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An Integral Involving Generalized Hypergeometric Function

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Abstract: *In this article, we have evaluated an integral involving a generalized hypergeometric function. This is achieved by employing a summation formula for the series* ${}_4F_3$ *obtained earlier by Choi and Rathie* [3]. *A few special cases have also been given. The result provided in this note is simple, interesting, easily established, and may be useful.*

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1. Introduction

The generalized hypergeometric function, denoted by ${}_pF_q(z)$, characterized by p numerator parameters and q denominators parameters, is defined by [7, 8]

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_p \end{matrix} \quad ; \quad z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_p)_n}{(b_1)_n (b_2)_n ... (b_p)_n} \frac{z^n}{n!} \tag{1} $$

where $(c)_n$ represent the Pochhammer symbol, which is defined as follows

$$ (c)_n = \begin{cases} c(c+1)\ldots(c+n+1); n \in \mathbb{N}, \\ 1 \qquad\qquad\qquad\qquad ; n = 0. \end{cases} $$

Express in terms of gamma function, we obtain

$$ (c)_n = \frac{\Gamma(c+n)}{\Gamma(c)}. $$

Here, p and q are non-negative integers and the parameters a_j $(1 \le j \le p)$ and b_j $(1 \le j \le q)$ can take arbitrary complex values with zero or negative integer values of b_j excluded [7]. The generalized hypergeometric function ${}_pF_q(z)$ converges for all

$$z < \infty \ (p \leq q), z < 1 \ (p = q + 1) \text{ and } |z| = 1 \ (p = q + 1 \text{ and } Res(s) > 0,$$

where s is the parametric excess defined by

$$s = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j$$

The generalized hypergeometric function plays a significant role across diverse disciplines due to its broad applicability. Its uses span fields such as mathematics, theoretical physics, engineering, and statistical sciences. For further information on this function, we refer the readers to the references [1,8].

Additionally, in the study of generalized hypergeometric functions, classical summation theorems hold significant importance. These results include Gauss's summation theorem and Gauss's second theorem. Additionally, they encompass Bailey's and Kummer's theorems for the series ${}_2F_1$. Furthermore, Watson's, Dixon's, Whipple's, and Saalschütz's theorems apply to the series ${}_3F_2$, among others. In this paper, we specifically highlight the classical Watson's theorem [1], stated as follows:

$${}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{a+b+1}{2}, \quad 2c \end{matrix} ; 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)}$$

$$= \text{s (say)} \tag{2}$$

Provided $Re(2c - a - b) > -1$.

In 1992, Lavoie, et al. [5] classical Watson's theorem (2) , then we obtain the explicit expression of

$${}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{a+b+i+1}{2}, \quad 2c+j \end{matrix} ; 1\right] \tag{3}$$

for $i, j = 0 \pm 1, \pm 2$.

Clearly for $i = j = 0$, the expression (3) reduces to the Watson's theorem (2). In the same paper, they have examined numerous fascinating special cases and limiting cases of their main results. In 2010, Kim, et al. [4] established the two interesting extensions of the classical Watson's summation theorem (2). In 2016, Choi and Rathie [3] established thirteen results including the two results obtained earlier by Kim, et al. [5] with the help of the contiguous results obtained earlier by Lavoie, et al. [5].

Here, we shall mention one result out of the thirteen results which will be essential for our current study.

$${}_4F_3\left[\begin{matrix} a, \quad b, \quad c \quad d+1 \\ \frac{a+b+1}{2}, \quad 2c+1, \quad d \end{matrix} ; 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)}$$

$$+\left(\frac{2c}{d}-1\right)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma\left(c-\frac{a}{2}+1\right)\Gamma\left(c-\frac{b}{2}+1\right)}$$

$$= s_1 \text{ (let)} \tag{4}$$

provided $d \notin \mathbb{Z}_0^-$ and $Re(2c - a - b) > -1$.

It is interesting to mention here that if we take $d = 2c$ in (4), we at once get the classical Watson's theorem (2). Thus, the result (4) is regarded as an extension of the classical Watson's summation theorem (2).

In 2024, Basnet, et al.[2] established a result on an integral involving product of two generalized hypergeometric functions. In this research, we have evaluated an interesting integral involving generalized hypergeometric function. This is achieved by employing the summation formula (4) due to the Choi and Rathie in the following integral due to the MacRobert [6]

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}d\theta = e^{i\frac{\pi}{2}(\alpha)}\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{5}$$

Provided $R(\alpha) > 0$ and $R(\beta) > 0$.

Several special cases have also been presented. The result in this paper is straightforward, intriguing, easy to derive, and potentially useful.

2. Main Result

In this section, we shall evaluate the following integral asserted in the theorem.

Theorem 2.1

$$\int_0^{\frac{\pi}{2}} e^{i(2c+1)\theta}(\sin\theta)^{c}(\cos\theta)^{c-1}\times {}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ \frac{a+b+1}{2}, & d & \end{matrix} ; \; e^{i\theta}\cos\theta\right]d\theta = e^{i\frac{\pi}{2}(c+1)}\,\frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)}\,s_1 \tag{6}$$

provided $d \notin \mathbb{Z}_0^-$, $Re(c) > 0$, $Re(2c - a - b) > -2$ and s_1 is the same as given in (4).

Proof:

To derive result (6) stated in Theorem 6, we follow the approach outlined below. Let II represent the left-hand side of equation (6), we have

$$I = \int_0^{\frac{\pi}{2}} e^{i(2c+1)\theta} (\sin\theta)^c (\cos\theta)^{c-1} \times {}_3F_2\left[\begin{matrix} a, \quad b, \quad d+1 \\ \frac{a+b+1}{2}, \quad d \end{matrix} \; ; \; e^{i\theta}\cos\theta\right] d\theta$$

Expressing the generalized hypergeometric function ${}_3F_2$ as a series, we have

$$I = \int_0^{\frac{\pi}{2}} e^{i(2c+1)\theta} (\sin\theta)^c (\cos\theta)^{c-1} \times \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (d+1)_n}{\left(\frac{a+b+1}{2}\right)_n (d)_n} . \frac{e^{in\theta}(\cos\theta)^n}{n!} d\theta$$

Rearranging the order of integration and summation is justified by the uniform convergence of the series and the absolute convergence of the integral, we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (d+1)_n}{\left(\frac{a+b+1}{2}\right)_n (d)_n} . \frac{1}{n!} \times \int_0^{\frac{\pi}{2}} e^{i(2c+n+1)\theta} (\sin\theta)^c (\cos\theta)^{c+n-1} d\theta$$

Evaluating the integral with the help of the known integral (5) due to MacRobert, we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (d+1)_n}{\left(\frac{a+b+1}{2}\right)_n (d)_n} \times e^{i\frac{\pi}{2}(c+1)} \frac{\Gamma(c+1)\Gamma(c+n)}{\Gamma(2c+n+1)}$$

After a little simplification, we have

$$I = e^{i\frac{\pi}{2}(c+1)} \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n (d+1)_n}{\left(\frac{a+b+1}{2}\right)_n (2c+1)_n (d)_n n!}$$

Summing up the series, we have

$$I = e^{i\frac{\pi}{2}(c+1)} \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)} {}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad d+1 \\ \frac{a+b+1}{2} \quad 2c+1 \quad d \end{matrix} \; ; \; 1\right]$$

We can now see that the ${}_4F_3$ can be evaluated using the known result (4), leading us to the right-hand side of equation (6). This concludes the proof of result (6) stated in the theorem.

We conclude this section by noting that in the following section, we will present several interesting special cases of our main results.

3. Special Cases

In this section, we shall mention a few special cases of our main findings.

Corollary 3.1

In (6), If we take $d = 2c$, then we get the following interesting results:

$$\int_0^{\frac{\pi}{2}} e^{i(2c+1)\theta}(\sin\theta)^c(\cos\theta)^{c-1} \times {}_3F_2\left[\begin{matrix} a, & b, & 2c+1 \\ \frac{a+b+1}{2}, & 2c & \end{matrix} ; \; e^{i\theta}\cos\theta\right]d\theta = e^{i\frac{\pi}{2}(c+1)} \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)} s \tag{7}$$

provided $R(c) > 0$, $Re(2c - a - b) > -1$ and s is the same as given in (2).

Corollary 3.2

In (6), first we let $b = -2n$ and replace a by $a + 2n$ and we let $b = -2n - 1$ and replace a by $a + 2n + 1$, where n is zero or a positive integer. In both cases, one of the two terms appearing on the right-hand side of the resulting integral (6) vanishes and we get the following interesting results:

$$\int_0^{\frac{\pi}{2}} e^{i(2c+1)\theta}(\sin\theta)^c(\cos\theta)^{c-1} \times {}_3F_2\left[\begin{matrix} -2n, & a+2n, & d+1 \\ \frac{a+1}{2}, & d & \end{matrix} ; \; e^{i\theta}\cos\theta\right]d\theta$$

$$= e^{i\frac{\pi}{2}(c+1)} \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{a}{2} - c + \frac{1}{2}\right)_n}{\left(\frac{a}{2} + \frac{1}{2}\right)_n \left(c + \frac{1}{2}\right)_n} \tag{8}$$

The beauty of this result is that the right-hand side of (8) is independent of d.

And

$$\int_0^{\frac{\pi}{2}} e^{i(2c+1)\theta}(\sin\theta)^c(\cos\theta)^{c-1} \times {}_3F_2\left[\begin{matrix} -2n-1, & a+2n+1, & d+1 \\ \frac{a+1}{2}, & d & \end{matrix} ; \; e^{i\theta}\cos\theta\right]d\theta$$

$$= \frac{e^{i\frac{\pi}{2}(c+1)}}{\Gamma(2c+1)} \left(1 - \frac{2c}{d}\right) \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(2c+1)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{a}{2} - c + \frac{1}{2}\right)_n}{\left(\frac{a}{2} + \frac{1}{2}\right)_n \left(c + \frac{3}{2}\right)_n} \tag{9}$$

Corollary 3.3

In (9), if we take $d = 2c$, we get the following elegant result

$$\int_0^{\frac{\pi}{2}} e^{i(2c+1)\theta}(\sin\theta)^c(\cos\theta)^{c-1} \times {}_3F_2\left[\begin{matrix} -2n-1, & a+2n+1, & 2c+1 \\ & \frac{a+1}{2}, & 2c \end{matrix} ; \; e^{i\theta}\cos\theta\right]d\theta = 0 \tag{10}$$

Similarly, other results can be obtained.

4. Conclusion

In this paper, we explored an integral representation involving the generalized hypergeometric function, shedding light on its utility and significance in mathematical analysis. We conclude this paper by remarking that in 2016, Choi and Rathie [3] established thirteen results for the series ${}_4F_3$ [1]. Other results similar to what we have obtained in this paper are under investigation and will form a part of the subsequent paper in this direction. Their contributions provide a foundational basis for examining related functions and their integrals in greater depth.

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