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Computational Analysis of Fractional Reaction-Diffusion Equations that Appear in Porous Media

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Abstract: *The Elzaki Transform Homotopy Perturbation Method (ETHPM), a modified computational technique, is used in this article to solve the time-fractional reaction-diffusion equation that emerges in porous media. Herein fractional-order derivatives are considered in Caputo sense. To show how simple and effective the suggested method is, some specific and understandable examples are provided. The numerical results produced by the suggested technique show that the method is accurate and easy to use. The graphical illustrations of the approximate solutions to the porous media equation for different particular cases are the key characteristics of the current research. The solution obtained is very useful and significant to analyze the many physical phenomena.*

Keywords: *Fractional calculus, Elzaki Transform Homotopy Perturbation Method (ETHPM), Fractional reaction-diffusion equations*

1. Introduction

Numerous problems in the real world have been solved using the theory and fundamental concepts of fractional calculus. As an extension of the conventional integer-order differential equations, fractional-order differential equations are being utilized more often to describe problems in the domains of engineering, mechanics, fluid flow, biology, and physics. Fractional partial differential equations (FPDEs) are widely used in science and engineering, and as a result, research on FPDEs has grown significantly over the past several decades. The theory of fractional partial differential equations can be used to more accurately and systematically translate real-world problems. A novel automated brain segmentation technique for magnetic resonance imaging was developed by Ahlgren et al. [1] employing fractional signal modeling of a spoiled gradient-recalled echo (SPGR) sequence acquired at different flip angles. Sun et al. [2] presented fractional and fractal derivative models for temporary anomalous diffusion. Here, four models are thoroughly compared with one another. In order to solve the time-fractional Navier-Stokes equation in a tube, Kumar et al. [3] devised a unique homotopy perturbation transform method. Murio [4] suggested an implicit unconditionally stable numerical strategy to address the one-dimensional linear time-fractional diffusion issue. The fractional-order diffusion equations were solved by Shah et al.

[5] using the Natural transform decomposition technique. The best approach for q-homotopy analysis was used by Darzi et al. [6] to solve partial differential equations with time-fractional derivatives. A space-time fractional order non-linear Cahn-Hilliard issue was resolved by Pandey et al. [7] using an operational matrix approach and Laguerre polynomials. Pandey et al. [8] recommended an effective Laguerre collocation technique to generate the approximate order non-linear reaction-advection-diffusion equations. The first basic solutions of general fractional-order diffusion equations within the negative Prabhakar kernel were taken into consideration by Yang et al. [9]. The symmetry analysis approach to determine the symmetry of the time-fractional diffusion equation has been covered by Liu et al. [10].

As a chemical moves from a zone of high concentration to one of low concentration, the diffusion process takes place. The dynamics of density profiles during the diffusion of a material are depicted by the diffusion type equation, which is a partial differential equation [11]. Fractional reaction-diffusion equations may be used to describe both shallow water waves in seas and ion-acoustic waves in plasma.

The present study deals with the following time- fraction reaction-diffusion equation which arises in porous media [12]

$$\frac{\partial^{\rho}}{\partial \tau^{\rho}} \chi(\zeta,\tau) = \mathrm{D} \frac{\partial^{\alpha_1}}{\partial \zeta^{\alpha_1}} \chi(\zeta,\tau) - \varsigma \chi^{\mathrm{m}} \frac{\partial^{\alpha_2}}{\partial \zeta^{\alpha_2}} \chi^{\mathrm{n}}(\zeta,\tau) + \mathrm{k}\chi(1-\chi) + \mathrm{f}(\zeta,\tau)\,, \tag{1}$$

Where $0 \le \zeta \le 1, 0 \le \tau \le 1,\ 0 < \rho \le 1, \alpha_1 > 1, \alpha_2 \ge 2$; with IC

$$\chi(\zeta,0) = \chi_0(\zeta). \tag{2}$$

Here, $\chi(\zeta,\tau)$ is a state variable and describes the concentration of a substance/solute profile, D denotes the diffusion coefficient, average velocity of fluid is denoted by $\varsigma > 0, k$ denotes the reaction coefficient and m, n are integers.

Here we will be applying ETHPM to find the approximate numerical solution of time-fractional reaction-diffusion equation (1)-(2). The correctness and effectiveness of the provided technique are demonstrated by the three test examples.

2. Basic definitions of fractional calculus and Elzaki Transform

In this section, we present some basic definitions of fractional calculus that will be incorporated into this study, as follows [13-15].

Definition 1. A real function $f(t)$, $t > 0$ is said to be in the space C_μ if $\mu \in R$, there exists a real number $p > \mu$ and the function $f_1(t) \in C[0,\infty)$ such that $f(t) = t^p f_1(t)$. Moreover, if

$f^{(n)} \in C_\mu$, then $f(t)$ is said to be in the space C_μ^n,$n \in N$.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha \ge 0$ for a function $f(t)$ is defined as

$$\mathrm{I}^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, & \alpha > 0 \\ f(t), & \alpha = 0 \end{cases}$$

Where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 3. The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $f(t)$ is defined as

$$\mathrm{D}^{\alpha} f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t),$$

$$= I^{n-\alpha} \frac{d^n}{dt^n} f(t), \quad n \in N, n-1 < \alpha \leq n.$$

Definition 4. The Caputo fractional derivative of order $\alpha > 0$ is defined as

$$\mathrm{D}^{\alpha} f(t) = \begin{cases} \frac{d^n f(t)}{dt^n}, & \alpha = n, \ n \in N \\ \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & 0 \leq n-1 < \alpha < n, \end{cases}$$

where n is an integer, $t > 0$ and $f(t) \in C_1^n$.

Definition 5. The Elzaki transform of $f(t)$ is defined [16] as

$$E[f(t)] = E[f(t), v] = T(v) = v \int_0^{\infty} f(t) e^{-\frac{t}{v}} dt, \ k_1 < v < k_2, k_1, k_2 > 0, 0 \leq t < \infty, \tag{3}$$

where $f(t)$ is taken from the set A, which is defined as

$$A = \left\{ f(t);\ \exists\, M, k_j > 0,\ j = 1,2, |f(t)| < M e^{\frac{|t|}{k_j}}, if\ t \in (-1)^j \times [0, \infty) \right\}, \tag{4}$$

here, constant M must be a finite number, k_1 and k_2 may be finite or infinite.

Using duality of Laplace [17], Elzaki transform of the Caputo fractional derivative (given in definition 4) of order $\alpha > 0$, can be obtained [18] and get as

$$E[D^{\alpha} f(t), v] = \frac{T(v)}{v^{\alpha}} - \sum_{k=0}^{n-1} v^{k-\alpha+2} f^{(k)}(0), \qquad n-1 < \alpha \leq n, \tag{5}$$

In Eq. (5), $T(v)$ is the Elzaki transform of the function $f(t)$.

Elzaki transform has many useful and important properties like linear property, scale property, shifting property, duality with Laplace transform ,and so forth. Further detail and properties about this transform can be found in [16-19].

3. Elzaki Transform Homotopy Perturbation Method

To illustrate the basic idea of this method, we consider a general form of nonlinear, non-homogeneous partial differential equation as follows:

$$D_t^{\alpha} u(x,t) = Lu(x,t) + Nu(x,t) + f(x,t), \ \alpha > 0 \tag{6}$$

With the following initial conditions

$$D_0^k u(x,0) = g_k, k = 0, \ldots, n-1, \ D_0^n u(x,0) = 0, \text{and } n = \lfloor \alpha \rfloor \tag{7}$$

In eq. (6), D_t^{α} denotes without loss of generality the Caputo fractional derivative operator, L represents a linear differential operator, N stands for nonlinear differential operator and $f(x,t)$ is a known function.

Taking Elzaki transform on both sides of eq. (6), to get

$$E[D_t^{\alpha} u(x,t)] = E[Lu(x,t)] + E[Nu(x,t)] + E[f(x,t)], \tag{8}$$

Using the differentiation property of Elzaki transform[16-19] and above initial conditions, we have

$$E[u(x,t)] = v^{\alpha} E[Lu(x,t)] + v^{\alpha} E[Nu(x,t)] + g(x,t) \tag{9}$$

Applying the inverse Elzaki transform on both sides of eq. (9), we obtain

$$u(x,t) = G(x,t) + E^{-1}\left[v^{\alpha} E[Lu(x,t)] + v^{\alpha} E[Nu(x,t)]\right] \tag{10}$$

Where $G(x,t)$ represents the term arising from the known function $f(x,t)$ and the prescribed initial condition.

Now, we implement the homotopy perturbation method, (see [20-22])

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{11}$$

And the nonlinear term can be decomposed as

$$N[u(x,t)] = \sum_{n=0}^{\infty} p^n H_n(u) \tag{12}$$

Where $H_n(u)$ are He's polynomials (see, [23-24]) and given by

$$H_n(u_0, u_1, \dots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \quad n = 0, 1, 2, \dots \tag{13}$$

Substituting equations (11) and (12) in equation (10), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) + p\{E^{-1}[v^{\alpha} E(L \sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u))]\}. \tag{14}$$

This is the coupling of the Elzaki transform and the Homotopy perturbation method using He's polynomials. Comparing the coefficients of like powers of p in eq. (14) on both sides, we obtain the following approximations as

$p^0: u_0(x,t) = G(x,t)$

$p^1: u_1(x,t) = E^{-1}\{v^{\alpha} E[Lu_0(x,t) + H_0(u)]\}$

$p^2: u_2(x,t) = E^{-1}\{v^{\alpha} E[Lu_1(x,t) + H_1(u)]\}$

$p^3: u_3(x,t) = E^{-1}\{v^{\alpha} E[Lu_2(x,t) + H_2(u)]\}$

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$p^n: u_n(x,t) = E^{-1}\{v^{\alpha} E[Lu_{n-1}(x,t) + H_{n-1}(u)]\}.$

Similarly, we can find rest of the terms and hence, we obtain the desired series solution. Thus, we approximate the analytical solution $u(x,t)$ as

$$u(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} u_n(x,t). \tag{15}$$

The series solution (15) converges very fast in a very few terms.

4. Solution of the time-fractional Reaction-Diffusion Equations

In this part of the article, we have solved some fractional order one-dimensional non-linear partial differential equations that originate in porous media by using ETHPM as mentioned in section 3.

Example 1. The following non-linear fractional order PDE has many uses in rotating flow of liquid in a tube, waves in plasma, etc.

$$\frac{\partial^\rho}{\partial\tau^\rho}\chi(\zeta,\tau)+\frac{\partial}{\partial\zeta}\left(\frac{\chi^2(\zeta,\tau)}{2}\right)-\frac{\partial^3}{\partial\zeta^2\partial\tau}\chi(\zeta,\tau)=0\,,\quad \tau>0,0\le\zeta\le 1,0<\rho\le 1 \tag{16}$$

with IC

$$\chi(\zeta,0)=\zeta. \tag{17}$$

On putting $\rho=1$, then exact solution of (16) is $\chi(\zeta,\tau)=\frac{\zeta}{1+\tau}$.

Applying the Elzaki transform on (16), get as

$$E\left[\frac{\partial^\rho}{\partial\tau^\rho}\chi(\zeta,\tau)\right]+E\left[\frac{\partial}{\partial\zeta}\left(\frac{\chi^2(\zeta,\tau)}{2}\right)-\frac{\partial^3}{\partial\zeta^2\partial\tau}\chi(\zeta,\tau)\right]=0,$$

By using the results of Elzaki transform and simultaneously using IC (17), we get

$$E[\chi(\zeta,\tau)]-v^2\zeta+v^\rho E\left[\frac{\partial}{\partial\zeta}\left(\frac{\chi^2(\zeta,\tau)}{2}\right)-\frac{\partial^3}{\partial\zeta^2\partial\tau}\chi(\zeta,\tau)\right]=0, \tag{18}$$

Employing inverse Elzaki transform on (18), it yields

$$\chi(\zeta,\tau)=\ \zeta-E^{-1}\left(v^\rho E\left[\frac{\partial}{\partial\zeta}\left(\frac{\chi^2(\zeta,\tau)}{2}\right)-\frac{\partial^3}{\partial\zeta^2\partial\tau}\chi(\zeta,\tau)\right]\right)=0, \tag{19}$$

Again incorporating the homotopy perturbation method, (see [20, 21, 22])

$$\chi(\zeta,\tau)=\ \sum_{n=0}^{\infty}p^n\chi_n(\zeta,\tau) \tag{20}$$

And the decomposition of nonlinear term as

$$N[\chi(\zeta,\tau)]=\ \sum_{n=0}^{\infty}p^nH_n(\chi) \tag{21}$$

Substituting (20) and (21), in (19), it reduces to

$$\sum_{n=0}^{\infty}p^n\chi_n(\zeta,\tau)=\ \ \zeta-p\left\{E^{-1}\left[v^\rho E\left(\sum_{n=0}^{\infty}p^nH_n(\chi)-\frac{\partial^3(\sum_{n=0}^{\infty}p^n\chi_n(\zeta,\tau))}{\partial\zeta^2\partial\tau}\right)\right]\right\}, \tag{22}$$

Where $H_n(\chi)$ are He's polynomials (see, [23, 24]). Some He's polynomials factors are

$$H_0(\chi)=\frac{\partial}{\partial\zeta}\left(\frac{{\chi_0}^2(\zeta,\tau)}{2}\right)$$

$$H_1(\chi)=\frac{\partial}{\partial\zeta}\left(\chi_0(\zeta,\tau)\chi_1(\zeta,\tau)\right)$$

$$H_2(\chi)=\frac{\partial}{\partial\zeta}\left(\chi_0(\zeta,\tau)\chi_2(\zeta,\tau)+\frac{{\chi_1}^2(\zeta,\tau)}{2}\right)$$

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Comparing the coefficients of like powers of p in eq. (22), its yields

p^0: $\chi_0(\zeta,\tau)=\zeta$,

p^1: $\chi_1(\zeta,\tau)=-E^{-1}\left\{v^\rho E\left[\frac{\partial}{\partial\zeta}\left(\frac{{\chi_0}^2(\zeta,\tau)}{2}\right)-\frac{\partial^3}{\partial\zeta^2\partial\tau}\chi_0(\zeta,\tau)\right]\right\}$,

On little simplification, we get

$p^1: \chi_1(\zeta,\tau) = -\frac{\zeta\tau^\rho}{\Gamma(\rho+1)}$,

$p^2: \chi_2(\zeta,\tau) = -E^{-1}\left\{v^\rho E\left[\frac{\partial}{\partial\zeta}\left(\chi_0(\zeta,\tau)\chi_1(\zeta,\tau)\right) - \frac{\partial^3}{\partial\zeta^2\partial\tau}\chi_1(\zeta,\tau)\right]\right\}$,

On putting previously obtained value and after that little simplification, get as

$p^2: \chi_2(\zeta,\tau) = \frac{2\zeta\tau^{2\rho}}{\Gamma(2\rho+1)}$,

Similarly

$p^3: \chi_3(\zeta,\tau) = -E^{-1}\left\{v^\rho E\left[\frac{\partial}{\partial\zeta}\left(\chi_0(\zeta,\tau)\chi_2(\zeta,\tau) + \frac{\chi_1{}^2(\zeta,\tau)}{2}\right) - \frac{\partial^3}{\partial\zeta^2\partial\tau}\chi_2(\zeta,\tau)\right]\right\}$,

On putting previously obtained value and after that little simplification, get as

$p^3: \chi_3(\zeta,\tau) = -\zeta\left[4 + \frac{\Gamma(2\rho+1)}{(\Gamma(\rho+1))^2}\right]\frac{\tau^{3\rho}}{\Gamma(3\rho+1)}$,

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Using the same procedure, we can extract more values, and by substituting the aforementioned values in (15), we get an approximate solution in the form of a series

$$\chi(\zeta,\tau) = \zeta - \frac{\zeta\tau^\rho}{\Gamma(\rho+1)} + \frac{2\zeta\tau^{2\rho}}{\Gamma(2\rho+1)} - \zeta\left[4 + \frac{\Gamma(2\rho+1)}{(\Gamma(\rho+1))^2}\right]\frac{\tau^{3\rho}}{\Gamma(3\rho+1)} + \ldots \tag{23}$$

Putting $\rho = 1$ in (23), we get

$$\chi(\zeta,\tau) = \zeta - \zeta\tau + \zeta\tau^2 - \zeta\tau^3 + \cdots. \tag{24}$$

This is identical to exact solution

$$\chi(\zeta,\tau) = \frac{\zeta}{1+\tau}. \tag{25}$$

Example 2. Taking a non-linear fractional order PDE which is a specific occurrence(**non-conservative case $\boldsymbol{k \neq 0}$**) of our concern equation i.e. (1).

On putting $D = \varsigma = \alpha_2 = k = 1$ and $\alpha_1 = 1.5$ in (1), it reduces into

$$\frac{\partial^\rho}{\partial\tau^\rho}\chi(\zeta,\tau) = \frac{\partial^{1.5}}{\partial\tau^{1.5}}\chi(\zeta,\tau) - \frac{\partial}{\partial\zeta}\left(\chi^2(\zeta,\tau)\right) + \left(1-\chi(\zeta,\tau)\right)\chi(\zeta,\tau), \tag{26}$$

with the initial condition $\chi(\zeta,0) = \zeta^2$. Jointly with this IC the exact solution of (26) is

$\chi(\zeta,\tau) = \zeta^2 + \tau^2$.

On using the computational technique (given in section 3) as applied for getting the solution of Example 1, obtain the coefficients of power of p as below

$p^0: \chi_0(\zeta,\tau) = \zeta^2$,

$p^1: \chi_1(\zeta,\tau) = \left(\frac{4\sqrt{\zeta}}{\sqrt{\pi}} + \zeta^2 - 4\zeta^3 - \zeta^4\right)\frac{\tau^\rho}{\Gamma(\rho+1)}$,

$p^2: \chi_2(\zeta,\tau) = \left(\frac{8\sqrt{\zeta}}{\sqrt{\pi}} - \frac{52}{\sqrt{\pi}}\zeta^{\frac{3}{2}} - \frac{24}{5\sqrt{\pi}}\zeta^{\frac{5}{2}} + \zeta^2 - 12\zeta^3 + 37\zeta^4 + 20\zeta^5 + 6\zeta^6\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}$,

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Similar obtain further values; on putting these obtained values in (15), get solution of (26), in series form

$$\chi(\zeta,\tau)=\zeta^2+\tau^2-\left(\frac{4\sqrt{\zeta}}{\sqrt{\pi}}+\zeta^2-4\zeta^3-\zeta^4\right)\frac{\tau^\rho}{\Gamma(\rho+1)}+\left(\frac{8\sqrt{\zeta}}{\sqrt{\pi}}-\frac{52}{\sqrt{\pi}}\zeta^{\frac{3}{2}}-\frac{24}{5\sqrt{\pi}}\zeta^{\frac{5}{2}}+\zeta^2-12\zeta^3+37\zeta^4+20\zeta^5+6\zeta^6\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}+\cdots. \quad (27)$$

Example 3. Taking a non-linear fractional order PDE which is a specific occurrence(**conservative case $\boldsymbol{k=0}$**) of our concern equation i.e. (1).

On putting $\mathrm{D}=\varsigma=\alpha_2=1$ and $\alpha_1=1.5,\ k=0, m=0, n=2$ in (1), it reduce into

$$\frac{\partial^\rho}{\partial\tau^\rho}\chi(\zeta,\tau)=\frac{\partial^{1.5}}{\partial\tau^{1.5}}\chi(\zeta,\tau)-\frac{\partial}{\partial\zeta}\left(\chi^2(\zeta,\tau)\right), \quad (28)$$

With the initial condition $\chi(\zeta,0)=\zeta-\zeta^2$.

On using the computational technique (given in section 3) as applied for solution of Example 1, get the coefficients of power of p as

$p^0: \chi_0(\zeta,\tau)=\zeta-\zeta^2,$

$p^1: \chi_1(\zeta,\tau)=\left(-\frac{4\sqrt{\zeta}}{\sqrt{\pi}}-2\zeta+6\zeta^2-4\zeta^3\right)\frac{\tau^\rho}{\Gamma(\rho+1)},$

$p^2: \chi_2(\zeta,\tau)=\left(\frac{36}{\sqrt{\pi}}\sqrt{\zeta}-\frac{52}{\sqrt{\pi}}\zeta^{\frac{3}{2}}+4\zeta-40\zeta^2+80\zeta^3-40\zeta^4\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)},$

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Similar obtain further values; on putting these obtained values in (15), get solution of (28), in series form

$$\chi(\zeta,\tau)=\zeta-\zeta^2+\left(-\frac{4\sqrt{\zeta}}{\sqrt{\pi}}-2\zeta+6\zeta^2-4\zeta^3\right)\frac{\tau^\rho}{\Gamma(\rho+1)}+\left(\frac{36}{\sqrt{\pi}}\sqrt{\zeta}-\frac{52}{\sqrt{\pi}}\zeta^{\frac{3}{2}}+4\zeta-40\zeta^2+80\zeta^3-40\zeta^4\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}+\cdots. \quad (29)$$

5. Graphical Analysis of the Approximate Results

In this section we are presenting some graphical analysis of the obtained approximate results as

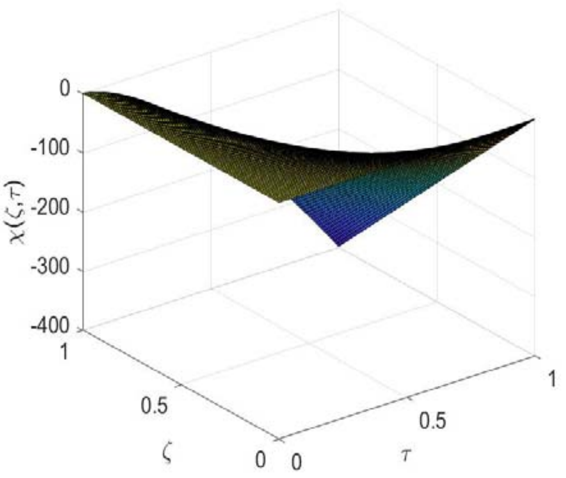

Fig. 1: The surface shows the ETHPM solution $\chi(\zeta,\tau)$ for Example 1, when $\rho=0.5$

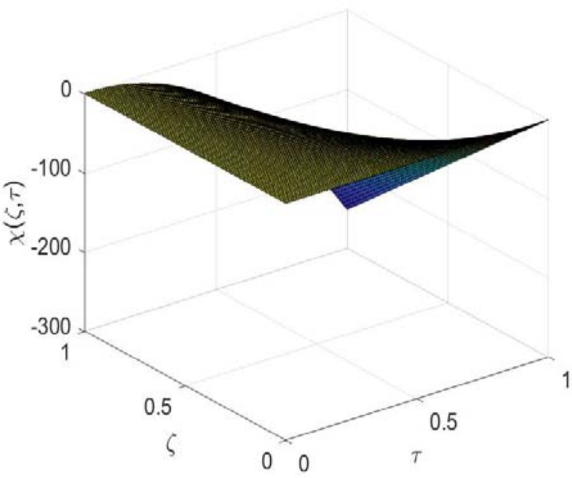

Fig. 2: The surface shows the ETHPM solution $\chi(\zeta,\tau)$ for Example 1, when $\rho=0.7$

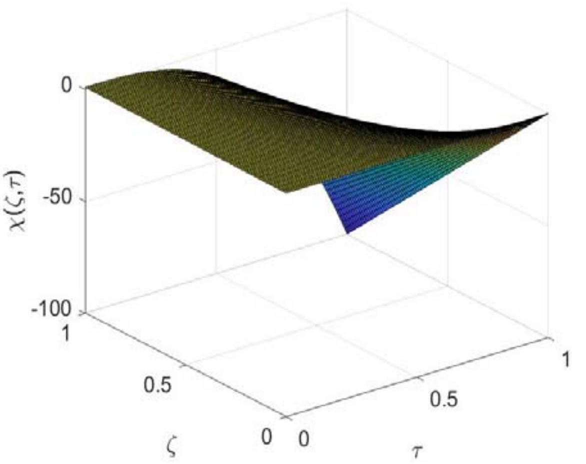

Fig. 3: The surface shows the ETHPM solution $\chi(\zeta,\tau)$ for Example 1, when $\rho = 1$

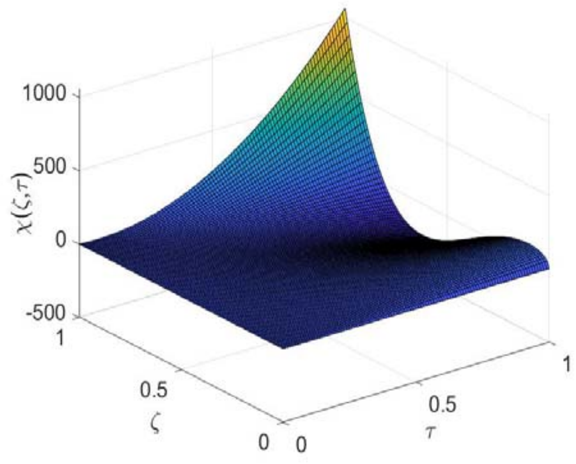

Fig. 4: The surface shows the ETHPM solution $\chi(\zeta,\tau)$ for Example 2, when $\rho = 1$

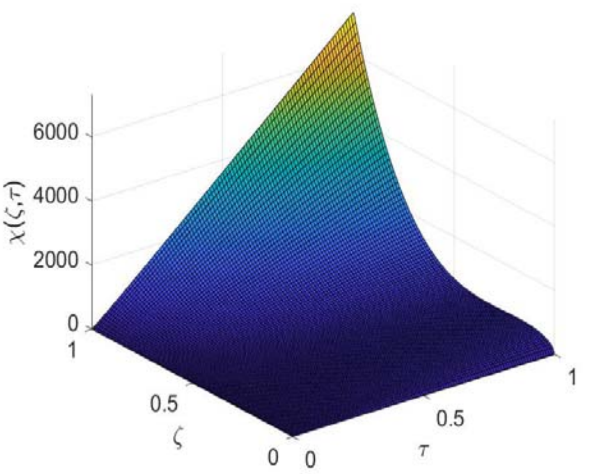

Fig. 5: The surface shows the ETHPM solution $\chi(\zeta,\tau)$ for Example 3, when $\rho = 0.5$

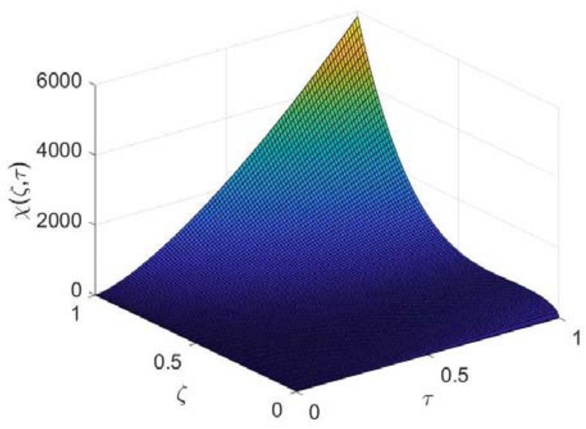

Fig. 6: The surface shows the ETHPM solution $\chi(\zeta,\tau)$ for Example 3, when $\rho = 0.7$

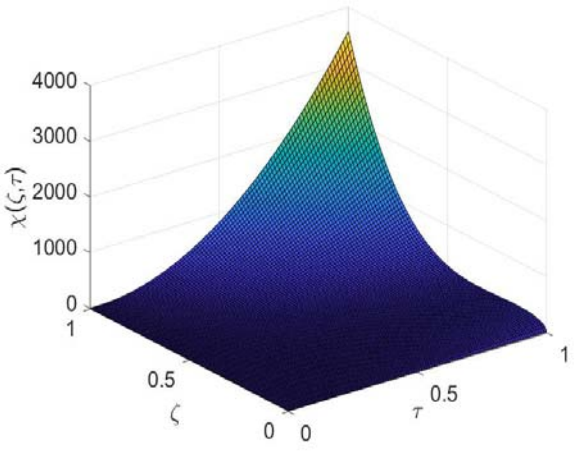

Fig. 7: The surface shows the ETHPM solution $\chi(\zeta, \tau)$ for Example 3, when $\rho = 1$

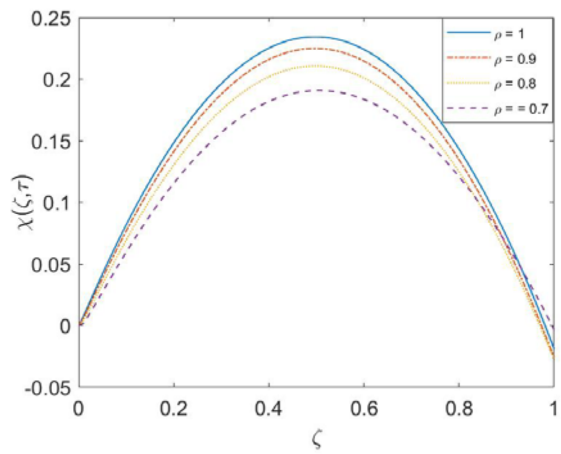

Fig. 8: The behavior of Solute concentration $\chi(\zeta, \tau)$ vs. ζ at $\tau = 1$, for different values of ρ for Example 3

It has been observed from all graphs that the fractional order is better to describe the solution of the time-fractional Reaction-Diffusion Equations, and give a free hand to adjust and control accordingly.

6. Conclusion

The major objective of this study is to demonstrate the usefulness of the combination of the homotopy perturbation technique and the novel integral transform "Elzaki transform" for obtaining both approximate and accurate solutions for nonlinear time-fractional reaction-diffusion equations. Graphs for different fractional order have been plotted to examine the various effects on solute concentration. The numerical result shows that the method used is very simple and straightforward to implement. Our findings provide interesting unifications and extensions of many results, hither to scattered in the literature. At the end, we can conclude that the ETHPM has nice refinement in all numerical methods and it can be used in solving many real world-problems.

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