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On Two Closed Form Evaluations for the Generalized Hypergeometric Functions ${}_3F_2\left(\frac{1}{16}\right)$

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Abstract: *The main objective of this note is to provide two closed-form evaluations for the generalized hypergeometric functions with the argument* 1/16. *This is achieved by means of separating a generalized hypergeometric function* ${}_3F_2$ *into even and odd components together with the use of two known sums involving reciprocal of the certain binomial coefficients obtained very recently by Gencev.*

Keywords: Generalized hypergeometric functions, Central binomial coefficients, and Combinatorial sum.

1. Introduction

The concept of hypergeometric and generalized hypergeometric functions is essential in mathematics, engineering mathematics, and mathematical physics. Many of the frequently encountered functions in analysis are special or limiting cases of these two functions. Prof. John Wallis [2] was the first person to use the term "hypergeometric" in his work *Arithmetica Infinitorum* (1655), to refer to any series that is extended beyond the ordinary series. In fact, he explored the series...

$$1+a+a(a+1)+a(a+1)(a+2)+...$$

Then Euler, Vandermode, Hidenberg, and other mathematicians studied similarly for the next hundred and fifty years.

In 1812, C.F. Gauss [3] used the symbol

$${}_2F_1\left[\begin{matrix} a, \quad b \\ c \end{matrix} ; z\right]$$

to define the infinite series given below:

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ;z\right] = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \dots \quad (1) $$

By using commonly used Pochhammer symbol for any $a \epsilon \mathbb{C}\ (\neq 0)$ and $n \in \mathbb{N}$ by

$$ (a)_n = \begin{cases} a(a+1)(a+2)\dots(a+n-1); & n \in \mathrm{N} \\ 1; & n = 0 \end{cases} $$

$$ = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad \text{where } \Gamma(a) \text{ is the gamma function.} $$

The series (1) can be written as

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ;z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (2) $$

The quantities a, b and c expressed in (2) are called the parameters (real or complex) of the series provided that $c \neq 0, -1, -2, \dots$. The term z is the variable of the series. By using the ratio tests, the series (2) is

(i) convergent for all values of z such that $|z| < 1$ and divergent for $|z| > 1$,

(ii) convergent for z =1 if $Re(c-a-b) > 0$ and divergent otherwise,

(iii) absolutely convergent for $z = -1$ if $Re(c-a-b) > 0$ and but

not absolutely convergent if $-1 < Re(c-a-b) \leq 0$ and divergent for $Re(c-a-b) \leq -1$.

The observations for the Gauss's series (4) are given below:

(i) the series (4) reduces to the geometric series if a =1 and $b = c$ or b =1 and $a = c$. From this fact, this series (4) is called the "Hypergeometric Series".

(ii) the series becomes unity if a and b or both are zero.

(iii) the series becomes a polynomial (the series containing the finite number of terms and the issue of convergence does not arise) if $a\ or\ b$ or both is a negative integer.

For the limiting case of (2), since $\frac{(b)_n}{b^n} z^n \to z^n$, and if we replace z by $\frac{z}{b}$ take the limit as $b \to \infty$, then we arrive at the infinite series denoted by the symbol ${}_1F_1\left[\begin{matrix} a, \\ c \end{matrix} ;z\right]$.

This is the Kummer's series or confluent hypergeometric function already defined in the literature viz.

$$ {}_1F_1\left[\begin{matrix} a, \\ c \end{matrix} ;z\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} \quad (3) $$

Gauss's hypergeometric function ${}_2F_1$ and its confluent hypergeometric function ${}_1F_1$ serve as foundational elements of special functions, encompassing many commonly used functions as special or limiting cases. These include the exponential function, trigonometric and inverse trigonometric functions, hyperbolic functions, Legendre's function, the incomplete Beta function, the complete elliptic functions of the first and second kinds, and most classical orthogonal polynomials; all of which are particular cases of the Gaussian hypergeometric function.

Similarly, the confluent hypergeometric function encompasses special cases such as Bessel's function, the parabolic cylinder function, and the Coulomb wave function. As previously noted, the Gaussian hypergeometric function is characterized by two numerator parameters a and b, along with one denominator parameter c. A natural generalization of this function is accomplished by introducing arbitrary number of p numerator and q number of denominator parameters. The resulting function is defined [2, 3, 9, 10, 12, 13, 16] as

$$ {}_pF_q\left[\begin{matrix}(a), \\ (c)\end{matrix} ;z\right] = {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p \\ c_1,\ldots,c_p\end{matrix} ;z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\ldots(a_p)_n}{(c_1)_n\ldots(c_q)_n}\frac{z^n}{n!} \tag{4} $$

It is also assumed that the numerator parameters a_j $(j = 1,2,\ldots,p)$, the denominator parameter c_j $(j = 1,2,\ldots,q)$ and the variable z, can take real or complex values provided that $c_j \neq 0,-1,-2,\ldots for\ j = 1,2,\ldots,q$.

Also, the series (4) is

(i) convergent for $|z| < \infty$ if $p < q$,

(ii) convergent for $|z| < 1$ if $p = q + 1$ and divergent for all $z,\ z \neq 0$ if $p > q + 1$.

Further if we set $\omega = \left(\sum_{j=1}^{q} c_j - \sum_{j=1}^{q} a_j\right)$ then the series (4) with $p = q + 1$ is

(i) convergent absolutely for $|z| = 1$ if $Re(\omega) > 0$,

(ii) conditionally convergent for $|z| = 1, z \neq 1$ if $-1 < Re(\omega) \leq 0$ and divergent $|z| = 1$ if $Re(\omega) \leq -1$.

Hypergeometric functions ${}_2F_1$ and generalized hypergeometric functions ${}_pF_q$ have wide range of applications in the field of mathematics, engineering mathematics and mathematical physics. For details, see [1, 13, 16].

One the other hand, the binomial coefficients for non-negative integers $n\ and\ k$ are defined by

$$ \binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & ;n \geq k \\ 0 & ;n < k \end{cases} \tag{5} $$

The central binomial coefficients are defined by

$$ \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \quad (n = 0,1,2,\ldots). \tag{6} $$

The binomial and the reciprocal of the binomial coefficients always plays an important role in many areas of mathematics (including number theory, probability and statistics). The sums containing the central binomial coefficients and the reciprocal of the central binomial coefficients are being studied since a long time ago.

The significant number of elegant results were published in the works of Kummer,et.al [7], Lehmer [8], Mansour [9], Pla [11], Sherman [15], Sprugnoli [17,18], Sury [19], Sury et.al [20], Trif [21], Wheelon [22], Zhng and J. [23] and Zhao and Wang [24]. Koshy [6] has mentioned the details on the central binomial coefficients and the reciprocal of the central binomial coefficients in his book. Gould [5] and Riordan [14] has collected numerous identities involving central binomial coefficients.

Recently, Gencev [4] studies the following interesting sums involving the reciprocal of the central binomial coefficients viz.

$$\sum_{k=1}^{\infty}\frac{1}{k^m\binom{2k}{k}} \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m\binom{2k}{k}}.$$

These sums are known as Aprey sums [3]. In particular, Uhl [9] obtained the following two interesting sums;

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)\binom{2k+2}{k+2}}=\frac{1}{2}-\frac{\sqrt{3}\pi}{18} \tag{7}$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)\binom{2k+4}{k+2}}=-\frac{1}{2}+\frac{3}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+1}{2}\right) \tag{8}$$

In terms of generalized hypergeometric functions, the results (7) and (8) can be written in the following manner;

$${}_3F_2\left[\begin{matrix}1, & 1, & 3 \\ & 2, & \frac{5}{2}\end{matrix};\frac{1}{4}\right]=3-\frac{\sqrt{3}\pi}{3} \tag{9}$$

and

$${}_3F_2\left[\begin{matrix}1, & 1, & 3 \\ & 2, & \frac{5}{2}\end{matrix};-\frac{1}{4}\right]=-3+\frac{18}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+1}{2}\right) \tag{10}$$

It is known that the new results can be obtained by resolving a generalized hypergeometric function ${}_pF_q(z)$ into odd and even components. We shall employ this procedure combined with the results (9) and (10) to obtain two new closed form evaluations of the series ${}_3F_2$ with argument $\frac{1}{16}$.

The same is given in the next section.

2.Two Closed Form Evaluations of ${}_3F_2\left(\frac{1}{16}\right)$

We shall establish the two new closed form evaluations for the generalized hypergeometric functions ${}_3F_2$ with the argument $\frac{1}{16}$ in this section via the theorem given below.

Theorem: The following two results hold true.

$${}_3F_2\left[\begin{matrix}\frac{1}{2}, & 1, & 2 \\ \frac{5}{4}, & \frac{7}{4} & \end{matrix};\frac{1}{16}\right]=\frac{9}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+1}{2}\right)-\frac{\sqrt{3}\pi}{6} \tag{11}$$

and

$$ {}_3F_2\left[\begin{matrix} 1, & 1, & \frac{5}{2} \\ \frac{7}{4}, & \frac{9}{4} & \end{matrix} ;\frac{1}{16}\right] = \frac{10}{3}\left\{6 - \frac{\sqrt{3}\pi}{3} - \frac{18}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+1}{2}\right)\right\} \tag{12} $$

Proof:

To prove the results (11) and (12), we shall use the following general results mentioned in

[12 $$ {}_{q+1}F_q\left[\begin{matrix} a_1,\dots,a_{q+1} \\ c_1,\dots,c_q \end{matrix} ;z\right] + {}_{q+1}F_q\left[\begin{matrix} a_1,\dots,a_{q+1} \\ c_1,\dots,c_q \end{matrix} ;-z\right] = 2\,{}_{2q+2}F_{2q+1}\left[\begin{matrix} \frac{a_1}{2},\frac{a_1}{2}+\frac{1}{2},\dots,\frac{a_{q+1}}{2}+\frac{1}{2} \\ \frac{1}{2},\frac{c_1}{2},\frac{c_1}{2}+\frac{1}{2}\dots,\frac{c_q}{2}+\frac{1}{2} \end{matrix} ;z^2\right] \tag{13} $$

and

$$ {}_{q+1}F_q\left[\begin{matrix} a_1,\dots,a_{q+1} \\ c_1,\dots,c_q \end{matrix} ;z\right] - {}_{q+1}F_q\left[\begin{matrix} a_1,\dots,a_{q+1} \\ c_1,\dots,c_q \end{matrix} ;-z\right] $$

$$ = \frac{2za_1a_2\dots a_{q+1}}{c_1c_2\dots c_q}\,{}_{2q+2}F_{2q+1}\left[\begin{matrix} \frac{a_1}{2}+\frac{1}{2},\frac{a_1}{2}+1,\dots,\frac{a_{q+1}}{2}+\frac{1}{2},\frac{a_{q+1}}{2}+1 \\ \frac{3}{2},\frac{c_1}{2}+\frac{1}{2},\frac{c_1}{2}+1,\dots,\frac{c_q}{2}+\frac{1}{2},\frac{c_q}{2}+1 \end{matrix} ;z^2\right] \tag{14} $$

These results (13) and (14) can be established by resolving a generalized hypergeometric function $ {}_{q+1}F_q\left[\begin{matrix} a_1,\dots,a_{q+1} \\ c_1,\dots,c_q \end{matrix} ;z\right] $ into even and odd components and making use of the following identities,

$$ (a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a}{2}+\frac{1}{2}\right)_n \quad \text{and} \quad (a)_{2n+1} = a2^{2n}\left(\frac{a}{2}+1\right)_n\left(\frac{a}{2}+\frac{1}{2}\right)_n $$

in (13) and (14), respectively.

Therefore, for the derivation of the results (11) and (12), we substitute in the results (9) and (10) by letting q = 2 and substituting $a_1 = a_2 = 1, a_3 = 3, b_1 = 2, b_2 = \frac{5}{2}$, z = $\frac{1}{4}$ in (3) and (4) respectively and we obtain the results (1) and (2) respectively after some simplification.

3. Conclusion

In this paper, two new and interesting closed-form evaluations of the generalized hypergeometric functions ${}_{q+1}F_q(z)$ for q = 2 with argument 1/16 have been established. This is done by separating the generalized hypergeometric function ${}_{q+1}F_q(z)$ into two components, even and odd, together with the use of two proven results by Gensev for the series involving reciprocals of the non-central binomial coefficients. We believe that the results established in this paper have not appeared in the literature before and represent a definite contribution in the area of generalized hypergeometric functions.

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