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A Study of Tsirelson Space

Bhuwan Prasad Ojha[1] and Dhruba Raj Sigdel[2]

[1] *Department of Mathematics, Apex College, Kathmandu, Nepal*

[2]*Department of Mathematics, Amrit Science Campus, Tribhuvan University, Kathmandu, Nepal*

Email: [1]*bhuwanp.ojha@apexcollege.edu.np*, [2]*drsigdelpentagon@gmail.com*

Abstract: This paper is a short review of some results in Tsirelson space. The concept of the paper is to include some well-known theorems on Tsirelson's space constructed by Figiel and Johnson together with results concerning holomorphic functions on the classical and modified mixed Tsirelson spaces.

Key Words: Tsirelson Space, Banach Space, Holomorphic Germs, Block Basic Sequence

AMS (MOS) Subject Classification: 46B25, 46E28.

1. Introduction

The first example of Banach space containing no subspaces isomorphic to any space of the convergent sequence or the space of all sequences converging to zero was introduced by B.S. Tsirel'son in 1974 [13]. He constructed reflexive Banach spaces which do not contain any infinite-dimensional subspace– linear and homeomorphic to a uniformly convex space. Figiel and Johnson in the paper [8] proved that " a super reflexive space with local unconditional structure can be equivalently normed in such a way that: the modulus of convexity of that space is of power type". To construct uniformly convex spaces, the convexity procedure to the norm of Tsirelson's space X was applied by Figiel and Johnson in the work [8].

Casazza and Shura in [3] studied a special behavior of block basic sequences in Tsirelson's space. They proved that "every bounded block sequence of $\{s_n\}$ spans a complemented subspace of X and it is equivalent to a subspace of $\{s_n\}$". This behavior is also compared with the statement that " a basis $\{y_n\}$ which is equivalent to its block basic sequence must be equivalent to the canonical unit vector basis of c_0 or the space l_p". Johnson in [9] gave an example of a reflexive complete normed linear space with an unconditional basis but that is not sufficiently Euclidean and this concept was studied by Casazza and shura in [4] by showing that Tsirelson's space is naturally isomorphic to the modified space, denoted by X_M.

The concept of isomorphism between subspace of Tsirelson's space spanned by sub-sequences of $\{s_n\}$ discussed in [5] and proved that "two sub-spaces of $\{s_n\}$ span isomorphic sub-space of the Tsirelson's space if and only if sub-sequences are equivalent. The unit vector basis of Tsirelson's space and its permutation discussed in [6] and two sub-sequences of any sequence are permutatively equivalent if and only if they are equivalent. For the complemented subspace of Tsirelson's space, a concept of unconditional basis was discussed in [12] by showing that every normalized unconditional basis for Tsirelson's space X has a sub-sequence which is permutatively equivalent to the original sequence. A uniformly convex complete normed linear space containing no isomorphic copies of the space of all sequences converging to zero or the space of convergent sequences was developed by Figiel and Johnson, as an example of Tsirelson-like space can be found in [8].

2. Introductory Results

An example of a complete normed linear space that contains no subspace isomorphic to c_0 or $l_p; 1 \leq p < \infty$ is called the original Tsirelson's space, which we denote by X and X^* denotes the dual of Tsirelson's space. Casazza et al. [7] proved that if every infinite-dimensional subspace of X contains in turn an isometric copy of X, then X is minimal. It is also proved that every quotient space of X embeds into X and different relations in relation to the structure of X^*, X, and related spaces are mentioned in the paper [7].

Definition 2.1. [7] Let $\mathbb{M}$ be a compact family and $1 < \lambda < 1$. The Tsirelson-type space is denoted by $X_{\mathbb{M}}^{\lambda}$ is the completion of c_{00} under the norm

$$\|x\| = \max\{\|x\|_\infty, \lambda \sup \sum_{j=1}^{k} \|F_j x\|\},$$

where the supremum is taken over all $\mathbb{M}-$admissible families $\{F_1, F_2, ..., F_k\}$.

Theorem 2.2. [7] *For any integer n and a vector $x \in X$, either*

$$\|x\|_{n+1} = \sup\{\frac{1}{2} \sum_{i=1}^{k} \|S_i x\|; k \leq S_1 < S_2 < ... < S_k, k = 1, 2, 3, ...\}$$

$$or \quad \|x\|_{n+1} = \|x\|_0$$

Lemma 2.3. [7] *Let $z_n = \sum_{j=q_n+1}^{q_{n+1}} b_j s_j; n = 1, 2, 3, ...$ be normalized block basic sequence of $\{s_n\}$. Then for any sequence of scalars $\{a_n\}$, we have*

$$\|\sum_{n=1}^{\infty} a_n s_{q_n+1}\| \leq \|\sum_{n=1}^{\infty} a_n z_n\|.$$

Proposition 2.4. [7] *Let $z_n = \sum_{j=q_n+1}^{q_n} b_j s_j; n = 1, 2, 3, ...$ be a normalized block basic sequence of $\{s_n\}$ in the dual of Tsirelson's original space X. Then for any choice of integers $q_n < k_n \leq q_{n+1}; n = 1, 2, 3, ...$ and scalars $\{a_n\}_{n=1}^{\infty}$, we have*

$$\frac{1}{3}\|\sum_{n=1}^{\infty} \|s_n x\| s_{k_n}\| \leq \|\sum_{n=1}^{\infty} a_n z_n\| \leq 18\|\sum_{n=1}^{\infty} a_n s_{n_k}\|, x \in X.$$

Corollary 2.5. [7] *Every block basic sequence of $\{s_n\}$ spans a complemented subspace of the dual of the original Tsirelson's space.*

Theorem 2.6 (The Blocking Principle)**.** [7] *Let $\{M_n\}_{n=1}^{\infty}$ be finite dimensional decomposition of subspace Z of a quotient space of a complete normed linear space Y, and assume that Y has an finite dimensional decomposition $\{B_n\}_{n=1}^{\infty}$ which is type X (resp.of type X^*). Them $\{M_n\}$ has a blocking $\{F_i\}_{i=1}^{\infty}$ which is a finite dimensional decomposition of type X (resp., of type X^*).*

3. Tsirelson's space contructed by Figiel and Johnson

Figiel and Johnson's construction [3] for Tsirelson's space allows to develop some sequential principles from which it is better to study subspaces of the canonical unit vector basis, complemented sub-spaces, and operators of the Tsirelson's space.

Definition 3.1. [3] Let F_1 and F_2 be non-empty finite subsets of the set of positive integers. If $\max F_1 \leq \min F_2$, then we say $F_1 \leq F_2$. Let $\{s_n\}$ be canonical unit vector basis of $\mathbb{R}^{\mathbb{N}}$ and for any $F_1 \geq 1$ and $x = \sum_n b_n s_n$, define $F_1(x) = \sum_{x\in F_1} b_n s_n$. The norm function $\{\|.\|_m\}$ defined on the space of all real scalar sequences with finite support, which is called the sequence norm, defined as follows:
Setting $x = \sum_{n=1}^{\infty} b_n s_n \in \mathbb{R}^{\mathbb{N}}$ and by taking inner maximum over all choices of finite subsets $\{F_j\}_{j=1}^{k}$ of the set of natural numbers as k varies and such that $k \leq F_1 < F_2 < \ldots\ldots < F_k$,

$$\|x\|_{m+1} = \max\{\|x\|_m, \frac{1}{2}\max[\sum_{i=1}^{k} \|F_i(x)\|_m]\}$$

$$\text{and} \quad \|x\|_0 = \max_n \mid a_n \mid \qquad \text{where } m \geq 0.$$

Any expression of the form $\frac{1}{2}\sum_{i=1}^{k} \|F_i(x)\|_m$ is called an admissible sum for a vector x.

Based on above definition, the following facts are true [3]:

(i) The sequence $\{s_n\}_{n=1}^{\infty}$ form a normalized 1-unconditional Schauder basis for the Tsirelson's space X.

(ii) If the inner supremum is taken over all choices $k \leq F_1 < F_2 < \ldots\ldots.. < F_k$, and all k, then for each $\|y\| = \sum_n b_n s_n \in X$,

$$\|y\| = \max\{\max_n \mid b_n \mid, \frac{1}{2}\sup\sum_{i=1}^{k} \|F_i(x)\|\}$$

(iii) For any positive integer k, and any k nomalized blocks $\{z_i\}_{i=1}^{k}$ such that

$$z_i = \sum_{n=q_j+1}^{q_{j+1}} b_n s_n, \quad \text{with} \begin{cases} 1 \leq j \leq k \text{ and} \\ k-1 \leq q_1 < q_2 < \ldots\ldots < q_{k+1} \end{cases} \quad \text{we have}$$

$$\frac{1}{2}\sum_{j=1}^{k} \mid c_j \mid \leq \|\sum_{j=1}^{k} c_j z_j\| \leq \sum_{j=1}^{k} \mid c_j \mid \quad \text{for all scalars } \{c_j\}_{j=1}^{k}.$$

Proposition 3.2. [3] *For any non-negative integer m and a vector belonging to Tsirelson's space, either*

$$\|x\|_{m+1} = \sup\Big\{\frac{1}{2}\sum_{i=1}^{k}\|F_j x\|_m : k \le F_1 < F_2 < ...... < F_k, k = 1, 2, 3, ...\Big\}$$

$$or \quad \|x\|_{m+1} = \|x\|_0.$$

Proposition 3.3. [3] *Tsirelson's space contains no infinite dimensional uniformly convexifiable subspaces.*

Definition 3.4. [12] Let X be an infinite-dimensional complete normed linear space. Then X is said to be minimal if it embeds into each of its infinite-dimensional subspaces.

Definition 3.5. [12] Let X_1 and X_2 be infinite-dimensional complete normed linear spaces. Then X_1 and X_2 are said to be totally incomparable if neither of X_1 and X_2 contains an infinite-dimensional subspace that is isometric to a subspace of the other.

Theorem 3.6. [12] *Tsirelson's space contains no sub-symmetric basic sequences.*

4. Holomorphic Germs on Tsirelson's space

J. Mujica and M. Valdivia studied the holomorphic germs on Tsirelson's space in the paper [10]. If F is a complex Frechet space and $\mathbb{K}$ is a compact subspace of F and $\{V_n\}$ is any decreasing fundamental sequence of an open neighborhood of K and $\mathbb{H}^\infty\{V_n\}$ is the complete normed linear space of all bounded holomorphic functions defined on $\{V_n\}$. Then the space of holomorphic germs on $\mathbb{K}$ is defined as $\mathbb{H}(\mathbb{K}) = ind\mathbb{H}^\infty\{V_n\}$. It has been proved that the inductive limit is compact if and only if F is Frechet Schwartz space. Mujica and Valdivia also proved that the inductive limit is weakly compact for each compact subset of complete normed linear space constructed by Tsirel'son in [13].

Theorem 4.1. [10] *For a reflexive Banach space F with* $\mathbb{P}(F) = \mathbb{P}_\omega(F)$*, the inductive limit* $\mathbb{H}(\mathbb{K}) = ind\mathbb{H}^\infty\{V_n\}$ *is weakly compact. In particular, for each compact subset* $\mathbb{K}$ *of F,* $\mathbb{H}(\mathbb{K})$ *is totally reflexive.*

Theorem 4.2. [10] *For any reflexive complete normed linear space F and open set V of F with* $\mathbb{P}(F) = \mathbb{P}_\omega(F)$*, the space* $(\mathbb{H}(V), \rho_\omega)$ *is semi-reflexive.*

5. Modified Tsirelson's Space

In 1967, B.S. Johnson in the paper [2] introduced the concept of modified Tsirelson's space. After that Casazza and Odel, [1] proved that modified Tsirelson's space is isomorphic to the original Tsirelson's space. In general, a relation between mixed Tsirelson's norm and the modified Mixed Tsirelson's norm is different from the one between Tsirelson's space and mixed Tsirelson's space. Argyros et al.[1] studied the modified and boundedly modified mixed Tsirelson's spaces defined by sub-sequences of the sequence of Schreir families. These are l_1 asymptotic spaces with an unconditional basis $\{e_j\}$ having the property that " every sequence of normalized disjointly supported vectors obtained in $\{e_j\}$ is equivalent to the basis of l_1". By proving the space of all sequences converging to zero is finitely disjoint represent-able in every block subspace of $T\{(F_n, \beta_n)_{n=1}^{\infty}\}$, Argyros et al. proved that " if $\lim \beta_n^{\frac{1}{n}} = 1$, then the space $T\{(F_n, \beta_n)_{n=1}^{\infty}\}$ and its modified variations $T_M\{(F_n, \beta_n)_{n=1}^{\infty}\}$ or $T_{M_{(s)}}\{(F_n, \beta_n)_{n=1}^{\infty}\}$ are totally incomparable. They also give an example of boundedly modified mixed Tsirelson's space which is arbitrarily distortable.

Let $\{\mathbb{M}_k\}_{k=1}^{\infty}$ be the compact families of finite subset of the set of natural number and the sequence $\{\beta_k\}_{k=1}^{\infty}$ of real numbers converging to zero, the mixed Tsirelson's space $T\{(\mathbb{M}_k, \beta_k)_{k=1}^{\infty}\}$ is defined as follows:

Definition 5.1. [1] Let $T\{(\mathbb{M}_k, \beta_k)_{k=1}^{\infty}\}$ is completion of linear space c_{00} of the sequences which are eventually zero under the norm defined by the following implicit formula: For $y \in c_{00}$,

$$(5.1) \quad \|y\| = \Big\{\|y\|_\infty, \quad \sup_k \beta_k \sup\Big\{\sum_{j=1}^{n} \|F_j y\| : n \in \mathbb{Z}^+, (F_j)_{j=1}^{\infty} \quad \text{is} \quad \mathbb{M}_k - \text{admissible}\Big\}\Big\}$$

where $\|Fy\|$ is the restriction of the vector y on the set F and $m-$admissible sequence is a sequence $\{F_j\}_{j=1}^{n}$ of a successive subset of the set of a positive integer such that the set $\Big\{\min F_1, \min F_2, ..., \min F_n\Big\}$ belongs to $\mathfrak{M}$, where $\mathfrak{M}$ is the family of subsets of the set of natural numbers and $F \subseteq \mathbb{N}$. The modified mixed Tsirelson's space $\mathbb{Z}_{\mathfrak{M}}$ corresponding to the Mixed Tsirelson's space $\mathbb{Z} = T\Big\{(\mathfrak{M}_k, \beta_k)_{k=1}^{\infty}\Big\}$ is complete normed linear space whose norm satisfies the implicit equation

$$(5.2) \quad \|z\| = \Big\{\|z\|_\infty, \quad \sup_k \beta_k \sup\Big\{\sum_{j=1}^{n} \|F_j z\| : n \in \mathbb{Z}^+, (F_j)_{j=1}^{\infty} \quad \text{is} \quad \mathbb{M}_k - \text{allowable}\Big\}\Big\}$$

With the help of relation (5.1) and (5.2), Argyros et al.[1] also studied boundedly modified Tsirelson's space, denoted by $\mathbb{Z}_{\mathbb{M}}(s)$ for some natural number n. The inner supremum is taken over all $\mathfrak{M}-$allowable families for $1 \leq k \leq s$ and over $\mathfrak{M}-$admissible families for $k \geq s + 1$.

Definition 5.2. [1] Let $\alpha > 0$. A complete normed linear space $\mathbb{Z}$ is said to be $\alpha-$distortable if there exists an equivalent norm $|\ .\ |$ on $\mathbb{Z}$ such that for every infinite-dimensional subspace $\mathbb{Y}$ of $\mathbb{Z}$

$$\sup\Big\{\frac{|\ u\ |}{|\ v\ |} : u, v \in \mathbb{Y} : \|u\| = \|v\| = 1\Big\} \geq \alpha$$

$\mathbb{Z}$ is said to be arbitrarily distort-able if it is $\alpha-$distortable for every $\alpha > 1$.

Theorem 5.3. [1] *Suppose that the sequence $\{\beta_n\}_{n=1}^{\infty}$ satisfies $\beta_{m+n} \geq \beta_m \beta_n$ for all natural numbers m, n and let $\beta = \lim \beta_n^{\frac{1}{n}}$. If $\beta_n \mid \beta^n \to 0$, then the space $T\Big\{(F_n, \beta_n)_{n=1}^{\infty}\Big\}$ is arbitrarily distort-able.*

Theorem 5.4. [1] *Let $\{\beta_n\}_{n=1}^{\infty}$ be a regular sequence with $\lim \beta_n^{\frac{1}{n}} = 1$ and $\mathbb{Z} = T\Big\{(F_n, \beta_n)_{n=1}^{\infty}\Big\}$. Then for every $\epsilon > 0$, every infinite dimensional block subspace $\mathbb{Y}$ of $\mathbb{Z}$ contains for every n a sequence of dis-jointly supported vectors $\{y_j\}_{j=1}^{n}$ which is $(1+\epsilon)-$equivalent to the canonical basis of l_∞^n.*

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