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# Maximal Monotone Operators on Product Spaces

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**Abstract:** *Let X and Y be real Banach spaces with duals* $X^*$ *and* $Y^*$, *respectively. Let* $T: X \to 2^{X^*}$ *and* $S: Y \to 2^{Y^*}$ *be maximal monotone operators. We discuss some known results on the maximality of the sum* $T + S$ *in the case* $X = Y$ *and an important characterization of maximal monotone operators in general Banach spaces. As our main result, we prove that the operator* $H: X \times Y \to 2^{X^* \times Y^*}$ *defined by* $H(x, y) = \{(x^*, y^*): x^* \in Tx, y^* \in Sy\}, x \in X, y \in Y\}$, *is maximal monotone.*

**Keywords:** Maximal monotone operators, Local boundedness, Upper semi-continuity

## 1. Introduction

In what follows, $X$ denotes a real Banach space with norm $\|.\|$ and $X^*$ denotes the dual space of $X$. The evaluation of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x^*, x\rangle$. Given a subset $B$ of $X$, we denote its strong closure and strong interior by cl $B$ and int $B$, respectively. The effective domain of a set-valued (multivalued) operator $T: X \to 2^{X^*}$ is defined as

$$D(T) = \{x \in X : Tx \neq \phi\}$$

and the graph of $T$ is defined as $G(T) = \{(x, x^*) \in X \times X^*: x^* \in Tx\}$.

A multivalued operator $T: X \to 2^{X^*}$ is said to be monotone if for any $x, y \in D(T)$

$$\langle x^* - y^*, x - y\rangle \geq 0 \text{ for all } x^* \in Tx \text{ , } y^* \in Ty.$$

We say that $T$ is maximal monotone if the graph of $T$ has no proper monotone extension when $X \times X^*$ is partially ordered by the set inclusion. Equivalently, $T$ is maximal monotone if for each $(x, x^*) \in X \times X^*$ such that

$$\langle x^* - y^*, x - y\rangle \geq 0 \text{ for all } (y, y^*) \in G(T), \text{we have } (x, x^*) \in G(T).$$

The theory of maximal monotone operators on real reflexive Banach spaces has been studied quite extensively. For example, sufficient conditions on the operators on reflexive Banach spaces have been studied for the sum of two maximal monotone to be maximal monotone (see, for example, [8]). We recall that the sum of two monotone operators is always monotone but this property may fail to hold for maximal monotonicity in general. The first result in this direction is given by Lescarret [6] for operators in Hilbert spaces. In reflexive Banach spaces, Browder [3, 4] established such results when at least one addendum is single-valued and maximal monotone. Specifically, if $T_1$ and $T_2$ are two monotone operators from $X$ to $X^*$ where $T_1$ is maximal, $D(T_2) = X$, and $T_2$ is bounded single-valued and hemi continuous, then $T_1 + T_2$ is maximal monotone. Rockafellar [9] generalized the results of [3, 4] by studying the following two conditions so that the sum $T_1 + T_2$ of two maximal monotone operators $T_1$ and $T_2$ from a reflexive real Banach space $X$ to its dual $X^*$ is maximal monotone:

$$D(T_1) \cap \text{int}\, D(T_2) = \phi, \text{or there exists } x \in \text{cl}\, D(T_1) \cap \text{cl}\, D(T_2) \text{ and } T_2 \text{ is locally bounded at } x.$$

In the same context, Attouch [1] gave weaker condition in Hilbert spaces; namely, if $0 \in \text{int}\big(D(T_1) - D(T_2)\big)$, then $T_1 + T_2$ is a maximal monotone. This condition is weaker than the Rockafellar condition $D(T_1) \cap \text{int}\, D(T_2) \neq \phi$ because

$$\text{int}\, D(T_1) - D(T_2) \subset \text{int}\,(D(T_1) - D(T_2)).$$

Heisler's result holds for nonreflexive Banach spaces. Specifically, the sum $T_1 + T_2$ of two maximal monotone operators $T_1$ and $T_2$ defined from nonreflexive Banach space $X$ to $X^*$ with $D(T_1) = D(T_2) = X$ is maximal monotone [11]. Several weaker sufficient conditions for such results in both reflexive and general Banach spaces have been studied in recent years. The theory of maximal monotone operators on nonreflexive Banach spaces is still developing. For further details on these topics, the reader is referred to [1, 2, 11].

In this paper, we study the maximality of an operator on the product space of two (possibly nonreflexive) Banach spaces in terms of maximal monotone operators that are defined on the individual Banach spaces. Specifically, given monotone operators $T: X \to 2^{X^*}$ and $S: Y \to 2^{Y^*}$, we study the monotonicity and maximal monotonicity of $H: X \times Y \to 2^{X^* \times Y^*}$ defined by

$$H(x, y) = \{(x^*, y^*): x^* \in Tx, y^* \in Sx\}, \quad x \in D(T), y \in D(S).$$

## 2. Preliminary Results

For any normed space $X$, a canonical mapping $c: X \to X^{**}$ is defined by $c(x) = g_x \ \forall x \in X$. Here, for each $x \in X$, the linear functional $g_x: X^* \to \mathbb{R}$ is given by $g_x(f) = f(x)$ for all $f \in X^*$. The canonical mapping $c = g_x$ is a linear isometry, and therefore c is an isomorphism from X onto its range $R(C) \subseteq X^{**}$. The weak* topology on $X^*$ is defined below.

**Definition 1.** The weak* topology is the smallest topology on $X^*$ which makes all the canonical mappings $g_x: X^* \to \mathbb{R}$ continuous. A set $B \subset X^*$ is an open set in the weak* topology of $X^*$ if and only if for every $g \in B$ there exist $\epsilon > 0$ and $x_1, x_2, \ldots, x_n \in X$ such that

$$\{f \in X^* : |\langle f - g, x_i \rangle| < \epsilon\} \subset B.$$

We start with a classical compactness theorem (see [10]).

**Theorem A (Banach-Alaoglu).** The closed unit ball $B_{X^*} = \{x^* \in X^*: ||x^*|| \leq 1\}$ in $X^*$ is compact in its weak* topology.

A net in $X$ is a function defined on a directed set $I$ with values in $X$. If $f : I \to X$ is a net, then for each $\alpha$ in $I$ the $\alpha^{th}$ term $f(\alpha)$ of the net is denoted by $x_\alpha$, and the entire net is often denoted by $(x_\alpha)_{\alpha \in I}$ or simply by $(x_\alpha)$.

**Example 1.** Every sequence is a net with the directed set being $\mathbb{N}$ in its natural order.

We next define the convergence of nets in topological spaces as in [7].

**Definition 2.** Let $I$ be a directed set with an ordering denoted by $\preceq$. Let $(x_\alpha)_{\alpha\in I}$ be a net in a topological space $X$ and let $x$ be an element of $X$. Then we say that $(x_\alpha)$ converges to $x$, and $x$ is called a limit point of $(x_\alpha)$ if for each neighborhood $U$ of $x$, there is an $\alpha_U$ in $I$ such that $x_\alpha \in U$ whenever $\alpha_U \preceq \alpha$. This convergence is expressed by writing $x_\alpha \to x$ or $lim\ _\alpha x_\alpha \to x$.

The following definition generalizes the notion of continuity in terms of net convergence in weak* topology.

**Definition 3.** Let $X$ and $Y$ be normed spaces and $\tau: X^* \to Y^*$ an operator. Then $\tau$ is said to be weak*-weak* continuous on $X^*$ if $lim_\alpha \tau(x_\alpha^*) = \tau(x^*)$ for every net $(x_\alpha^*)$ in $X^*$ that converges to $x^* \in X^*$.

**Definition 4.** An operator $T: X \to 2^{X^*}$ is said to be locally bounded at $x \in D(T)$ if there exists an open neighborhood $V$ of $x$ such that $T(V)$ is bounded. If $T$ is locally bounded at each point of $U \subset D(T)$, then $T$ is said to be locally bounded on.

The following theorem addresses the local boundedness of monotone operators (see [9]).

**Theorem B.** Let $T: X \to 2^{X^*}$ be monotone. Then $T$ is locally bounded at each $x \in \text{int}\, D(T)$.

For multivalued operators, the notion of continuity is generally replaced with that of the upper semicontinuity as defined below (e.g., see [5]).

**Definition 5 (Upper Semi-continuity).** Let $X$ and $Y$ be two linear topological spaces. A multivalued operator $T: X \to 2^Y$ is said to be upper semi-continuous if the set $\{x \in X: Tx \subset U\}$ is open in $X$ whenever $U$ is open in $Y$ .

Note that $T : X \to \mathbb{R}$ is upper semicontinuous if and only if $\{x \in X: Tx \geq a\}$ is closed in $D(T)$ for every $a \in \mathbb{R}$.

We recall the following result from [5] that discusses sufficient conditions for a monotone operator to be maximal monotone.

**Theorem C.** Let $X$ be real Banach space and $T : X \to 2^{X^*}$ a monotone operator.

(i) If $T$ is maximal monotone, then for each $x \in D(T), Tx$ is convex and weak* closed, and $T$ is norm-weak* upper semicontinuous.

(ii) If for each $x \in X, Tx$ is nonempty, convex, and weak* closed subset of $X^*$ , and if $T$ is norm - weak* upper semicontinuous, then $T$ is maximal monotone.

We next discuss some basic results about maximal monotone operators (e.g., see [11]) that will be used in the proofs of our main results.

**Lemma 1** (Simons, **[11]**). Let $X$ be a Banach space and $T: X \to 2^{X^*}$ a maximal monotone operator. Then the following statements hold.

1. $Tx$ is convex and weak* compact for any $x \in \text{int}\, D(T)$.
2. Assume that $((x_\alpha, x_\alpha^*))$ is a bounded and norm-weak* converges to $(x, x^*)$ in $G(T)$. Then $(x, x^*) \in G(T)$.
3. Let $y \in X$. Define $T_y: D(T) \to \mathbb{R} \cup \{\infty\}$ by

$$T_y(x) = \sup\{\langle y, T_x\rangle : x \in \text{int}\, D(T)\}.$$

Then $T_y$ is real-valued and upper semicontinuous on int $D(T)$.

We have the following theorem as a characterization of the maximal monotone operators with full domain.

**Theorem 2.1 (**Simons, **[11]**). Let $D(T) = X$ and let $T: X \to 2^{X^*}$ be monotone. Then $Tx$ is convex and weak* compact for all $x \in X$, and $T_y: X \to \mathbb{R} \cup \{\infty\}$ is upper semicontinuous for all $y \in X$ if and only if $T$ is maximal monotone.

**Proof.**

Suppose $T$ is maximal. Then from 1 and 3 of previous lemma, the if part is straightforward.

For the converse part, let $(z, z*) \in X \times X^*$ and

$$inf_{(s,s^*)\in G(T)}\langle s - z, s^* - z^*\rangle \geq 0. \quad (1)$$

We have to show that $(z, z^*) \in G(T)$.

Let $y$ be an arbitrary element of $X$. Let $\lambda > 0$. Since $D(T) = X$, there exists

$$(T_\lambda, T_\lambda^*) \in G(T): T_\lambda = z + \lambda y.$$

Then, $\langle \lambda y, T_\lambda^* - z^*\rangle = \langle T_\lambda - z, s_\lambda^* - z^*\rangle \geq 0$

and as a result, $\langle y, s_\lambda^* - z^*\rangle \geq 0$.

From the definition of $T_\lambda$, $T_\lambda(s\lambda) \geq \langle y, z^*\rangle$.

As λ →0+, $s\lambda \to z$ then as $T_y$ is upper semicontinuous $T_y(z) \geq \langle y, z^*\rangle$.

This implies $\forall y \in X, \sup\langle y, T(z)\rangle \geq \langle y, z^*\rangle$.

As $T_y$ is compact, $z^* \in Tz. i.e. (z, z^*) \in G(T)$.

Now we present the Heisler's result.

**Theorem 2.2 (**Simons, **[11]**). Assume, for two maximal operators

$T_1: X \to 2^{X^*}$ and $T_2: X \to 2^{X^*}$, $D(\mathrm{T}_1) = D(\mathrm{T}_2) = X$ . Then $T_1 + T_2$ is maximal monotone.

Proof.

Denote $T = T_1 + T_2$ . Then

$$D(T) = D(T_1) \cap D(T_2) = X.$$

As we know the sum of two weak* compact convex sets are also weak* compact and convex, for each $y \in X, T_y$ is weak* compact and convex. Also, the sum of two upper semicontinuous mappings $T_y = (T_1)_y + (T_2)_y$ is upper semi-continuous. Then $T = T_1 + T_2$ is maximal monotone from theorem 2.1.

## 3. Main Results

Let $X$ and $Y$ be two reals valued Banach spaces and $X^*$ and $Y^*$ be their duals. Let $T : X \to 2^{X^*}$ and $S : Y \to 2^{Y^*}$ be monotone operators. Define $H : X \times Y \to 2^{X^* \times Y^*}$ by

$$H(x,y) = \{(x^*,y^*): x^* \in Tx, y^* \in Sx\}, (x,y) \in X \times Y .$$

The monotonicity of $H$ is given in the following theorem.

**Theorem 3.1.** The mapping $H : X \times Y \to 2^{X^* \times Y^*}$ defined by

$$H(x,y) = \{(x^*,y^*) : x^* \in Tx, y^* \in Sx\} \text{ for all } (x,y) \in X \times Y \text{ is monotone.}$$

**Proof.**

Let $a, b \in X$. Since $T: X \to 2^{X^*}$ is monotone,

$$\langle a^* - b^*, a - b\rangle \geq 0, a^* \in Ta \text{ and } b^* \in Tb. \quad (2)$$

Similarly, from the monotonicity of $S: Y \to 2^{Y^*}$, for all $c, d \in Y$

$$\langle c^* - d^*, c - d\rangle \geq 0, c^* \in Sc \text{ and } d^* \in Sd. \quad (3)$$

Clearly $(a,c), (b,d) \in X \times Y$ . Then from the definition of $H$

$$(a^*, c^*) \in H(a,c) \text{ and } (b^*, d^*) \in H(b,d).$$

We observe that

$$\langle (a^*,c^*) - (b^*,d^*), (a,c) - (b,d)\rangle = \langle (a^* - b^*, c^* - d^*), (a - b, c - d)\rangle.$$

Using the duality in $(X \times Y) \times (X^* \times Y^*)$ defined by

$$\langle (x^*,y^*), (x,y)\rangle = \langle x^*, x\rangle + \langle y^*, y\rangle, (x,y) \in X \times Y, (x^*,y^*) \in X^* \times Y^*,$$

we obtain $\langle a^* - b^*, a - b\rangle + \langle c^* - d^*, c - d\rangle \geq 0,$

from the monotonicity of $T$ and $S$. Thus $H$ is monotone. □

The following result addresses the maximality of $H$ whenever $T$ and $S$ are maximal monotone.

**Theorem 3.2.** Let $X$ and $Y$ be two real Banach spaces and $T : X \to 2^{X^*}$ and $S : Y \to 2^{Y^*}$ are maximal monotone operators. Then $H : (X \times Y) \to 2^{X^* \times Y^*}$ by

$$H(x,y) = \{(x^*,y^*) : x^* \in Tx, y^* \in Sx\} \text{ for } (x,y) \in X \times Y \text{ is maximal monotone.}$$

**Proof.**

By Theorem 3.1, $H$ is monotone. To show that $H$ is maximal monotone, we will show in view of Theorem $C$ that

(i) for each $(x,y) \in X \times Y, H(x,y)$ is non-empty convex and weak*− closed, and

(ii) $H$ is norm-to-weak* upper semicontinuous.

For each $(x,y) \in D(H)$, it follows from the maximal monotonicity of $T$ and $S$ that

$$\text{H(x,y)} = \bigcap_{((x_1,y_1),(x_1^*,y_1^*)\in G(H)} \{(x^*,y^*) \in X^* \times Y^*: \langle (x^*,y^*) - (x_1^*,y_1^*), (x,y) - (x_1,y_1)\rangle \geq 0\},$$

Which implies

$$\text{H(x,y)} = \bigcap_{((x_1,y_1),(x_1^*,y_1^*)\in G(H)} \{(x^*,y^*) \in X^* \times Y^*: \langle x^* - x_1^*, x - x_1\rangle + \langle y^* - y_1^*, y - y_1\rangle \geq 0\},$$

Since $T$ and $S$ are maximal monotone, it follows that $Tx$ and $Sy$ are nonempty, convex and weak* closed, and therefore, $H(x,y)$ also shares all these properties. This proves (i). Since $T$ is maximal monotone, $T$ is norm-to-weak* upper semicontinuous. Then for each $x \in X$, each weak* neighborhood $U$ of $Tx$ in $X*$, and each sequence $\{x_n\}$ in $X$ such that $x_n^* \in Tx_n$ and $x_n \to x$, we have $x_n^* \in U$ for all sufficiently large $n \in \mathbb{N}$. Similarly, from the maximality of $S$, it follows that for each $y \in Y$, each weak* neighborhood $V$ of $Sx$ in $Y^*$, and each sequence $\{y_n\}$ in Y such that $y_n^* \in S_{y_n}$ and $y_n \to y$, we have $y_n^* \in V$ for all sufficiently large $n \in \mathbb{N}$ Suppose, on the contrary, that $H$ is not weak* upper semicontinuous. Then there exists a point

$(x, y) \in X \times Y$, a weak* neighborhood $W$ of $H(x, y)$ $in$ $X^* \times Y^*$, and a sequence $\{(x_n, y_n)\} \subseteq X \times Y$ such that $(x_n^*, y_n^*) \in H(x_n, y_n)$ and $(x_n, y_n) \to (x, y)$ and $(x_n^*, y_n^*) \notin W$ for infinitely many values of $n \in \mathbb{N}$. Consequently, there exist weak* neighborhoods $U$ and $V$ of $x$ in $X$ and $y$ in $Y$, respectively, such that $(x_n^*, y_n^*) \notin U \times V$ for infinitely many values of $n$. This is a contradiction. Therefore, $H$ is norm-to-weak* upper semicontinuous, which proves (ii). It follows from the part (ii) of Theorem $C$ that $H : X \times Y \to 2^{X^* \times Y^*}$ is maximal monotone.

## 4. Conclusion

In this paper, we discussed some requirements for the sum $T + S$ of two maximal monotone mappings $T$ and $S$ to be again a maximal monotone in the case $X = \mathrm{Y}$ and an important characterization of maximal monotone operators in general Banach spaces. As our main result, we proved that the operator $\mathrm{H}: \mathrm{X} \times \mathrm{Y} \to 2^{\mathrm{X}^* \times \mathrm{Y}^*}$ defined by $\mathrm{H(x, y)} = \{(x^*, \mathrm{y}^*): \mathrm{x}^* \in \mathrm{T}x, \mathrm{y}^* \in \mathrm{Sy}\}, x \in \mathrm{X}, \mathrm{y} \in \mathrm{Y}$, is maximal monotone.

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