ (-a)^n}{\Gamma(\alpha n+\beta)n!}\ S[t^{\alpha n}]$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_n\ (-a)^n}{\Gamma(\alpha n+\beta)n!}\ \Gamma(\alpha n+1)\upsilon^{\alpha n-1}$$

$$= \upsilon^{-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\ \Gamma(\alpha n+1)\ (-a\upsilon^{\alpha})^n}{\Gamma(\alpha n+\beta)\ n!}$$

Corollary 3.7. Inserting $\beta = \gamma = 1$in Theorem (3.6), determine the known result as obtained by Wadi et.al. [12] in terms of Mittag-Leffler function or Wiman's function [13]:

$$S[E_{\alpha}\,(-at^{\alpha})]\ = \upsilon^{-1}\sum_{n=0}^{\infty}(-a\upsilon^{\alpha})^n = \frac{1}{\upsilon(1+a\upsilon^{\alpha})}$$

4. Conclusion

In this paper, the Sawi transform of the generalized Wright hypergeometric function was examined from a new perspective. We found the basic result that connects the transform to Mittag-Leffler functions and hypergeometric functions. The Sawi transform is a useful tool for more scientific research because it is easy to use and works well. These methods can be used in a lot of different areas of applied sciences (continuum mechanics and thermodynamics).

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