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Numerical Solution of Electrostatic Potential Distribution on a Unit Circular Disc Using Possion's Equation

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Abstract: *In this paper, the two-dimensional electrostatic potential distribution problem has been solved numerically using the finite difference approach using Poisson's equation in polar coordinates with Dirichlet's boundary condition inside a unit circular disc. We also use Gauss elimination method to solve numbers of linear equations to obtain solutions of unknowns at each grid point. The numerical solution to the same problem is contrasted with the analytic solution. Finally, we examine the absolute errors throughout a range of iterations to evaluate the accuracy of the schemes.*

Keywords: *Finite difference method, Poisson's equation, Numerical method, Circular disc, Polar coordinates.*

1 Introduction

The $Poission's$ equation is the elliptic partial differential equation (PDE), which bears the name of the French mathematician and physicist '$Simeon\ Denis\ Poission$ (1781 - 1840)'. It is a generalization of Laplace equation and most significant PDE in physics. This equation appears in a wide range of physical circumstances. The electrostatic field may be calculated after the potential field is known, which is the result of Poisson's equation for an electric charge distribution. The Laplace equation in two dimension with electrostatic potential p is [6, 8, 9]

$$\nabla^2 p = 0 \tag{1}$$

$Poisson's$ equation is the in-homogeneous counterpart of Laplace equation, that is

$$\nabla^2 p = \phi \tag{2}$$

where p and ϕ are real or complex-valued functions. Then two dimensional Poisson's equation in Cartesian coordinate is [16]

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \phi(x, y) \tag{3}$$

where $\phi(x, y)$ is the source term and $p(x, y)$ is the electrostatic potential.
The $Poisson's$ equation is reduced into polar coordinates, replacing (x, y) by $x = r\cos\theta,\ y = r\sin\theta$. Then $Poisson's$ equation in polar coordinates form is [7, 14, 15]

$$\frac{1}{r}(rp_r)_r + \frac{1}{r^2}p_{\theta\theta} = \psi(r,\theta) \quad \text{that is} \quad \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = \psi(r,\theta) \tag{4}$$

Two dimensional $Poisson's$ equation in polar coordinate (4) is to describe the electrostatic field with homogeneous boundary condition caused by a given electric charge in a unit disc with $Dirichlet's$ boundary condition $p(1, \theta) = g(\theta)$, $0 \le \theta \le 2\pi$.
Now source term $\psi(r, \theta)$ and boundary condition $g(\theta)$ are given that is $\psi(r, \theta) = -3\cos\theta$ and $g(\theta) = 0$. Then the $Poisson's$ equation on a unit circular disc with homogeneous boundary condition is given by [7]

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = -3\cos\theta, \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

and

$$p(1, \theta) = 0, \ \ 0 \le \theta \le 2\pi \tag{5}$$

1.1 Electrostatic Potential Distribution Inside of Unit Circular Disc:

$Poisson's$ equation arises as the basic equilibrium equation in a remarkable variety of physical systems. It describes the static electric charge of an object. We may interpret $p(r, \theta)$ as the electric charge and the in-homogeneity i.e. source term $\psi(r, \theta) = -3 \cos\theta$ represents an external forcing of potential field. Homogeneous $Dirichlet's$ boundary condition $p(1, \theta) = 0, \ \ \theta \in [0, 2\pi]$ specify the value of $p(r, \theta)$ on the boundary where $r = 1$.
Poisson's equation (5) has an analytical solution along the interior of the unit circle with the provided boundary condition, which is [1, 8]

$$p(r, \theta) = r(1 - r) \ \cos\theta, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi \tag{6}$$

1.2 Numerical method: Finite difference scheme

To solve mathematical problems,generally we used two approaches, they are analytical and numerical methods. Numerical methods are the numerical algorithms, they are finite difference method, finite element method, finite volume method et al. These methods are useful in engineering,physics, and finance where analytical solutions are often not feasible or even possible. Numerical methods find the approximate solutions close to the exact solutions. They are used to solve a wide range of differential equations,like ODEs and PDEs. All numerical methods used to solve $PDEs$ should have consistency,stability and convergence. A numerical method is consistent if all the approximation of the derivatives approaches the exact value as the step size tends to zero. It is stable if the error does not grow with time. Then it is convergent if it is both stable and consistent. The accuracy of the numerical solution depends on the algorithms used and the numbers of iterations performed. In this paper, we use finite difference method and $Gauss$ elimination method [7, 15]. Finite difference method is used to approximate the solutions of PDEs by discretizing the domain and approximating the derivatives using finite differences [6, 7, 15].

1.2.1 Finite difference grid of unit circular disc

The most common discritization technique for $PDEs$ is the finite difference method. In order to solve PDEs, the finite difference method discretizes the continuous physical domain into a discrete finite difference grid. At the points of intersections of these grid curves (lines), numerical solutions to the PDE is obtained by using finite difference method. Assume that the grid curves be uniformly spaced along r and θ directions, with $\triangle r = h$ and $\triangle\theta = k$. Then the set of grid points are denoted by $(r_i, \theta_j), \ i = 0, 1, 2, ..., M$ and $j = 0, 1, 2, ..., N$, where $\theta_N = 2\pi$ and $r_M = 1$. On the grid point (r_i, θ_j), a continuous function $p(r, \theta)$ which is changing on (r_i, θ_j) is denoted by the discrete function $p_{i,j}$, as shown in figure (1)[6].

1.2.2 Finite difference schemes of partial derivatives

A. Thom invented the finite difference method, sometimes known as "the method of square," in the 1920s to solve non-linear hydrodynamic equations. A set of grid points in $r\theta-$plane will approximate the values of the smooth function $p(r, \theta)$ if partial derivatives of $PDEs$ are replaced by finite difference schemes. Finite difference method is used to approximate the solutions of PDEs by discretizing the domain and approximating the derivatives using finite differences. Some basic finite difference schemes of first and order partial derivatives in polar coordinate form are [6, 7, 14, 15]

i) Forward schemes

$$\frac{\partial p}{\partial r}(r_i, \theta_j) \approx \frac{p_{i+1,j} - p_{i,j}}{\Delta r}$$

and

$$\frac{\partial p}{\partial r}(r_{i+1/2}, \theta_j) \approx \frac{p_{i+1,j} - p_{i,j}}{\Delta r}$$

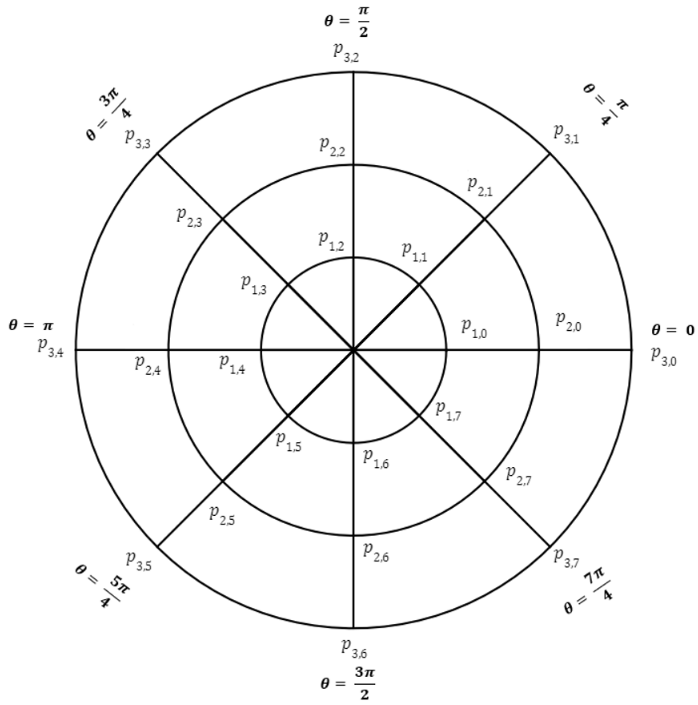

Figure 1: Finite Difference Grid Points in a Unit Circular Disc

ii) Backward schemes

$$\frac{\partial p}{\partial r}(r_i, \theta_j) \approx \frac{p_{i,j} - p_{i-1,j}}{\Delta r}$$

and

$$\frac{\partial p}{\partial r}(r_{i-1/2}, \theta_j) \approx \frac{p_{i,j} - p_{i-1,j}}{\Delta r}$$

iii) Central scheme

$$\frac{\partial p}{\partial r}(r_i, \theta_j) \approx \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta r}$$

iv) Second-order difference scheme

$$\frac{\partial^2 p}{\partial r^2}(r_i, \theta_j) \approx \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{(\Delta r)^2}$$

2 Numerical Solution of Poisson's Equation Inside Unit Circular disc

The Poisson's equation must be solved in polar coordinates from inside a unit circular disc since there are many circles in real world. From (5), the model problem in unit circular disc with $Dirichlet's$ boundary condition is [2, 6, 15]

$$\frac{1}{r}(rp_r)_r + \frac{1}{r^2}p_{\theta\theta} = -3\cos\theta$$

That is

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = \psi(r,\theta), \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

and

$$p(1,\theta) = 0, \quad 0 \le \theta \le 2\pi \tag{7}$$

Now,

$$\frac{1}{r}(rp_r)_r \approx \frac{1}{r_i(\Delta r)}[(rp_r)_{i+1/2,j} - (rp_r)_{i-1/2,j}$$

$$= \frac{1}{r_i(\Delta r)}[r_{i+1/2}(\frac{p_{i+1,j} - p_{i,j}}{\Delta r}) - r_{i-1/2}(\frac{p_{i,j} - p_{i-1,j}}{\Delta r})]$$

$$= \frac{1}{r_i(\Delta r)^2}[r_{i+1/2}(p_{i+1,j} - p_{i,j}) - r_{i-1/2}(p_{i,j} - p_{i-1,j})]$$

$$\frac{1}{r^2}p_{\theta\theta} \approx \frac{1}{(r_i)^2(\Delta\theta)^2}(p_{i,j+1} - 2p_{i,j} + pi, j - 1)$$

Now substituting in above $Poisson's$ model problem (7) , we obtain

$$\frac{1}{r_i(\Delta r)^2}[r_{i+1/2}(p_{i+1,j} - p_{i,j}) - r_{i-1/2}(p_{i,j} - p_{i-1,j})] + \frac{1}{(r_i)^2(\Delta\theta)^2}(p_{i,j+1} - 2p_{i,j} + pi, j - 1) = -3\cos\theta_j, \quad i = 1,2,3,...,M, \;\; j = 1,2,3,...,N-1. \tag{8}$$

Due to periodic boundary and periodic relationship, $j = 0$ and $j = N$ represent the same angle, likewise for $j = 1$ and $j = N-1$ that is $p_{i,0} = p_{i,N}$ and $p_{i,1} = p_{i,N-1}$ $, i = 0,1,2,...,M,$

$$\frac{1}{r_i(\Delta r)^2}[r_{i+1/2}(p_{i+1,0} - p_{i,0}) - r_{i-1/2}(p_{i,0} - p_{i-1,0})] + \frac{1}{(r_i)^2(\Delta\theta)^2}(p_{i,1} - 2p_{i,0} + p_{i,N-1}) = -3\cos\theta_0 = -3, \quad i = 1,2,3,...,M-1 \tag{9}$$

For the case of center of the circle, the set of grid points associated with $i = 0, \;\; j = 0,1,2,...,N$ is really only one point, it is denoted by $p_{0,j}$ for any j, there is only one function value that is p_0. Using control volume approach [15],

$$\frac{4}{(\Delta r)^2}p_0 - \frac{2\Delta\theta}{\pi(\Delta r)^2}\sum_{j=1}^{N-1} p_{1,j} = F_0 = 0, \quad j = 0,1,...,N. \tag{10}$$

We arrange the like terms in (8), we obtain

$$\frac{r_{i+1/2}}{r_i(\Delta r)^2}p_{i+1,j} - [\frac{1}{r_i(\Delta r)^2} + \frac{r_{i-1/2}}{r_i(\Delta r)^2} + \frac{2}{(r_i)^2(\Delta\theta)^2}]p_{i,j} + \frac{r_{i-1/2}}{r_i(\Delta r)^2}p_{i-1,j} + \frac{1}{(r_i)^2(\Delta\theta)^2}[p_{i,j+1} + p_{i,j-1}] = -3\cos\theta_j.$$

Let $\Delta r = h, \quad \Delta\theta = k, \quad \frac{r_{i+1/2}}{r_i h^2} = a_i, \quad \frac{r_{i-1/2}}{r_i h^2} = b_i, \quad \frac{1}{(r_i)^2 k^2} = c_i.$. Then

$$a_i p_{i+1,j} - (a_i + b_i + 2c_i)p_{i,j} + b_i p_{i-1,j} + c_i(p_{i,j+1} + p_{i,j-1}) = -3\cos\theta_j \tag{11}$$

Let $h = \frac{1}{3}, \quad k = \frac{\pi}{4}$ and discrete boundary condition for unit circular disc given in (7), is $p_{M,j} = 0, \quad j = 0,1,...,N-1$, that is $p_{3,0} = p_{3,1} = p_{3,2} = p_{3,3} = p_{3,4} = p_{3,5} = p_{3,6} = p_{3,7} = p_{3,8} = 0$ and for $i = 0$, at the center $p_{0,j}, \quad j = 1,2,...,N$ that is $p_{0,1} = p_{0,2} = p_{0,3} = p_{0,4} = p_{0,5} = p_{0,6} = p_{0,7} = p_{0,8} = p_0$.
Each grid point has one unknown and need one algebraic equation, so that numbers of equations and unknown must be equal. For $i = 0,1,2, \quad j = 0,1,2,...,7$, there are 17 unknowns, hence there are 17

algebraic equations [2].

When $i = 0$, from (10), we obtain

$$36p_0 - 4.5p_{1,1} - 4.5p_{1,2} - 4.5p_{1,3} - 4.5p_{1,4} - 4.5p_{1,5} - 4.5p_{1,6} - 4.5p_{1,7} = 0 \tag{12}$$

When $i = 1, j = 8$ and using (11), we obtain
$a_1 = \frac{r_{3/2}}{r_1 h^2} = \frac{0.5}{(1/3)(1/9)} = 13.5$, $b_1 = \frac{r_{1/2}}{r_1 h^2} = \frac{1/6}{(1/3)(1/9)} = 4.5$, $c_i = \frac{1}{(r_1)^2 k^2} = \frac{1}{(1/9)(\pi^2/16)} = 14.5903$

Using (11), we obtain

$$13.5p_{2,8} - 32.5903p_{1,8} + 4.5p_{0,8} + 14.5903(p_{1,9} + p_{1,7}) = -3\cos 2\pi$$

Due to periodic boundary and periodic relationship, using (9), $p_{2,8} = p_{2,0}, p_{1,8} = p_{1,0}, p_{1,9} = p_{1,1}$. Then

$$4.5p_0 - 32.5903p_{1,0} + 13.5p_{2,0} + 14.5903p_{1,1} + 14.5903p_{1,7} = -3 \tag{13}$$

Similarly, when $i = 1, j = 2, 3, 4, 5, 6, 7$ we obtain

$$4.5p_0 + 14.5903p_{1,0} - 32.5903p_{1,1} + 14.5903p_{1,2} + 13.5p_{2,1} = -3\cos\pi/4 = -2.1213 \tag{14}$$
$$4.5p_0 + 14.5903p_{1,1} - 32.5903p_{1,2} + 14.5903p_{1,3} + 13.5p_{2,2} = -3\cos\pi/2 = 0 \tag{15}$$
$$4.5p_0 + 14.5903p_{1,2} - 32.5903p_{1,3} + 14.5903p_{1,4} + 13.5p_{2,3} = -3\cos 3\pi/4 = 2.1213 \tag{16}$$
$$4.5p_0 + 14.5903p_{1,3} - 32.5903p_{1,4} + 14.5903p_{1,5} + 13.5p_{2,4} = -3\cos\pi = 3 \tag{17}$$
$$4.5p_0 + 14.5903p_{1,4} - 32.5903p_{1,5} + 14.5903p_{1,6} + 13.5p_{2,5} = -3\cos 5\pi/4 = 2.1213 \tag{18}$$
$$4.5p_0 + 14.5903p_{1,5} - 32.5903p_{1,6} + 14.5903p_{1,7} + 13.5p_{2,6} = -3\cos 3\pi/2 = 0 \tag{19}$$
$$4.5p_0 + 14.5903p_{1,0} + 14.5903p_{1,6} - 32.5903p_{1,7} + 13.5p_{2,7} = -3\cos 7\pi/4 = -2.1213 \tag{20}$$

When $i = 2, j = 1, 2, 3, 4, 5, 6, 7$, and $a_2 = 11.25, b_2 = 6.75, c_2 = 3.6476$. Then

$$6.75p_{1,1} + 3.6476p_{2,0} - 21.6476p_{2,1} + 3.6476p_{2,2} = -2.1213 \tag{21}$$
$$6.75p_{1,2} + 3.6476p_{2,1} - 21.6476p_{2,2} + 3.6476p_{2,3} = 0 \tag{22}$$
$$6.75p_{1,3} + 3.6476p_{2,2} - 21.6476p_{2,3} + 3.6476p_{2,4} = 2.1213 \tag{23}$$
$$6.75p_{1,4} + 3.6476p_{2,3} - 21.6476p_{2,4} + 3.6476p_{2,5} = 3 \tag{24}$$
$$6.75p_{1,5} + 3.6476p_{2,4} - 21.6476p_{2,5} + 3.6476p_{2,6} = 2.1213 \tag{25}$$
$$6.75p_{1,6} + 3.6476p_{2,5} - 21.6476p_{2,6} + 3.6476p_{2,7} = 0 \tag{26}$$
$$6.75p_{1,7} + 3.6476p_{2,0} + 3.6476p_{2,6} - 21.6476p_{2,7} = 2.1213 \tag{27}$$
$$6.75p_{1,0} - 21.6476p_{2,0} + 3.6476p_{2,1} + 3.6476p_{2,7} = -3 \tag{28}$$

From above system of linear equations (12)- (28, we obtain a linear sparse system [14, 15]

$$AP = B$$

$$P = \begin{pmatrix} p_0 & p_{1,0} & p_{1,1}\dots & p_{1,7} & p_{2,0} & p_{2,1}\dots & p_{2,7} \end{pmatrix}^T$$
$$B = \begin{pmatrix} 0 & -3 & -2.1213\dots & 0 & -2.1213\dots & -3 \end{pmatrix}^T$$

Using Gaussian elimination method in $MATLAB$, we obtain
$p_0 = 0.0231$, $p_{1,0} = 0.8901$, $p_{1,1} = 0.5564$, $p_{1,2} = -0.0175$, $p_{1,3} = -0.4854$, $p_{1,4} = -0.5832$, $p_{1,5} = -0.2792$, $p_{1,6} = .2312$, $p_{1,7} = 0.7625$, $p_{2,0} = 0.4934$, $p_{2,1} = 0.2353$, $p_{2,2} = -0.1267$, $p_{2,3} = -0.3731$, $p_{2,4} = -0.3670$, $p_{2,5} = -0.1442$, $p_{2,6} = 0.0280$, $p_{2,7} = 0.4642$.

$$A = \begin{bmatrix}
36 & 0 & -4.5 & -4.5 & -4.5 & -4.5 & -4.5 & -4.5 & -4.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.5 & -32.59 & 14.59 & 0 & 0 & 0 & 0 & 0 & 14.59 & 13.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.5 & 14.59 & -32.59 & 14.59 & 0 & 0 & 0 & 0 & 0 & 0 & 13.5 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.5 & 0 & 14.59 & -32.59 & 14.59 & 0 & 0 & 0 & 0 & 0 & 0 & 13.5 & 0 & 0 & 0 & 0 & 0 \\
4.5 & 0 & 0 & 14.59 & -32.59 & 14.59 & 0 & 0 & 0 & 0 & 0 & 0 & 13.5 & 0 & 0 & 0 & 0 \\
4.5 & 0 & 0 & 0 & 14.59 & -32.59 & 14.59 & 0 & 0 & 0 & 0 & 0 & 0 & 13.5 & 0 & 0 & 0 \\
4.5 & 0 & 0 & 0 & 0 & 14.59 & -32.59 & 14.59 & 0 & 0 & 0 & 0 & 0 & 0 & 13.5 & 0 & 0 \\
4.5 & 0 & 0 & 0 & 0 & 0 & 014.59 & -32.59 & 14.59 & 0 & 0 & 0 & 0 & 0 & 0 & 13.5 & 0 \\
4.5 & 14.59 & 0 & 0 & 0 & 0 & 0 & 14.59 & -32.59 & 14.59 & 0 & 0 & 0 & 0 & 0 & 0 & 13.5 \\
0 & 6.75 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -21.65 & 3.65 & 0 & 0 & 0 & 0 & 0 & 3.65 \\
0 & 0 & 6.75 & 0 & 0 & 0 & 0 & 0 & 0 & 3.65 & -21.65 & 3.65 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6.75 & 0 & 0 & 0 & 0 & 0 & 0 & 3.65 & -21.65 & 3.65 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6.75 & 0 & 0 & 0 & 0 & 0 & 0 & 03.65 & -21.65 & 3.65 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6.75 & 0 & 0 & 0 & 0 & 0 & 0 & 3.65 & -21.65 & 3.6476 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6.75 & 0 & 0 & 0 & 0 & 0 & 0 & 3.6476 & -21.65 & 3.65 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.75 & 0 & 0 & 0 & 0 & 0 & 0 & 3.65 & -21.65 & 3.65 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.75 & 3.65 & 0 & 0 & 0 & 0 & 0 & 3.65 & -21.65
\end{bmatrix}$$

3 Error Analysis

We require approximate values since it is not always possible to utilize the exact and precise values while doing mathematical calculations. The result is unreliable as a result of approximation, and we can say that error was introduced throughout the calculations.The distinction between the problem's analytic solution and its approximation solution is, in fact, absolute error. There are three primary sources of computational errors, which are truncation error, rounding error and human error. A truncation error is the difference between the exact solution and the value that was truncated. In general, computational error is the truncation error incorporating with rounding error. The expansion for the truncation error of the forward difference is

$$E(r,\theta : h,k) = p'(r,\theta) - \frac{p(r+h,\theta)-p(r,\theta)}{h} = p'(r,\theta) - \frac{p(r,\theta)+hp'(r,\theta)+h^2/2 \quad p''(\xi,\theta)-p(r,\theta)}{h} = -\frac{h}{2}p''(\xi,\theta)$$

for some $\xi \in (r, r+h)$ (first order approximation).

Similarly, the expansion for the truncation error of the backward difference is $E(r,\theta : h,k) = \frac{h}{2}p''(\xi,\theta)$ for some $\xi \in (r-h, r)$ (first order approximation).

The expansion for the truncation error of the backward difference is

$$\begin{aligned} E(r,\theta : h,k) &= p'(r,\theta) - \frac{p(r+h,\theta)-p(r-h,\theta)}{2h} \\ &= p'(r,\theta) - \frac{1}{2h}[p(r,\theta) + hp'(r,\theta) + h^2/2 \quad p''(r,\theta) + h^3/6 \quad p'''(\xi_1,\theta) \\ &\text{- } p(r,\theta) + hp'(r,\theta) - h^2/2 \quad p''(r,\theta) + h^3/6 \quad p'''(\xi_2,\theta)] \\ &= -\frac{h^2}{6}[p''(\xi_1,\theta) + p''(\xi_2,\theta] \end{aligned}$$

$= -\frac{h^2}{6} \quad p''(\eta,\theta)$ for some $\eta \in (r-h, r+h)$ (second order approximation).

Similarly, the expansion for the truncation error of second order partial derivative is also second order approximation. Therefore forward, backward, central and second-order schemes respectively are [14, 15]

$$\frac{\partial p}{\partial r}(r_i,\theta_j) = \frac{p_{i+1,j} - p_{i,j}}{\Delta r} + \circ(h)$$

$$\frac{\partial p}{\partial r}(r_i,\theta_j) = \frac{p_{i,j} - p_{i-1,j}}{\Delta r} + \circ(h)$$

$$\frac{\partial p}{\partial r}(r_i,\theta_j) = \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta r} + \circ(h^2)$$

$$\frac{\partial^2 p}{\partial r^2}(r_i,\theta_j) = \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{(\Delta r)^2} + \circ(h^2)$$

Using above approximations, truncation error of the $Poisson's$ equation is $[\circ(h^2) + \circ(k^2)]$.
If there exists a constant M independent of h and k and order of an approximation are $p = 2$ and $q = 2$. Then error in $Poission$ equation is

$$|E(r,\theta : h,k)| \leq Mh^2k^2$$

for sufficiently small $h > 0, \quad k > 0$.
The error is useful for evaluating the accuracy of numerical methods when solution is known. When the precise solution to the problem is unknown, the true absolute error cannot be estimated, in this situation different methods are used to assess the accuracy of a numerical solution [3, 4]. Truncation error can be minimized by performing repeated iteration and incorporating as many as terms in the approximation as possible. In this paper, absolute errors of the analytic and numerical solutions of the above model problem of $Poisson's$ equation when $i = 0, 1, 2 \quad j = 0, 1, 2, ..., 7$, as shown in the following table.

Table 1: Absolute errors

Values of (r, θ)	Analytic solutions	Numerical solutions	Absolute Errors
$(0, 0)$	0	0.0231	0.0231
$(1/3, 0)$	0.2222	0.8901	0.6679
$(1/3, \pi/4)$	0.1571	0.5564	0.4047
$(1/3, \pi/2)$	0	-0.0175	0.0175
$(1/3, 3\pi/4)$	-0.1571	-0.4854	0.3283
$(1/3, \pi)$	-0.2222	-0.5832	0.3610
$(1/3, 5\pi/4)$	-0.1571	-0.2792	0.1221
$(1/3, 3\pi/2)$	0	0.2312	0.2312
$(1/3, 7\pi/4)$	0.1571	0.7625	0.6054
$(2/3, 0)$	0.2222	0.4934	0.2712
$(2/3, \pi/4)$	0.1571	0.2353	0.0782
$(2/3, \pi/2)$	0	-0.1267	0.1267
$(2/3, 3\pi/4)$	-0.1571	-0.3731	0.2160
$(2/3, \pi)$	-0.2222	-0.3670	0.1448
$(2/3, 3\pi/4)$	-0.1571	-0.1442	0.0129
$(2/3, 3\pi/2)$	0	0.0280	0.0280
$(2/3, 7\pi/4)$	0.1571	0.4642	0.3071

4 Comparative Study of Results

4.1 Two dimensional and three Dimensional figures of analytical solution

The following figures (2) are the electrostatic potential distribution in a unit circular disc with given $Dirichlet's$ boundary values analytically [3, 4, 12];

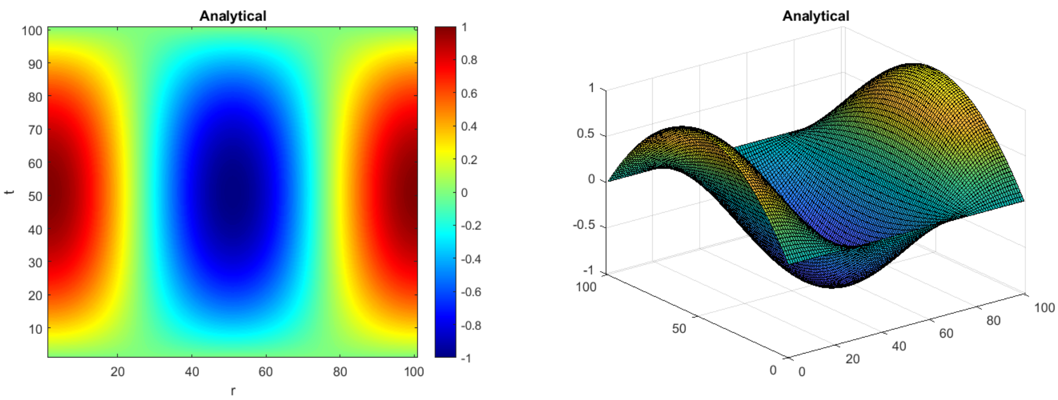

Figure 2: Potential Distribution on a Unit Circular Disc in 2D and 3D Analytically

4.2 Two dimensional and three dimensional figures of numerical solution and error

The following figures (3, 4) represent the electrostatic potential distribution in a unit circular disc with $Dirichlet's$ boundary condition including errors. We performed 25 iterations, as shown in Figure (3), and we can see that there is some errors. The error further decreases as we increase the number of iterations to 50, as shown in the figures (4). For the figure (4), we performed 50 iterations, and the electric charge distribution in a unit circular disc is nearly identical to the analytical distribution shown in figure (2), with an error that goes toward zero[3, 7, 4, 12].

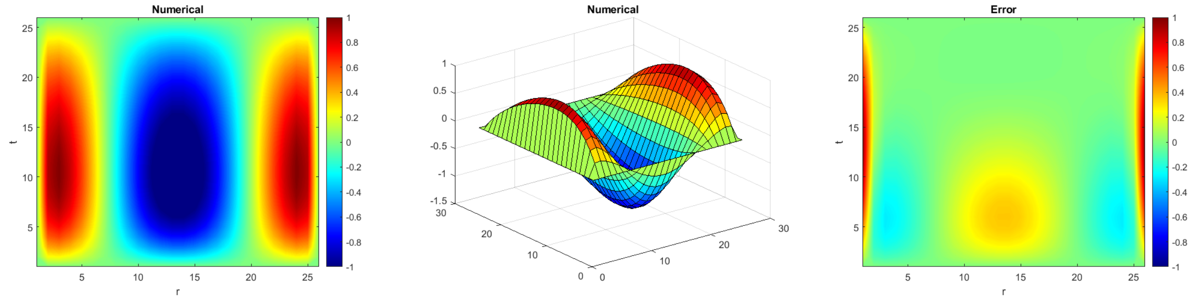

Figure 3: Potential Distribution With Error in a Unit Circular Disc of Iteration 25

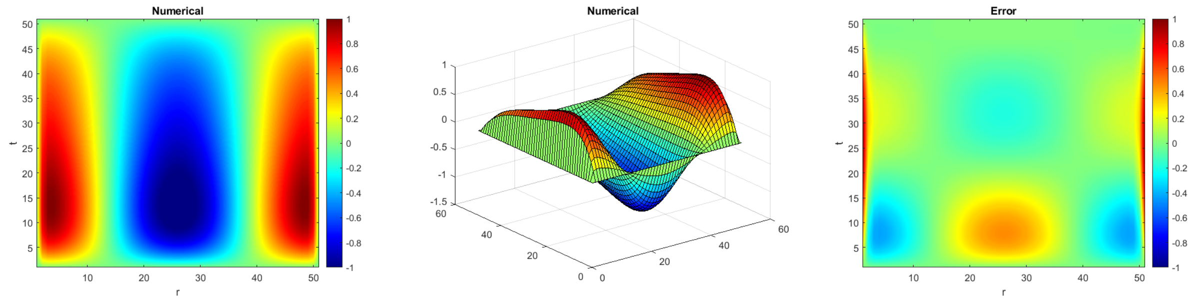

Figure 4: Potential Distribution With Error in a Unit Circular Disc of Iteration 50

5 Conclusions

In real life, we are facing many circular phenomena so that electrostatic potential distribution inside a unit circle plays a leading role in the field of application. Because of the several complicated theories involved, including the variable separation method, the $Sturm-Liouville$ equation, and the $Cauchy-Euler$ equation, finding an analytical solution to $Poisson's$ equation in a polar coordinate system is not only challenging but also time-consuming. In contrast, the numerical approach is fast and simple for the same problem. For the numerical solution of the given model problem in this study, which is the polar coordinate form of the Poisson's equation inside a unit circular disc with the Dirichlet boundary condition, we used the Gauss-elimination procedure in $MATLAB$. We identify the errors at each node by comparing the analytical solution with the corresponding numerical solution. Actually, we did not find minimum errors, we took $i = 0, 1, 2,\ j = 0, 1, 2, ..., 7$. But we are confident that if we increase the numbers of iterations, surely absolute error nearly tends to zero, as shown in figures (3) and (4). Hence, if increase the numbers of iterations sufficiently large, numerical solution of the given model problem is more accurate and sufficiently near to the exact solutions.

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