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Study of Common Fixed Point Theorems for Interpolative Contraction in Metric Space

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Abstract: *This paper aims to establish common fixed point results that can be addressed using an interpolative contraction condition proposed by Karapinar et al. [6] and Karapinar et al. [7] within a complete metric space. We have developed both the H-R type contraction and R-R-C–Rus–type contraction in the context of metric spaces, and we have proved related interpolation common fixed point theorem. Furthermore, we provide examples to illustrate the significance of our findings.*

Keywords: Metric space, Common fixed point, Interpolation, Hardy–Rogers contraction, Reich–Rus–Ciric contraction.

1. Introduction and preliminaries

The Banach contraction mapping principle (BCP) was developed by the Polish mathematician Stefan Banach [1] in 1922 and focuses on contraction mappings with unique fixed point results on metric spaces. Due to its importance, several authors have extended and generalized this principle. Researchers have been inspired to explore alternative forms of contraction based on Banach's FPT. A notable early response came from Kannan [2, 3] in 1968 and 1969, who introduced a new type of contraction mapping that does not require continuity.

Definition 1.1(see [2, 3]): A mapping $\mathcal{G}: \mathcal{W} \to \mathcal{W}$ is called Kannan type contraction, if there exists $\mathscr{k} \in [0, \frac{1}{2})$ such that

$$d(\mathcal{G}\mathscr{p}, \mathcal{G}\mathscr{q}) \leq \mathscr{k}[d(\mathscr{p}, \mathcal{G}\mathscr{p}) + d(\mathscr{q}, \mathcal{G}\mathscr{q}) \text{ for all } \mathscr{p}, \mathscr{q} \in \mathcal{W}. \quad (1)$$

Kannan [2, 3] established the following theorem:

Theorem 1.2 (see [2, 3]): If $(\mathcal{W}, d)$ is a complete metric space, then every Kannan contraction on $\mathcal{W}$ has a unique fixed point.

A notable recent generalization of the Kannan theorem was published by Karapinar E. [4] in 2018. He presented a new type of contraction obtained from interpolation of the Kannan contraction as follows:

Definition 1.3 (see [4]): A mapping $\mathcal{G}: \mathcal{W} \to \mathcal{W}$ is called interpolative Kannan type contraction on metric space $(\mathcal{W}, d)$, if there exists $\mathscr{k} \in [0, \frac{1}{2})$ such that

$$d(\mathcal{G}\mathscr{p}, \mathcal{G}\mathscr{q}) \leq \mathscr{k}[d(\mathscr{p}, \mathcal{G}\mathscr{p})]^{\alpha}[d(\mathscr{q}, \mathcal{G}\mathscr{q})]^{1-\alpha} \text{ for all } \mathscr{p}, \mathscr{q} \in \mathcal{W} \text{ with } \mathscr{p} \neq \mathcal{G}\mathscr{p} \text{ and } \mathscr{q} \neq \mathcal{G}\mathscr{q}. \quad (2)$$

Karapinar, E. [4] established the following theorem:

Theorem 1.4 (see [4]): If $(\mathcal{W}, d)$ is a complete metric space, then every interpolative Kannan type contraction on $\mathcal{W}$ has unique fixed point.

However, the theorem 1.4 has been generalized by Noorwali [15] who obtained a common fixed point for two maps as follows:

Theorem 1.5: Suppose $(\mathcal{W}, d)$ be a metric space and $\mathcal{G}, \mathscr{H}: \mathcal{W} \to \mathcal{W}$ be self mappings. Take over that $\exists\ \sigma \in [0,1)$ and $\alpha \in (0,1)$ with $\alpha + \beta + \gamma < 1,$ satisfied the condition

$$d(\mathcal{G}a, \mathcal{H}b) \le \sigma[d(a, \mathcal{G}a)]^{\alpha}.[b, \mathscr{H}b)]^{1-\alpha} \tag{3}$$

for all $a, b \in \mathcal{W}$ such that $\mathcal{G}a \ne a$ whenever $\mathscr{H}b \ne b$. Then $\mathcal{G}$ and $\mathscr{H}$ have a unique common fixed point.

In sequel, Karapinar, et al. [6] introduced the notion of interpolative Hardy- Rogers's type contraction by using the well known contraction of Hardy and Rogers [5].

Definition1.5 (see [6]): A self-mapping $\mathcal{G}: \mathcal{W} \to \mathcal{W}$ is called an interpolative H-R type contraction metric space $(\mathcal{W}, d)$, if $\exists\ k \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ where $\alpha + \beta + \gamma < 1,$ such that

$$d(\mathcal{G}p, \mathcal{G}q) \le k[d(p, q)]^{\beta}.[d(p, \mathcal{G}p)]^{\alpha}.[q, \mathcal{G}q)]^{\gamma}\ .\left[\frac{1}{2}\ (d((p, \mathcal{G}q) + d(q, \mathcal{G}p)\right]^{1-\alpha-\beta-\gamma} \tag{4}$$

for all $p, q \in \mathcal{W}\backslash Fix\ (\mathcal{G})$.

Karpinar et al. [6] established the following theorem:
Theorem 1.6 (see [6]): Let $(\mathcal{W}, d)$ be a complete metric space and $\mathcal{G}$ be an interpolative Hardy-Rogers type contraction. In that case, $\mathcal{G}$ is fixed point of $\mathcal{W}$.

Very recently, Karapinar et al. [7] introduced the notion of Interpolative Riech-Rus- Ciric type contraction by using the well known contraction of Riech-Rus-Ciric [8-14].

Definition 1.7 (see [7]): Let $(\mathcal{W}, d)$ be a metric space. Then a self mapping $\mathcal{G}: \mathcal{W} \to \mathcal{W}$ is called interpolative Riech-Rus-Ciric type contraction if there exists $k \in [0; 1)$, $\alpha_1, \alpha_2 \in [0,1)$ with $\alpha_1 + \alpha_2 < 1$ such that

$$d(\mathcal{G}p, \mathcal{G}q) \le k[d(p, q)]^{\alpha}.[d(p, \mathcal{G}p)]^{\alpha_2}.[d(q, \mathcal{G}q)]^{1-\alpha_2-\alpha_3} \quad \text{for all } p, q \in \mathcal{W}. \tag{5}$$

Karapinar et al. [7] proved the following theorem

Theorem 1.8 (see [7]): Let $(\mathcal{W}, d)$ be a complete metric space and $\mathcal{G}$ be an interpolative Reich-Rush-Ciric type contraction. In that case, $\mathcal{G}$ is fixed point of $\mathcal{W}$.

Very Recently, Zahid et al. [16] introduced Reich-Rus-Ciric type contraction in rectangular $\mathcal{M}$-metric spaces and obtained fixed point theorems in these spaces. In the same year, Edraoui M. et *al*. [17] presented some fixed point results of Hardy-Rogers-type for cyclic mappings on complete metric space. Later, many authors continued their investigations and more results were obtained, such as, [18-27].

2. Main Result

In this section, we extend and generalize the result of Karapinar et al. [6] of theorem 1.6 and Karapinar et al. [7] of theorem 1.8 to obtain common fixed point results. First, we extend the theorem 1.6 as follows:

Theorem 2.1: Suppose $\mathcal{G}, \mathscr{H}: \mathcal{W} \to \mathcal{W}$ be any two self-interpolative Hardy- Rogers type contraction on metric space $(\mathcal{W}, d)$ and if $\exists r \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ while $\alpha + \beta + \gamma < 1$, satisfied the condition by definition 1.5

$$d(\mathcal{G}a, \mathscr{H}b) \leq r[d(a, b)]^{\beta}.[d(a, \mathcal{G}a)]^{\alpha}.[b, \mathscr{H}b)]^{\gamma} . \left[\frac{1}{2}\left(d((a, \mathscr{H}b) + d(b, \mathcal{G}a)\right)\right]^{1-\alpha-\beta-\gamma} \quad (6)$$

for all $a, b \in \mathcal{W}$ such that $\mathcal{G}a \neq a$ whenever $\mathscr{H}b \neq b$. Then $\mathcal{G}$ and $\mathscr{H}$ have a unique common fixed point.

Proof:

Consider $a_0 \in \mathcal{W}$ with sequence $\{a_{2\eta}\}$ such as

$$a_{2\eta+1} = \mathcal{G}a_{2\eta} \text{ and } a_{2\eta+2} = \mathscr{H}a_{2\eta+1}, \forall\, \eta \in \{0,1,2, \dots\}.$$

Even if $\exists\, \eta \in \{0,1,2,\dots\}$ and $a_{2\eta} = a_{2\eta+1} = a_{2\eta+2}$, also, $a_{2\eta}$ is a common fixed point of $\mathcal{G}$ and $\mathscr{H}$; so let us suppose that there does not exist three consecutive identical terms in the sequence $\{a_{2\eta}\}$ and that $a_0 \neq a_1$.

By substituting the values $a = a_{2\eta}$ and $b = a_{2\eta+1}$ in (6), we get

$$\begin{aligned} d(a_{2\eta+1}, a_{2\eta+2}) &= d(\mathcal{G}a_{2\eta}, \mathscr{H}_{2\eta+1}) \\ &\leq r[d(a_{2\eta}, a_{2\eta+1})]^{\beta}.[d(a_{2\eta}, \mathcal{G}a_{2\eta})]^{\alpha}.\left[d(a_{2\eta+1}, \mathscr{H}a_{2\eta+1})\right]^{\gamma}. \\ &\quad \left[\frac{1}{2}\left(d(a_{2\eta}, \mathscr{H}a_{2\eta+1}) + d(a_{2\eta+1}, \mathcal{G}a_{2\eta})\right)\right]^{1-\alpha-\beta-\gamma} \\ &\leq r[d(a_{2\eta}, a_{2\eta+1})]^{\beta}.[d(a_{2\eta}, a_{2\eta+1})]^{\alpha}.\left[d(a_{2\eta+1}, a_{2\eta+2})\right]^{\gamma}. \\ &\quad \left[\frac{1}{2}\left(d(a_{2\eta}, a_{2\eta+2}) + d(a_{2\eta+1}, a_{2\eta+1})\right)\right]^{1-\alpha-\beta-\gamma} \\ &\leq r[d(a_{2\eta}, a_{2\eta+1})]^{\beta}.[d(a_{2\eta}, a_{2\eta+1})]^{\alpha}.\left[d(a_{2\eta+1}, a_{2\eta+2})\right].^{\gamma} \\ &\quad \left[\frac{1}{2}\left(d(a_{2\eta}, a_{2\eta+1}) + d(a_{2\eta+1}, a_{2\eta+2})\right)\right]^{1-\alpha-\beta-\gamma}. \end{aligned} \quad (7)$$

Suppose that $d(a_{2\eta}, a_{2\eta+1}) < d(a_{2\eta+1}, a_{2\eta+2})$, for $r \geq 1$.

Thus

$$\left[\frac{1}{2}\left(d(a_{2\eta}, a_{2\eta+1}) + d(a_{2\eta+1}, a_{2\eta+2})\right)\right] \leq d(a_{2\eta+1}, a_{2\eta+2}).$$

Consequently, the inequality (7), yields that

$$[d(a_{2\eta+1}, a_{2\eta+2})]^{\beta+\gamma} \leq r[d(a_{2\eta}, a_{2\eta+1})]^{\beta+\gamma} . \quad (8)$$

So, we conclude that $d(a_{2\eta}, a_{2\eta+1}) \geq d(a_{2\eta+1}, a_{2\eta+2})$, which is conflict. Accordingly

$$d(a_{2\eta+1}, a_{2\eta+2}) \leq d(a_{2\eta}, a_{2\eta+1})\ \forall r \geq 1.$$

Where, $d(a_{2\eta}, a_{2\eta+1})$ is a positive term and non increasing sequence .Consequently a non negative constant ℓ such as $\lim_{\eta\to\infty} d(a_{2\eta}, a_{2\eta+1}) = \ell$.

We have

$$\left[\frac{1}{2}\, d\big(a_{2\eta}, a_{2\eta+1}\big) + d(a_{2\eta+1}, a_{2\eta+2})\right] \le d(a_{2\eta}, a_{2\eta+1}), \text{ for all } \eta \ge 1.$$

By the inequality (7), we get

$$[d(a_{2\eta+1}, a_{2\eta+2})]^{1-\alpha} \le r[d\big(a_{2\eta}, a_{2\eta+1}\big)]^{1-\alpha} \text{ , for all } \eta \ge 1. \tag{9}$$

We deduce that

$$d\big(a_{2\eta+1}, a_{2\eta+2}\big) \le r d\big(a_{2\eta}, a_{2\eta+1}\big) \le \cdots \le r^{2\eta} d(a_0, a_1) \tag{10}$$

Now using (10), and claim that $\{a_{2\eta}\}$ having Cauchy sequence. Let $\eta, \ell \in \{0,1,2, \dots\}$

$$\begin{aligned} d\big(a_{2\eta}, a_{2\eta+2l}\big) &\le d(a_{2\eta}, a_{2\eta+1}) + d\big(a_{2\eta+1}, a_{2\eta+2}\big) + \cdots . + d(a_{2\eta+2l-1}, a_{2\eta+2\ell}) \\ &\le [r^{2\eta} + r^{2\eta+1} + \cdots . + r^{2\eta+2\ell-1}] d(a_0, a_1) \\ &\le \frac{r^{2\eta}}{1-r}\, d(a_0, a_1). \end{aligned} \tag{11}$$

Letting $\eta \to \infty$, we deduce that $\{a_{2\eta}\}$ is a Cauchy sequence in the complete metric space $(\mathcal{W}, d)$ and $\exists\, u \in \mathcal{W}$ such that

$$\lim_{\eta\to\infty} a_{2\eta} = u.$$

Now, prove that u is a common fixed point of $\mathcal{G}$ and $\mathscr{H}$. Now consider,

$$\begin{aligned} d\big(\mathcal{G}u, a_{2\eta+2}\big) &= d(\mathcal{G}u, \mathscr{H}a_{2\eta+1}) \\ &\le r[d(u, a_{2\eta+1)}]^{\beta}. [d(u, \mathcal{G}u)]^{\alpha}. [d\big(a_{2\eta+1}, \mathscr{H}a_{2\eta+1}\big)]^{\gamma}. \\ &\quad [\frac{1}{2}\, (d\big(a_{2\eta+1,}\mathcal{G}u\big) + d\big(\, a_{2\eta+1}, a_{2\eta+2}\big))]^{1-\alpha-\beta-\gamma}. \end{aligned}$$

Letting $\eta \to \infty$, we get $d(\mathcal{G}u, u) = 0 \Rightarrow \mathcal{G}u = u$.

Similarly, we can prove that $\mathscr{H}u = u$. Since $\mathcal{G}u = u = \mathscr{H}u$. Hence, u is a common fixed point of $\mathcal{G}$ and $\mathscr{H}$.

Now, we claim that u is the unique common fixed point theorem of $\mathcal{G}$ and $\mathscr{H}$. Suppose that v is another common fixed point of $\mathcal{G}$ and $\mathscr{H}$, then

$$\begin{aligned} d(u, v) &= d(\mathcal{G}u, \mathscr{H}v) \\ &\le r[d(u, v)]^{\beta}. [d(u, \mathcal{G}u)]^{\alpha}. [d(v, \mathscr{H}v)]^{\gamma}. \left[\frac{1}{2}(d(u, \mathscr{H}v) + d(v, \mathcal{G}u))\right]^{1-\alpha-\beta-\gamma} \\ &= 0. \end{aligned}$$

Hence $u = v$. Thus, u is the unique common fixed point theorem of $\mathcal{G}$ and $\mathscr{H}$.

Example 2.2: Consider $\mathcal{W} = \{0,1,2,3,5\}$ endowed with $d(a, b) = |a - b|$.
Now let

$$\alpha = \frac{1}{3}, \quad \beta = \frac{1}{2} \quad \text{and } \gamma = \frac{1}{7}.$$

It is obvious that

$$d(\mathcal{G}a,\mathcal{G}b) \le r[d(a,b)]^{\beta}.[d(a,\mathcal{G}a)]^{\alpha}.[b,\mathcal{H}b)]^{\gamma}.\ \left[\frac{1}{2}\ (d((a,\mathcal{H}b)+d(b,\mathcal{G}a)\right]^{1-\alpha-\beta-\gamma}$$

for all $a,b \in \mathcal{W}$ such that $\mathcal{G}a \neq a$ whenever $\mathcal{H}b \neq b$, that is (7) hold.

All the hypotheses of Theorem 2.1 are satisfied, and so 0 and 1 are common fixed points.

Next, we will extend and generalize the Theorem 1.8 as follows:

Theorem 2.3: Suppose $\mathcal{G},\mathcal{H}: \mathcal{W} \to \mathcal{W}$ be any two self-interpolative R-R-C type contraction metric space $(\mathcal{W},d)$ and satisfied the condition by definition 1.7, if $\exists\ t \in [0,1)$ with $\alpha_1,\alpha_2, \in (0,1)$ where $\alpha_1+\alpha_2<1$, such that

$$d(\mathcal{G}a,\mathcal{H}b) \le t[d(a,b)]^{\alpha_1}.[d(a,\mathcal{G}a)]^{\alpha_2}.[b,\mathcal{H}b)]^{1-\alpha_1-\alpha_2} \tag{12}$$

for all $a,b \in \mathcal{W}$ such that $\mathcal{G}a \neq a$ whenever $\mathcal{H}b \neq b$. Then $\mathcal{G}$ and $\mathcal{H}$ have a unique common fixed point.

Proof:

Consider $a_0 \in \mathcal{W}$ with sequence $\{a_{2\eta}\}$ such as

$$a_{2\eta+1}=\mathcal{G}a_{2\eta} \text{ and } a_{2\eta+2}=\mathcal{H}a_{2\eta+1},\ \forall\ \eta \in \{0,1,2,\dots\}.$$

Since $\eta \in \{0,1,2,\dots\}$ and $a_{2\eta}=a_{2\eta+1}=a_{2\eta+2}$, hence $a_{2\eta}$ is a common fixed point of $\mathcal{G}$ and $\mathcal{H}$; so let us suppose that there does not exist $a_0 \neq a_1$.

By substituting the values $a=a_{2\eta}$ and $b=a_{2\eta+1}$ in (12), we get

$$\begin{aligned} d\left(a_{2\eta+1},a_{2\eta+1}\right) &= d(\mathcal{G}a_{2\eta},\mathcal{H}a_{2\eta+1}) \\ &\le t\left[d\left(a_{2\eta},a_{2\eta+1}\right)\right]^{\alpha_1}.\left[d\left(a_{2\eta},\mathcal{G}a_{2\eta}\right)\right]^{\alpha_2}.\left[d\left(a_{2\eta+1,}\mathcal{H}a_{2\eta+1}\right)\right].^{1-\alpha_1-\alpha_2} \\ &\le t\left[d\left(a_{2\eta},a_{2\eta+1}\right)\right]^{\alpha_1}.\left[d\left(a_{2\eta},a_{2\eta+1}\right)\right]^{\alpha_2}\left[d\left(a_{2\eta+1,}a_{2\eta+2}\right)\right].^{1-\alpha_1-\alpha_2} \end{aligned}$$

This implies that

$$[d(a_{2\eta+1},a_{2\eta+2})]^{\alpha_1+\alpha_2} \le t[d\left(a_{2\eta},a_{2\eta+1}\right)]^{\alpha_1+\alpha_2}$$

or

$$d\left(a_{2\eta+1},a_{2\eta+2}\right) \le t d\left(a_{2\eta},a_{2\eta+1}\right)$$

Hence

$$d\left(a_{2\eta+1},a_{2\eta+2}\right) \le t d\left(a_{2\eta},a_{2\eta+1}\right) \le \cdots \dots \le t^{2\eta} d(a_0,a_1)\dots \tag{13}$$

Similarly, we can show that

$$[d\left(a_{2\eta+1},a_{2\eta}\right)]^{1-\alpha_2} \le t[d\left(a_{2\eta},a_{2\eta-1}\right)]^{1-\alpha_2}$$

or

$$\left[d\left(a_{2\eta+1},a_{2\eta}\right)\right] \le t[d\left(a_{2\eta},a_{2\eta-1}\right)].$$

Hence $\quad d\left(a_{2\eta+1},a_{2\eta}\right) \le t d\left(a_{2\eta-1},a_{2\eta}\right) \le \cdots \le t^{2\eta} d(a_0,a_1)\dots \quad (14)$

From (13) and (14), we can deduce that

$$d\left(a_{2\eta},a_{2\eta+1}\right) \le\ t^{2\eta} d(a_0,a_1) \tag{15}$$

Now using (15), and show that $\{a_{2\eta}\}$ is a Cauchy sequence. Let $\eta, \ell \in \{0,1,2,\ldots\}$, we have

$$d(a_{2\eta}, a_{2\eta+2l}) \le d(a_{2\eta}, a_{2\eta+1}) + d(a_{2\eta+1}, a_{2\eta+2}) + \cdots . + d(a_{2\eta+2l-1}, a_{2\eta+2\ell})$$
$$\le [t^{2\eta} + t^{2\eta+1} + \cdots . + t^{2\eta+2\ell-1}] d(a_0, a_1)$$
$$\le \frac{t^{2\eta}}{1-t} d(a_0, a_1). \qquad (16)$$

Letting $\eta, l \to \infty$, $i.e \lim_{\eta,\ell\to\infty} d(a_{2\eta}, a_{2\eta+2l}) = 0$.

Therefore, $\{a_{2\eta}\}$ is a Cauchy sequence in $(\mathcal{W}, d)$ and $\exists\ \tau \in \mathcal{W}$ such that

$$\lim_{\eta\to\infty} a_{2\eta} = \tau.$$

Now consider,

$$d(\mathcal{G}\tau, a_{2\eta+2}) = d(\mathcal{G}\tau, \mathcal{H}a_{2\eta+1})$$
$$\le t[d(\tau, a_{2\eta+1)}]^{\alpha_1}. [d(\tau, \mathcal{G}\tau)]^{\alpha_2}. [d(a_{2\eta+1}, \mathcal{H}a_{2\eta+1})]^{1-\alpha_1-\alpha_2}.$$

Letting $\eta \to \infty$, we get $d(\mathcal{G}\tau, \tau) = 0 \Rightarrow \mathcal{G}\tau = \tau$.

Similarly, $d(a_{2\eta+1}, \mathcal{H}\tau) = d(\mathcal{G}a_{2\eta}, \mathcal{H}\tau)$

$$\le t[d(a_{2\eta}, \mathcal{H}\tau]^{\alpha_1}. [d(a_{2\eta}, \mathcal{H}a_{2\eta})]^{\alpha_2}. [d(\tau, \mathcal{H}\tau)]^{1-\alpha_1-\alpha_2}.$$

Letting $\eta \to \infty$, we get $d(\tau, \mathcal{H}\tau) = 0$, hence $\mathcal{H}\tau = \tau$.

Since $\mathcal{G}\tau = \tau = \mathcal{H}\tau$. So, τ is common fixed point of $\mathcal{G}$ and $\mathcal{H}$.

Next, we show that, common fixed point τ is the unique of $\mathcal{G}$ and $\mathcal{H}$. Suppose that, ς is another common fixed point of $\mathcal{G}$ and $\mathcal{H}$, then

$$d(\tau, \varsigma) = d(\mathcal{G}\tau, \mathcal{H}\varsigma)$$
$$\le t[d(\tau, \varsigma)]^{\alpha_1}. [d(\tau, \mathcal{G}\tau)]^{\alpha}. [d(\varsigma, \mathcal{H}\varsigma)]^{1-\alpha_1-\alpha_2}.$$
$$= 0.$$

Hence $\tau = \varsigma$. As follows, common fixed point τ is a unique of $\mathcal{G}$ and $\mathcal{H}$.
If we take $\alpha_1 = 0$ in Theorem 2.3, then we get the following Corollary

Corollary 2.4: Suppose $\mathcal{G}, \mathcal{H}: \mathcal{W} \to \mathcal{W}$ be any two self-interpolative R-R-C type contraction metric space$(\mathcal{W}, d)$ and satisfied the condition by definition 1.7, if $\exists\ t \in [0,1)$ with $\alpha_1, \alpha_2, \in (0,1)$ where $\alpha_1 + \alpha_2 < 1$, such that

$$d(\mathcal{G}a, \mathcal{H}b) \le t[d(a, \mathcal{G}a)]^{\alpha_2}.[b, \mathcal{H}b)]^{1-\alpha_2} \qquad (17)$$

for all $a, b \in \mathcal{W}$ such that $\mathcal{G}a \ne a$ whenever $\mathcal{H}b \ne b$. Accordingly, $\mathcal{G}$ and $\mathcal{H}$ have a unique common fixed point.

Example 2.5: Consider $\mathcal{W} = \{1,2,3,4\}$.and $d(a, b) = \max\{a, b\} + |a - b|$ that is

$d(a,b)$	1	2	3	4
1	1	3	5	7
2	3	2	4	6
3	5	4	3	5
4	7	6	5	4

Now, a self mappings of $\mathcal{G}$ and $\mathscr{H}$ on $\mathcal{W}$ as

$$\mathcal{G} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix} \text{ and } \mathscr{H} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix} \text{ as shown in Figure 1.}$$

Choose $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{3}$ and $t = \frac{7}{10}$.

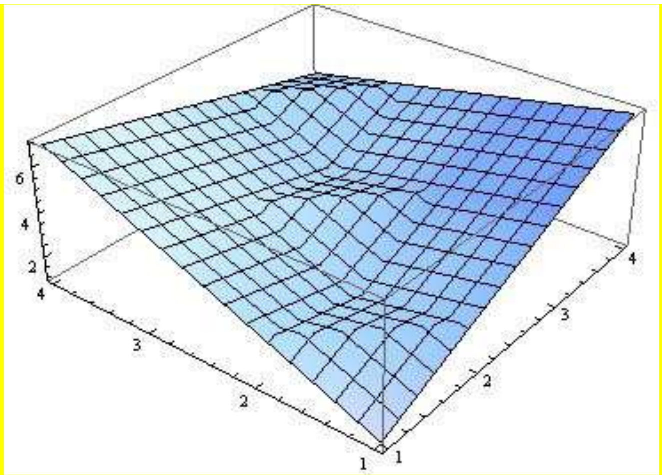

Figure1. 1 is the common fixed point of $\mathcal{G}$ and $\mathscr{H}$.

Case 1.Presume $(a, b) = (3,4)$, we have

$$d(\mathcal{G}a, \mathscr{H}b) \leq t[d(a, b)]^{\alpha_1}.[d(a, \mathcal{G}a)]^{\alpha_2}.[b, \mathscr{H}b)]^{1-\alpha_1-\alpha_2}$$

$$d(\mathcal{G}3, \mathscr{H}4) = 1$$

$$\leq t[d(3,4)]^{1/3}.[d(3, \mathcal{G}3)]^{1/2}.[4, \mathscr{H}4)]^{1/6}.$$

Case 2. Let $(a, b) = (1,4)$, $d(\mathcal{G}1, \mathscr{H}4) = 1$

$$\leq t[d(1,4)]^{1/3}.[d(1, \mathcal{G}1)]^{1/2}.[4, \mathscr{H}4)]^{1/6}.$$

Therefore, 1 is the common fixed point of $\mathcal{G}$ and $\mathscr{H}$ in the setting of the interpolative R-R-C ´ type contraction.

3. Conclusion

This article examines a significant contraction that demonstrates a unique common fixed point for both the Interpolative H-R contraction and the Interpolative R-R-C type contraction mappings within a metric space. Our main results build upon and extend the earlier research conducted by Karapinar et al. [6] and Karapinar et al. [7]. Furthermore, we present a relevant example to support these findings.

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