Nepal Journal of Mathematical Sciences (NJMS)
ISSN: **2738-9928 (*online*), 2738-9812 (*print*)**
Vol. 4, No. 1, 2023 (February): 11-20
DOI: https://doi.org/*10.3126/njmathsci.v4i1.53153*
©School of Mathematical Sciences,
Tribhuvan University, Kathmandu, Nepal

Research Article
Received Date: September 10, 2022
Accepted Date: February 9, 2023
Published Date: February 28,2023

Laplace Transform of Some Hypergeometric Functions

Madhav Prasad Poudel[1], Harsh Vardhan Harsh[2], Narayan Prasad Pahari[3]

[1]*School of Engineering, Central Campus, Pokhara University, Pokhara-30, Kaski, Nepal*
[2]*Faculty of Sci. & Tech., ICFAI Tech. School, ICFAI University Jaipur, Agra Road, Jamdoli, , Jaipur, Rajasthan , India*
[3]*Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal.*

E-mail: [1]*pdmadav@gmail.com,* [2]*harshvardhanharsh@gmail.com,* [3]*nppahari@gmail.com*

Corresponding Author: Madhav Prasad Poudel

Abstract: *The hypergeometric functions are one of the most important and special functions in mathematics. They are the generalization of the exponential functions. Particularly the ordinary hypergeometric function* ${}_2F_1(a,b;c;z)$ *is represented by hypergeometric series and is a solution to a second order differential equation. Similarly, Laplace transform is a form of integral transform that converts linear differential equations to algebraic equations. This paper aims to study the convergence of hypergeometric function and Laplace transform of some hypergeometric functions. Moreover, some relationships between Laplace transformation and hypergeometric functions is established in the concluding section of this paper.*

Keywords: *Hypergeometric function, Laplace transformation, Gamma function*

1. Introduction

Hypergeometric functions are one of the oldest transcendental functions. Normally exponential functions are generalized in terms of hypergeometric functions. They can be manipulated analytically as well [1].The hypergeometric series plays a significant role in the number system, partition theory, graph theory, Lie algebra, etc. [10]. According to Rao [9], John Wallis (1616-1703) extended the ordinary geometric series

$$1 + x^2 + x^3 + x^4 + x^5 + x^6 + ... \quad (1)$$

The above expression can be expressed in the Hommer's series[9] of the form

$$1 + x(1 + x(1 + x(1 + x) + ...))\,) \quad (2)$$

to the hypergeometric series of the form

$$1 + a + a(a+b) + a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) + ... \quad (3)$$

Whose nth term is given by

$$(a)_0 = 1 + a + a(a+b) + a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) + ... \quad (4)$$

At b =1, the representation (4) can be written in the form

$$(a)_0 = 1 + a + a(a+1) + a(a+b)(a+2) + a(a+b)(a+2)(a+3) + ...$$

$$= \prod_{k=1}^{n} (a+k-1) \tag{5}$$

For *n* is a non-negative integer. The equation (5) is called the Pochhammer function (see.[10],[13]). In 1707-83 Leonhard Euler introduced the power series expansion of the form

$$1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + ... \tag{6}$$

The series (6) Can be represented in the Hommer's form as

$$1 + \frac{ab}{c}\frac{z}{1}\left[1 + \frac{(a+1)(b+1)z}{(c+1)2}\left[1 + \frac{(a+2)(b+2)z}{(c+2)3} ....\right]\right] \tag{7}$$

The Pochhamer symbol (5) can be written in terms of gamma function as

$$(a)_n = a(a+1)(a+2)...........\frac{\Gamma(a+n)}{\Gamma(a)_n}; (a)_0 = 1 \tag{8}$$

The equation (6) is called the hypergeometric function with two numerator parameters a, and b and the denominator parameter c. This can be expressed as

$$\text{F(a, b; c; z)} = {}_2F_1\left[\begin{matrix} a & b; & \\ & c; & z \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!}$$

$$= 1 + \frac{ab}{c}\frac{z}{1}\left[1 + \frac{(a+1)(b+1)}{(c+1)}\left[1 + \frac{(a+2)(b+2)}{(c+2)} ....\right]\right] \tag{9}$$

or equivalently[4],

$${}_2F_1\left[\begin{matrix} a,b & \\ c & ;1 \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{10}$$

Hypergeometric functions are not only expressed as the Euler Hypergeometric functions but are also the solutions of the second order differential equation

$$z(1-z)\frac{d^2y}{dz^2} + [c-(a+b+1)z]\frac{dy}{dz} - aby = 0 \tag{11}$$

In term of operator the above equation can be expressed as

$$[\theta(\theta + c - 1) - z(\theta + a)(\theta + b)]w = 0 \tag{12}$$

where $\theta = z\frac{d}{dz}$. The series is convergent if $|z| < 1$ and divergent for $|z| > 1$ and z =1 for $\boldsymbol{R}(c-a-b) > 0$

Some classical summation for the hypergeometric series and generalized hypergeometric series as mentioned by Rainville [8] are as follows:

Gauss's theorem

$${}_2F_1\left[\begin{matrix} a, & b; & \\ & c; & 1 \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{13}$$

Kummer's theorem

$$ {}_2F_1\left[\begin{matrix} a, & b; & -1 \\ & 1+a-b; & \end{matrix}\right] = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)} \tag{14} $$

Gauss's second theorem

$$ {}_2F_1\left[\begin{matrix} a, & b; & \frac{1}{2} \\ & \frac{1}{2}(1+a+b); & \end{matrix}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left[\frac{1}{2}a+\frac{1}{2}\right]\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{15} $$

In 1908, Barn used the generalized hypergeometric function with p numerator parameters and q denominator (p and q) being non negative integers) with the notation of the form. (see,[8][11])

$$ {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} ;z\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{n!(b_1)_n\ldots(b_q)_n}z^n = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{n!\prod_{j=1}^{q}(b_j)_n}z^n \tag{16} $$

is convergent for all $|z| < \infty$ if $p \le q$ and for $|z| < 1$ if $p = q + 1$ while it is divergent for all $z \neq 0$ if $p > q + 1$. When $|z| = 1$ with $p = q + 1$, the series (5) converges absolutely if

$$ \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0 \tag{17} $$

Where p and q are integers in the application such that $q=p+1$.

2. Laplace Transformation of Hypergeometric Functions

Transform theory in mathematics relates a function in a domain to the other function in the next domain. The transformation is done to yield the mathematical solution of the complex problems from a simple function. To start with the transformation, it is known that

$$ (1-z)^{-a} = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{n!} \tag{18} $$

Now

$$ \begin{aligned} (1-z)^{-a}\,{}_2F_1\left[\begin{matrix} a, & c-b; & -\frac{z}{1-z} \\ & c; & \end{matrix}\right] &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a)_k(c-b)_k(-1)^k z^k}{(c)_k(1-z)^{k+a}k!} \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_k(c-b)_k(a+k)_n(-1)^k z^k}{(c)_n k!n!} \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{n+k}(c-b)_k(-1)^k z^{n=k}}{(c)_n k!n!} \end{aligned} \tag{19} $$

Since $(a)_k(a+k)_n = (a)_{n+k}$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(a)_n(c-b)_n(-1)^k z^{n+k}}{(c)_n k!(n-k)!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-n)_k(c-b)_n}{(c)_n k!}\frac{(a)_n z!}{n!} \tag{20}$$

The sum on the right hand side of (20) is a terminating hypergeometric series so from (13) we get,

$$(1-z)^{-a}{}_2F_1\left[\begin{matrix}a & c-b; & -\dfrac{z}{1-z}\\ & c & \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{\Gamma(c)\Gamma(b+n)(a)_n z^n}{\Gamma(c+n)\Gamma(b)n!}$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n n!}=F(a,b;c;z) \tag{21}$$

Definition 2.1 : Let $f(t)$ be the function of a variable t which is defined for all value of t. Then the Laplace transform of $f(t)$ is defined as

$$\mathcal{L}[f(t)]=F(s)=\int_0^{\infty}e^{-st}f(t)dt \tag{22}$$

provided that the integral exists in the Lebesgue sense. In this case, the inverse of Laplace transform is represented as $\mathcal{L}^{1}[F(s)]=f(t)$. Laplace transformation is used to solve the ordinary differential equations having the constant coefficients. Laplace transforms are tested and evaluated according to the criteria of application of the problems to various types of functions for numerical accuracy, computational efficacy and ease of programming and implementation [6]

Definition 2.2 : The Laplace Transform, in terms of Gamma function, is expressed as

$$\int_0^{\infty}e^{-st}t^{\alpha-1}dt=\Gamma(\alpha)s^{-\alpha} \tag{23}$$

provided that $\boldsymbol{R}(s)>0$ and $\boldsymbol{R}(\alpha)>0$.

Many researches are done to find out the Laplace transformation of the hypergeometric functions. Herz used a Laplace transformation (and inverse Laplace transformation) to transform from ${}_pF_q$ to ${}_{p+1}F_q$ hypergeometric functions. Rathie studied the Laplace transform for the generalized hypergeometric function ${}_2F_2$ and ${}_3F_3$ (see, [6],[7]). The convergence of the hypergeometric series, Laplace transforms and their relations are briefly introduced/ reviewed in this paper.

3. Main Results

The hypergeometric series in terms of power series is given in (2) then the nth term is given by

$$a_n=\frac{a(a+1)...(a+n-1)b(b+1)...(b+n-1)}{1.2.3....n.c(c+1)...(c+n-1)}z^n \tag{24}$$

Then the $(n+1)$th term is given by

$$a_{n+1}=\frac{a(a+1)...(a+n-1)b(b+1)...(b+n)}{1.2.3....n.c(c+1)...(c+n)}z^{n+1} \tag{25}$$

For the test of convergence, using De Alembert's ratio test[5] from (24) and (25)

$$\frac{a_n}{a_{n+1}} = \frac{1}{(a+n)(b+n)}(c+1)(c+n).\frac{1}{z} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}.\frac{1}{z} \to \frac{1}{z} \text{ as } n \to \infty \quad (26)$$

The series $\sum_{n=0}^{\infty} a_n$ converges if 0 < z < 1 and diverges as $z > 1$

If $z = 1$, then from (26), we get

$$\frac{a_n}{a_{n+1}} = \frac{1}{(a+n)(b+n)}(c+1)(c+n). = \frac{n^2+(c+1)n+c}{n^2+(a+b)n+ab} = 1+\frac{c+1-a-b}{n}+\frac{\beta_n}{n^2} \quad (27)$$

where $\beta_n = \dfrac{A+\frac{B}{n}}{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}, A = c-ab-(a+b)(c+1-a-b)$ and $B = -ab(c+1-a-b)$

Thus $\beta_n \to A$ and β_n is bounded. Hence if n =1 the hypergeometric series converges if $(c+1-a-b) > 1$ i. e. if $c > a + b$ and diverges if $c \le 1$, Otherwise

$$\frac{a_n}{a_{n+1}} = \left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)\left(1+\frac{a}{n}\right)^{-1}\left(1+\frac{b}{n}\right)^{-1}$$

$$= \left\{1+\frac{c+1}{n}+O\left(\frac{1}{n^2}\right)\right\}\left\{1-\frac{a}{n}+O\left(\frac{1}{n^2}\right)\right\}\left\{1-\frac{b}{n}+O\left(\frac{1}{n^2}\right)\right\}$$

where O represents the order notation.

$$= \left\{1+\frac{c+1}{n}+O\left(\frac{1}{n^2}\right)\right\}.\left\{1-\frac{a+b}{n}+O\left(\frac{1}{n^2}\right)\right\} = 1+\frac{c+1-a-b}{n}+O\left(\frac{1}{n^2}\right) \quad (28)$$

Hence $\sum_{n=0}^{\infty} a_n$ converges if $c + 1 - a - b > 1$ and diverges if $c + 1 - a - b \le 1$.

3.1 The convergence of Laplace transform

For a complex exponential function $f(t) = e^{-at}\mu(t)$, for $|\mu(t)| = 1$ we have

$$F(s) = \int_0^{\infty} e^{-at}e^{-st}dt = \int_0^{\infty} e^{-(a+s)t}e^{-st}dt = -\frac{1}{s+a}\left[\lim_{t\to\infty} e^{-(s+a)t} - 1\right] \quad (29)$$

The relation (20) will tend to infinity when $\lim_{t\to\infty} e^{-(s+a)t}$ tends to infinity.

Let $s = \sigma + jw$. Then $\lim_{t\to\infty} e^{-(s+a)t} = \lim_{t\to\infty} e^{-(\sigma+jw+a)t} = \lim_{t\to\infty} e^{-(\sigma+a)t}.e^{-jwt}$ (30)

Here e^{-jwt} is sinusoidal $\sigma + a$ is positive. The exponential will be a negative power which will tend to infinity cause the function tends to zero as t tends to infinity If $\sigma + a$ is negative or zero, the exponential will not be a negative power which will prevent it from tending to zero and the system will not converge. So the condition of convergence is $Re(s) > -a$ and the condition of anti-casual convergence is $Re(s) < -a$.

3.2 Laplace transforms of some Hypergeometric functions

Now we will present the results involving the hypergeometric function and the Laplace transforms.

Now using the Laplace transform in (10), Rathie[12] has obtained the results which are given below.

$$\int_0^\infty e^{-st}t^{v-1}\,{}_pF_q\left[\begin{matrix} a_1,a_2,\ldots..a_{p;} \\ b_1,b_2,\ldots.a_{q;} \end{matrix}\; wt\right]=\Gamma(v)s^{-v}\,{}_{p+1}F_q\left[\begin{matrix} v & a_1,a_2,\ldots..a_{p;} & \frac{w}{s} \\ & b_1,b_2,\ldots.a_{q;} & \end{matrix}\right] \tag{31}$$

provided that (i) *If* $p < q$, $\boldsymbol{R}(v) > 0$, $\boldsymbol{R}(s) > \boldsymbol{R}(w)$ and w is the arbitrary or

(ii) $p = q > 0$ $\boldsymbol{R}(v) > 0$, *and* $\boldsymbol{R}(s) > \boldsymbol{R}(w)$

Especially (iii) *If* $p = q > 0$, $s = w$, $\boldsymbol{R}(v) > 0$, $\boldsymbol{R}(s) > 0$ *and* $R(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j - v) > 0$

Laplace transform for Kummer's confluent Function $_1F_1$

In particular if $p = q = 1$ in (31), The Laplace transformation of the Kummer's confluent hypergeometric function $_1F_1$, is

$$\int_0^\infty e^{-st}t^{v-1}\,{}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix}\; wt\right]=\Gamma(v)s^{-v}\,{}_2F_1\left[\begin{matrix} a, & b; & \frac{w}{s} \\ & c; & \end{matrix}\right] \tag{32}$$

provided that (i) $\boldsymbol{R}(b) > 0$, $\boldsymbol{R}(s) > max\{\boldsymbol{R}(w); 0\}$ and w is the arbitrary or

(ii) $s = w$, $\boldsymbol{R}$ (s) > 0, and $\boldsymbol{R}(c - a - b) > 0$

Laplace transform for generalized Hypergeometric Function $_2F_2$

When $p = q = 2$ in (22) then the Laplace transformation of the generalized hypergeometric function is given by

$$\int_0^\infty e^{-st}t^{v-1}\,{}_2F_2\left[\begin{matrix} a_1, & a_2; \\ b_1, & b_2; \end{matrix}\; wt\right]=\Gamma(v)s^{-v}\,{}_3F_2\left[\begin{matrix} v, & a_1, & a_{2;} & \frac{w}{s} \\ & b_{1,} & b_{2;} & \end{matrix}\right] \tag{33}$$

Provided that (i) $\boldsymbol{R}(v) > 0$, *and* $\boldsymbol{R}(s) > max\{\boldsymbol{R}(w); 0\}$ and w is the arbitrary or

(ii) $s = w$, $\boldsymbol{R}(s) > 0$, *and* $\boldsymbol{R}(b_1+b_2-a_1-a_2-v) > 0$.

Laplace transform for generalized Hypergeometric Function $_3F_3$

Now replacing $p = q = 3$ in (22) then the Laplace transformation of the generalized hypergeometric function is given by

$$\int_0^\infty e^{-st}t^{v-1}\,{}_3F_3\left[\begin{matrix} a_{1,} & a_{2,} & a_{3;} \\ b_1, & b_2, & b_{3;} \end{matrix}\; wt\right]dt=\Gamma(v)s^{-v}\,{}_4F_3\left[\begin{matrix} v, & a_1, & a_2\,a_{3;} & \frac{w}{s} \\ & b_{1,} & b_{2,}\,b_3; & \end{matrix}\right] \tag{34}$$

provided that (i) $\boldsymbol{R}(v) > 0$, *and* $\boldsymbol{R}(s) > max\{\boldsymbol{R}(w); 0\}$ and w is the arbitrary or

(ii) $s = w$, $\boldsymbol{R}(s) > 0$, *and* $\boldsymbol{R}(b_1+b_2\,b_3-a_1-a_2-a_3-v) > 0$.

By using these theorems Kim et.al [12] has obtained a large number of Laplace transform s for the confluent hypergeometric functions $_1F_1$ and generalized Hypergeometric function $_2F_2$. These are listed as follows;

Laplace transform for Guass Theoem for Hypergeometric Function $_1F_1$

$$\int_0^\infty e^{-st} t^{b-1} {}_1F_1\left[\begin{matrix} a; & \\ \frac{1}{2}(a+b+c); & \frac{1}{2}wt \end{matrix}\right] dt = s^{-b} \frac{\Gamma(b)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{35}$$

provided that $\boldsymbol{R}(v) > 0$, *and* $\boldsymbol{R}(s) > 0$,

$$\text{and} \int_0^\infty e^{-st} t^{-a} {}_1F_1\left[\begin{matrix} a; & \frac{1}{2}ts \\ c; & \end{matrix}\right] dt = s^{a-1} \frac{\Gamma(1-a)\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)} \tag{36}$$

provided that $\boldsymbol{R}(1-a) > 0$, *and* $\boldsymbol{R}(s) > 0$

Laplace transform for Kummer Function $_1F_1$

$$\int_0^\infty e^{-st} t^{-a} {}_1F_1\left[\begin{matrix} a; & \\ 1+a-b; & -ts \end{matrix}\right] dt = s^{-b} 2^{-a} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma(1+a-b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-b\right)} \tag{37}$$

provided that $\boldsymbol{R}(b) > 0$, *and* $\boldsymbol{R}(s) > 0$

Laplace Transform for $_2F_1$ (*a*, b; c ; –t)

The Laplace transformation for the Hypergeometric series $_2F_1$ as mentioned by Weinstein[14], is

$$L\left[{}_2F_1\left[\begin{matrix} a & b & \\ & c & -t \end{matrix}\right]\right](z) = \frac{\Gamma(1-a)\Gamma(b-a)\Gamma(c)z^{a-1}}{\Gamma(b)\Gamma(c-a)} {}_1F_1\left[\begin{matrix} a-c; & \\ a-b+1; & z \end{matrix}\right]$$

$$+\frac{\Gamma(1-b)\Gamma(a-b)\Gamma(c)z^{b-1}}{\Gamma(a)\Gamma(c-b)} {}_1F_1\left[\begin{matrix} b-c; & \\ 1-b+a; & z \end{matrix}\right]$$

$$+\frac{c-1}{(a-1)(b-1)} {}_2F_2\left[\begin{matrix} 1 & 2-c; & \\ 2-a & 2-b; & z \end{matrix}\right]; / \operatorname{Re}(z) > 0 \tag{38}$$

Laplace Transform of $_2F_1$(*a, b*; c ; t)in terms of Gamma function

From definition,

$${}_2F_1\left[\begin{matrix} a & b; & \\ & c; & t \end{matrix}\right] = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}$$

Then $$\int_0^\infty e^{-st} t^{\gamma-1} F(a,b;c;t) dt = \int_0^\infty e^{-st} t^{\gamma-1} \left[\sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}\right] dt = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{n!(c)_n} \left[\int_0^\infty e^{-st} t^{n+\gamma-1} dt\right]$$

$$= \sum_{n=0}^\infty \frac{(a)_n (b)_n}{n!(c)_n} \frac{\Gamma(n+\gamma)}{s^{n+\gamma}} \tag{39}$$

$$\text{where} \left[\int_0^\infty e^{-st} t^{n+\gamma-1} dt\right] = \frac{\Gamma(n+\gamma)}{s^{n+\gamma}}$$

This relation is valid if $n+\gamma-1 > -1$.

Laplace transform on Generalized Hypergeometric Function

The generalized form of the hypergeometric function ${}_pF_q$ is

$$ {}_pF_q\left[\begin{matrix}(a_1)_n(a_2)_n...............(a_p)_n\\(b_1)_n(b_2)_n...............(b_p)_n\end{matrix}\right]\frac{t^n}{n!} \tag{40}$$

Now replacing $p = 3$, $q = 1$, $a_1 = a$, $a_2 = b$, $a_3 = \gamma$, $b_1 = c$, $t = \frac{1}{s}$, then the above series(30)reduces to

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}\frac{\Gamma(n+\gamma)}{s^{n+\gamma}} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+\gamma)\Gamma(\gamma)}{n!\Gamma(c+n)\Gamma(\gamma)s^{n+\gamma}}$$

$$= \frac{\Gamma(c)\Gamma(\gamma)}{\Gamma(a)\Gamma(b)}s^{\gamma}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{n!\Gamma(c+n)s^n}$$

$$= \frac{\Gamma(c)\Gamma(\gamma)}{\Gamma(a)\Gamma(b)}s^{\gamma}\,{}_3F_1(a,b,\gamma;c;\frac{1}{s}) \tag{41}$$

Laplace transform for Guass Function ${}_2F_2$

The Laplace Transform for the Guass function [13] is

$$\int_0^{\infty} e^{-st}t^{c-1}\,{}_2F_2\left[\begin{matrix}a & b; & \\ \frac{1}{2}(a+b+1); & 2c; & ts\end{matrix}\right]dt$$

$$= s^{-c}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}b+\frac{1}{2}\right)} \tag{42}$$

provided that (i) $\boldsymbol{R}(c) > 0$, and $\boldsymbol{R}(s) > 0$

(*ii*) $\boldsymbol{R}(2c-a-b)) > -1$.

Also, $\int_0^{\infty} e^{-st}t^{c-1}\,{}_2F_2\left[\begin{matrix}a & b; & \\ (1+a+b); & 1+a-c; & ts\end{matrix}\right]dt$

$$= s^{-c}\frac{\Gamma(c)\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)} \tag{43}$$

provided that (i) $\boldsymbol{R}(c) > 0$, *and* $\boldsymbol{R}(s) > 0$ (*ii*) $\boldsymbol{R}(a-2b-2c) > -2$

Also,

$$\int_0^{\infty} e^{-st}t^{c-1}\,{}_2F_2\left[\begin{matrix}a & b; & \\ d; & e; & st\end{matrix}\right]dt = s^{-c}\frac{\pi\Gamma(c)\Gamma(d)\Gamma(e)}{2^{2c-1}\Gamma\left(\frac{1}{2}a+\frac{1}{2}d\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}d\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}e\right)} \tag{44}$$

provided that(i) $\boldsymbol{R}(c) > 0$, *and* $\boldsymbol{R}(s) > 0$

(*ii*) $1+b=1$, $d + e=1+2c$

4. Application of Shifting Theorems

Shifting theorem of Laplace transform on $_2F_1$

The Laplace transform of a generalized hypergeometric function is

$$\mathcal{L}\left[{}_2F_1\left[\begin{matrix} a & b; \\ & 1 \end{matrix} \quad z(1-e)^{-t}\right]\right] = \frac{1}{s}F\left[\begin{matrix} a & b; \\ & s+1; \end{matrix} \quad z\right] \tag{45}$$

$$\text{Let A}= \frac{1}{s}F\left[\begin{matrix} a & b; \\ 1+s; & \end{matrix}; \quad z\right] = \frac{1}{s}\sum_{n=o}^{\infty}\frac{(a)_n(b)_n z^n}{(s+1)_n n!} = \frac{1}{s}\sum_{n=o}^{\infty}\frac{(a)_n(b)_n z^n}{n!}\frac{1}{s(1+s)_n} \tag{46}$$

Taking Inverse Laplace transform to both sides

$$\mathcal{L}^{-1}(\text{A}) = \mathcal{L}^{-1}\left[\frac{1}{s}\sum_{n=o}^{\infty}\frac{(a)_n(b)_n z^n}{n!}\frac{1}{s(1+s)_n}\right] \tag{44}$$

Now for
$$\frac{1}{s(s+1)_n} = \frac{\Gamma(1+s)\Gamma(1+n)}{s.n!\Gamma(1+s+n)\Gamma(1)} = \frac{1}{n!s}F\left[\begin{matrix} -n & s; \\ 1+s; & \end{matrix} \quad 1\right] = \frac{1}{s.n!}\sum_{k=0}^{n}\frac{(-n)_n(s)_k}{k!(1+s)_k}$$

implies
$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)_n}\right] = \frac{1}{n!}\sum_{k=0}^{n}\frac{(n)_k e^{-kt}}{k!} = \frac{1}{n!}(1-e^{-t})^n \tag{45}$$

Therefore from (44) and (45)

$$L\left\{{}_2F_1\left[\begin{matrix} a & b \\ & 1 \end{matrix} \quad z(1-e)^{-t}\right]\right\} = \frac{1}{s}F\left[\begin{matrix} a & b; \\ & s+1; \end{matrix} \quad z\right] \tag{46}$$

Laplace Transform of $t^n \sin at$ **in terms of Hypergeometric series**

Here
$$\sin at = \sum_{k=0}^{\infty}\frac{(-1)^k a^{2k+1}t^{2k+1}}{(2k+1)!} \tag{50}$$

Then from (50)
$$t^n \sin at = \sum_{k=0}^{\infty}\frac{(-1)^k a^{2k+1}t^{n+2k+1}}{(2k+1)!}$$

We know,
$$\mathcal{L}(t^m) = \frac{\Gamma(m+1)}{s^{m+1}}$$

$$\text{Hence } \mathcal{L}(t^m \sin at) = \sum_{k=0}^{\infty}\frac{(-1)^k a^{2k+1}\Gamma(n+2k+2)}{(2k+1)!S^{n+2k+2}} = \frac{a\Gamma(n+2)}{s^{n+2}}\sum_{k=0}^{\infty}\frac{(-1)^k(n+2)_{2k}a^{2k}}{(2)_{2k}s}$$

$$= \frac{a\Gamma(n+2)}{s^{n+2}}\sum_{k=0}^{\infty}\frac{(-1)^k\left[\frac{n+2}{2}\right]_k\left[\frac{n+3}{2}\right]_k a^{2k}}{k!\left[\frac{3}{2}\right]s^{2k}}$$

$$= \frac{a\Gamma(n+2)}{s^{n+2}}F\left[\begin{matrix} \frac{n+2}{2} & \frac{n+3}{2}; \\ & \frac{3}{2}; \end{matrix} \quad -\frac{a^2}{s^2}\right]$$

$$\text{Hence, } \mathcal{L}(t^n \sin at) = \left[\frac{a\Gamma(n+2)}{s^{n+2}}\right]{}_2F_1\left[\begin{matrix} 1+\frac{n}{2} & \frac{3}{2}+\frac{n}{2}; \\ & \frac{3}{2}; \end{matrix} \quad -\frac{a^2}{s^2}\right] \tag{51}$$

5. Conclusion:

This paper aims to establish the Laplace transformation of some hypergeometric functions. The nature of convergence of Laplace Transform and Hypergeometric function is observed in 3.1.The Laplace transform of Hypergeometric series of Kummer's confluent function ${}_1F_1$, Generalized Hypergeometric Function ${}_2F_2$ and ${}_3F_3$, Kummer's Hypergeometric function ${}_1F_1$ and ${}_2F_1$ in 3.2.It has also shown the relationship between Laplace transform and Hypergeometric function. The application of first shifting theorem of Laplace transform to Hypergeometric Function ${}_2F_1$ and the Laplace transform of $t^n \sin at$ are well illustrated in section 4. The list of formulae presented here are applicable in Mathematics, Engineering, Biology and Applied Physics.

References

[1]. Abadir, K. M. **(1999).** An introduction to hypergeometric functions for economists. *Econometric Reviews*, 1*8*(3), 287-330.

[2]. Aomoto, K., Kita, M., Kohno, T., & Lohara, K. **(2011).** *Theory of hypergeometric functions*. Tokyo: Springer.

[3]. Davies, B., & Martin, B. **(1979).** Numerical inversion of the Laplace transform: a survey and comparison of methods. *Journal of computational physics*, 33(1), 1-32.

[4]. Deepthi, P., Prajapati, J. C., & Rathie, A. K. **(2017).** New Laplace Transforms For The 2F2 Generalised Hypergeometric Functions. *Journal of Fractional Calculus and Applications*, 8(2), 150-155.

[5].Ghosh, R. K. and Maity, K. C.**(1993).** An Introduction to Analysis and differential Calculus, Part I. New Central Book Agency (p) Ltd. Calcutta, India

[6].Kim, Y. S., & Lee, C. H. **(2017).** Evaluation of Some NEW Laplace Transform For The Generalized Hypergeometric Function pFp. *Honam Mathematical Journal*, 39(1), 73-82.

[7]. Kim, Y. S., Rathie, A. K., & Cvijović, D. **(2012).** New Laplace transforms of Kummer's confluent hypergeometric functions. *Mathematical and Computer Modelling*, 55(3-4), 1068-1071.

[8].Rainville, E.D.**(1960).**Special Functions, Macmillan, new York , Reprinted by Chelsea Publishing, Bronx, new York

[9].Rao, K. S., & Lakshmi Narayanan, V. **(2018).** *Generalized hypergeometric functions*. IOP Publishing.

[10].Sharma, B. R.,& Malla,K. R.**(2020).**On a certain transformation and summation Formulae involving hypergeometric series*, Gandaki Journal of Mathematics,* 1*,*27-40

[11].Slater, L. J.**(1966)**.Generalized Hypergeometric Functions, Cambridge University Press. Cambridge.

[12].Wang, X., & Rathie, A. K. **(2015).** New Laplace transforms for the generalized hypergeometric functions 2F2 and 3F3. *arXiv preprint arXiv:*1505.07134.

[13]. Weinstein, E. W. **(2008).** *q*-Pochhammer symbol. *https://mathworld. wolfram. com/*.

[14].Weinstein, E. W. **(2008).***https://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F*1/