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Bi-Univalent Condition Associated with the Modified Sigmoid Function

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Abstract: *In the present work, the authors define and determine the bounds on the first few coefficients of the function* $f(z)$ *belonging to a new class of analytic functions with complex order associated with modified sigmoid function in the open unit disk.*

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1. Introduction

Let $\Gamma(\omega)$ be the class of functions $f(z)$ of the form:

$$f(z)=(z-\omega)+\sum_{k=2}^{\infty}a_k(z-\omega)^k \tag{1}$$

which are analytic and univalent in the open unit disk $U=\{z:z\in C,|z|<1\}$ and normalized with $f(\omega)=0$ and $f'(\omega)-1=0$, where ω is an arbitrary fixed point in U. Let S denote the class of analytic function that are univalent in U. Also, let $\Gamma_p(\omega)$ denote the class of analytic p-valent functions having the form:

$$f_p(z)=(z-\omega)^p+\sum_{k=1}^{\infty}a_{k+p}(z-\omega)^{k+p} \tag{2}$$

in the unit disk U and satisfy the condition that $f_p(z)=0$, $|f_p(z)|<1$ and $z\in U$. Seker andEker [20] introduced and studied the following differential operator $D^{n+p}f_p(z)$, for $f_p(z)\in\Gamma_p(\omega)$ such that

$$D^0 f_p(z)=f_p(z)$$

$$D^1 f_p(z) = D\left(f_p(z)\right) = \frac{(z-\omega)}{p} f'(z) = (z-\omega)^p + \sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right) a_{k+p}(z-\omega)^{p+k}$$

$$D^2 f_p(z) = \frac{(z-\omega)}{p} D^1\left(f_p(z)\right) = (z-\omega)^p + \sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2 a_{k+p}(z-\omega)^{p+k}$$

and in general

$$D^{n+p} f_p(z) = D\left(D^{n+p-1} f_p(z)\right) = (z-\omega)^p + \sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^n a_{k+p}(z-\omega)^{p+k} \quad (3)$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Similarly, we can write for functions $f_p(z) \in \Gamma_p(\omega)$ using Aoul *et al.* [1] such that

$$I^0_{\omega,p}(\lambda,l) f_p(z) = f_p(z),$$

$$I^1_{\omega,p}(\lambda,l) f_p(z) = \left(\frac{1-\lambda+l}{1+l}\right) I^0_{\omega,p}(\lambda,l) f_p(z) + \frac{\lambda(z-\omega)}{1+l}\left(I^0_{\omega,p}(\lambda,l) f_p(z)\right)'$$

and

$$I^n_{\omega,p}(\lambda,l) f_p(z) = \left(\frac{1-\lambda+l}{1+l}\right) I^{n-1}_{\omega,p}(\lambda,l) f_p(z) + \frac{\lambda(z-\omega)}{1+l}\left(I^{n-1}_{\omega,p}(\lambda,l) f_p(z)\right)'. \quad (4)$$

It follows from equation (4) that

$$I^n_{\omega,p}(\lambda,l) f_p(z) = \left(\frac{1+\lambda(p-1)+l}{1+l}\right)^n (z-\omega)^p + \sum_{k=1}^{\infty}\left(\frac{1+\lambda(k+p-1)+l}{1+l}\right)^n a_{p+k}(z-\omega)^{k+p} \quad (5)$$

$n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda \geq 0$ *and* $l \geq 0$.

Trivially, one can show that

$$I^n_{0,p}(1,0) f_p(z) = p^n D^{n+p} f_p(z). \quad (6)$$

It is noted here that the function $f \in S$ has an inverse f^{-1} which is given by

$$f^{-1}\left(f(z)\right) = (z-\omega), \; z \in U$$

and

$$f\left(f^{-1}(\mu)\right) = (\mu-\omega), \; \left\{|\mu| < r_0(f) : r_0(f) \geq \frac{1}{4}\right\}.$$

We can also write that

$$g(\mu) = f^{-1}(\mu) = (\mu-\omega) - a_2(\mu-\omega)^2 + (2a_2^2 - a_3)(\mu-\omega)^3 - (5a_2^3 - 5a_2a_3 + a_4)(\mu-\omega)^4 + \ldots \quad (7)$$

Here, let $g_p(\mu)$ be defined such that

$$g_p(\mu) = f_p^{-1}(\mu) = (\mu-\omega)^p + \sum_{k=1}^{\infty} b_{p+k}(\mu-\omega)^{p+k},$$

where

$$b_{p+1} = -a_{p+1},\; b_{p+2} = 2a_{p+1}^2 - a_{p+2},\; \ldots \quad .$$

A function f is said to be bi-univalent in U if both f and its inverse, f^{-1}, are univalent in U. Suppose that Σ denote the class of all analytic bi-univalent functions in U, several authors have studied the class Σ from different perspective and their results authenticated diversely in literatures, see [2], [3], [4], [5], [8], [11], [12], [13], [14], [16], [21], [22]) among others. However, their results seem to lack full stamina in addressing the coefficient problems for functions in Σ associated with sigmoid function. Consequently, the present work aim at investigating the bi-univalent condition for analytic p-valent function with some fixed points as related to modified sigmoid function in the open unit disk. Few examples of bi-univalent functions are given below:

1. $\frac{z}{1-z}$ and its corresponding inverse is $\frac{\mu}{1-\mu}$.
2. $\frac{1}{2}\log\frac{1+z}{1-z}$ and its corresponding inverse is $\frac{e^{2\mu}-1}{e^{2\mu}+1}$.
3. $\log\frac{1}{1-z}$ and its corresponding inverse is $\frac{e^{\mu}-1}{e^{\mu}}$.

So, the class of bi-univalent functions is non-empty.

For the purpose of this work, we shall consider the following Lemmas.

Lemma 1.1 [18]: Let a function $p \in P$ be given by

$$p(z)=1+\sum_{k=1}^{\infty}p_k z^k, \quad z\in U. \tag{8}$$

Then

$$|p(z)|\leq 2 \qquad k\in\mathbb{N}, \tag{9}$$

where p is the family of function analytic in U for which

$$p(0)=1,\ \Re e\left[p(z)\right]>0, \quad z\in U. \tag{10}$$

Lemma 1.2 [6, 19]: Let the function $r(z)$ be given by

$$r(z)=1+\sum_{k=1}^{\infty}c_k z^k, \quad z\in U \tag{11}$$

be convex in U. Also, let the function $l(z)$ given by

$$l(z)=1+\sum_{k=1}^{\infty}l_k z^k, \quad z\in U \tag{12}$$

be holomorphic in u. If

$$l(z)\prec r(z), \quad z\in U$$

then

$$|l_k|\leq|c_1|, \qquad k\in\mathbb{N}\ .$$

2. Sigmoid Function

Sigmoid function is referred to as special logistic function and defined by

$$g(z)=\frac{1}{1+e^{-z}}.$$

A sigmoid function is a bounded differentiable real function that is defined for all real input values and has a positive derivative at each point. It is perfectly useful in geometric function theory because of the following properties:

1. It outputs real numbers between 0 and 1.
2. It maps a large domain to a small range.
3. It is a one to one function hence, the information is well-preserved.
4. It increases monotonically.

Just of recent, precisely in 2013, Fadipe-Joseph *et al.* [7] defined the modified sigmoid function as $\phi(z)=2g(z)$. They show among others that $\phi(z)$ is a function with the positive real part and that $\phi(z)$ belongs to the class P of Caratheodory functions.

Fortunately, $\phi(z)$ has the following series expansion

$$\phi(z)=1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-... \tag{13}$$

see also Hamzat and Makinde [10], Murugusundaramoorthy and Janani [15], Oladipo and Gbolagade [17].

Definition 2.1: Let $\gamma: U \to \mathbb{C}$ be a convex univalent function in U and satisfying thefollowing conditions:

$$\gamma(0) = 1 \text{ and } \Re e\,\{\gamma(z)\} > 0 \ (z \in U).$$

Further, let $\gamma(z)$ be defined such that

$$\gamma(z)=1+\sum_{k=1}^{\infty}B_k z^k . \tag{14}$$

Now the function $f_p(z)$ is said to belongs to the class $\Sigma^{n,p}(b,l,\alpha,\lambda,\omega,\zeta)$, if and only if

$$1+\frac{1}{b}\left\{\frac{e^{i\zeta}(z-\omega)\left(\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)f_p(z)\right)'}{\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)f_p(z)}-p\,e^{i\zeta}\right\}\prec\frac{1+A(z-\omega)}{1+B(z-\omega)} \tag{15}$$

and

$$1+\frac{1}{b}\left\{\frac{e^{i\zeta}(\mu-\omega)\left(\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)g_p(\mu)\right)'}{\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)g_p(\mu)}-p\,e^{i\zeta}\right\}\prec\frac{1+A(\mu-\omega)}{1+B(\mu-\omega)} , \tag{16}$$

where b is any non-zero complex number, $\prec$ denotes the subordination sign, $\lambda \geq 0$,

$l \geq 0$, $-1 \leq B < A \leq 1$, $|\zeta| < \frac{\pi}{2}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, ω is arbitrary fixed point in U and $\mu, z \in U$.

Hence by the definition of subordination, it follows that

$$1+\frac{1}{b}\left\{\frac{e^{i\zeta}(z-\omega)\left(\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)f_p(z)\right)'}{\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)f_p(z)}-pe^{i\zeta}\right\}=\frac{1+Ah(z-\omega)}{1+Bh(z-\omega)}=\alpha\, p(z)+(1-\alpha)\phi(z),\ \ \alpha\in[0,1] \tag{17}$$

and

$$1+\frac{1}{b}\left\{\frac{e^{i\zeta}(\mu-\omega)\left(\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)g_p(\mu)\right)'}{\frac{1}{p^n}I^n_{\omega,p}(\lambda,l)g_p(\mu)}-pe^{i\zeta}\right\}=\frac{1+Ah(\mu-\omega)}{1+Bh(\mu-\omega)}=\alpha\, q(\mu)+(1-\alpha)\phi(\mu),\ \ \alpha\in[0,1] \tag{18}$$

where $p(z), q(z), \phi(z)\in P(class\ of\ Caratheodory\ functions)$ such that

$$p(z)=1+\sum_{k=1}^{\infty}p_k z^k,\ \ z\in U \tag{19}$$

and

$$q(\mu)=1+\sum_{k=1}^{\infty}q_k \mu^k,\ \ \mu\in U \tag{20}$$

while $\phi(z)$ is as earlier defined in (13).

Special Remarks:

1. Suppose that $\zeta = 0$ and $\alpha = 1$ in the above definition then, we immediately have the definition given by Hamzat and Adeleke [9].
2. Following the linear combination of $p(z)$ and $\phi(z)$, it is obvious that if letting $\alpha=0$ in (17) and (18), then the bi-univalent results obtained would be associated purely, with the modified Sigmoid function $\phi(z)$ and when $\alpha=1$, the results obtained would be associated purely with the usual $p(z)\in P$.

3. Main Results

Theorem 3.1: Let $f_p(z)\in\Sigma^{n,p}(b,l,\alpha,\lambda,\omega,\zeta)$, then for $\lambda\geq 0,\ l\geq 0,\ \alpha\in[0,1]$,
$n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $z\in U$,

$$|a_{p+1}| \leq \sqrt{\frac{\alpha |b||B_1| \left(\frac{1+\lambda(p-1)+l}{1+l}\right)^{2n}}{2\left(\frac{1+\lambda(p-1)+l}{1+l}\right)^{n}\left(\frac{1+\lambda(p+1)+l}{1+l}\right)^{n} - \left(\frac{1+\lambda p+l}{1+l}\right)^{2n}}}$$

and

$$|a_{p+2}| \leq \frac{\alpha |b||B_1|}{2}\left(\frac{1+\lambda(p-1)+l}{1+\lambda(p+1)+l}\right)^{n} + \frac{|b|^2}{4}\left(1+\alpha(2|B_1|-1)\right)^2\left(\frac{1+\lambda(p-1)+l}{1+\lambda p+l}\right)^{2n}.$$

Proof:

Let $f_p(z) \in \Sigma^{n,p}(b,l,\alpha,\lambda,\omega,\zeta)$. Then from (17) and (18), it follows that

$$2e^{i\zeta}\left(\frac{1+\lambda p+l}{1+\lambda(p-1)+l}\right)^{n} a_{p+1} = b\left(1+\alpha(2p_1-1)\right) \tag{21}$$

and

$$2e^{i\zeta}\left(\frac{1+\lambda(p+1)+l}{1+\lambda(p-1)+l}\right)^{n} a_{p+2} - e^{i\zeta}\left(\frac{1+\lambda p+l}{1+\lambda(p-1)+l}\right)^{2n} a_{p+1}^2 = \alpha b\, p_2. \tag{22}$$

Also

$$2e^{i\zeta}\left(\frac{1+\lambda p+l}{1+\lambda(p-1)+l}\right)^{n} a_{p+1} = -b\left(1+\alpha(2q_1-1)\right) \tag{23}$$

and

$$e^{i\zeta}\left[4\left(\frac{1+\lambda p+l}{1+l}\right)^{n} - \left(\frac{1+\lambda p+l}{1+\lambda(p-1)+l}\right)^{2n}\right]a_{p+1}^2 - 2e^{i\zeta}\left(\frac{1+\lambda(p+1)+l}{1+\lambda(p-1)+l}\right)^{n} a_{p+2} = \alpha b q_2 \tag{24}$$

since, $b_{p+1} = -a_{p+1}$ and $b_{p+2} = 2a_{p+1}^2 - a_{p+2}$. Furthermore, from (21) and (23), it is obvious that

$$a_{p+1} = \frac{b}{2}e^{i\zeta}\left(1+\alpha(2p_1-1)\right)\left(\frac{1+\lambda p+l}{1+l}\right)^{n} = -\frac{b}{2}e^{i\zeta}\left(1+\alpha(2q_1-1)\right)\left(\frac{1+\lambda p+l}{1+l}\right)^{n}, \tag{25}$$

which implies that

$$p_1 = q_1. \tag{26}$$

If we square both side (21) and (23) and then add, we have

$$a_{p+1}^2 = \frac{b^2}{8e^{2i\zeta}}\left[\left(1+\alpha(2p_1-1)\right)^2 + \left(1+\alpha(2q_1-1)\right)^2\right]\left(\frac{1+\lambda(p-1)+l}{1+\lambda p+l}\right)^{2n}. \tag{27}$$

Also, add equations (22) and (24), then

$$2e^{i\zeta}\left[2\left(\frac{1+\lambda(p+1)+l}{1+\lambda(p-1)+l}\right)^{n}-\left(\frac{1+\lambda p+l}{1+\lambda(p-1)+l}\right)^{2n}\right]a_{p+1}^{2}=\alpha b(p_2+q_2) \tag{28}$$

which implies that

$$a_{p+1}^{2}=\frac{\alpha\, b\, e^{i\zeta}(p_2+q_2)}{2\left[2\left(\frac{1+\lambda(p+1)+l}{1+\lambda(p-1)+l}\right)^{n}-\left(\frac{1+\lambda p+l}{1+\lambda(p-1)+l}\right)^{2n}\right]}. \tag{29}$$

Recall that $p(z), q(\mu)\subset h(U)$. With reference to equations (14), (19), (20) and Lemma (1.2), we have

$$|p_k|=\left|\frac{p^{k}(0)}{k!}\right|\leq |B_1|, \quad k\in\mathbb{N} \tag{30}$$

and

$$|q_k|=\left|\frac{q^{k}(0)}{k!}\right|\leq |B_1|, \quad k\in\mathbb{N}. \tag{31}$$

Therefore, applying equations (30) and (31) in (29), we obtain

$$\left|a_{p+1}\right|^{2}=\frac{\alpha|b||B_1|}{2\left(\frac{1+\lambda(p+1)+l}{1+\lambda(p-1)+l}\right)^{n}-\left(\frac{1+\lambda p+l}{1+\lambda(p-1)+l}\right)^{2n}} \tag{32}$$

which readily yields the expected bounds on the coefficient of a_{p+1} as contained in Theorem 3.1. Also, suppose that equation (24) is subtracted from (22), then

$$4e^{i\zeta}\left(a_{p+2}-a_{p+1}^{2}\right)\left(\frac{1+\lambda(p+1)+l}{1+\lambda(p-1)+l}\right)=\alpha\, b\,(p_2-q_2). \tag{33}$$

Using equation (27) in (33), we have

$$a_{p+2}=\frac{\alpha\, b\, e^{-i\zeta}(p_2-q_2)}{4\left(\frac{1+\lambda(p+1)+l}{1+\lambda(p-1)+l}\right)^{n}}+\frac{b^{2}e^{-2i\zeta}\left(\left(1+\alpha(2p_1-1)\right)^{2}+\left(1+\alpha(2q_1-1)\right)^{2}\right)}{8\left(\frac{1+\lambda p+l}{1+\lambda(p-1)+l}\right)^{2n}}. \tag{34}$$

The application of equations (30), (31) and Lemma 1.1 in (34) yields

$$\left|a_{p+2}\right|\leq\frac{\alpha|b||B_1|}{2\left(\frac{1+\lambda(p+1)+l}{1+\lambda(p-1)+l}\right)^{n}}+\frac{|b|^{2}\left(\left(1+\alpha(2|B_1|-1)\right)^{2}\right)}{4\left(\frac{1+\lambda p+l}{1+\lambda(p-1)+l}\right)^{2n}}, \tag{35}$$

which is the required bound on a_{p+2} as seen in Theorem 3.1 and this obviously completes the proof.

Now with various choices of the parameters l, n, p, α and λ in Theorem 3.1, several corollaries are obtained. Few of them are stated below.

Let $p=1$ in Theorem 3.1, and then the following corollary is obtained.

Corollary 3.2: Let $f_1(z)\in\Sigma^{n,1}(b,l,\alpha,\lambda,\omega,\zeta)$, then for $\lambda\geq 0,\ l\geq 0,\ \alpha\in[0,1]$,

$n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $z\in U$,

$$|a_2|\leq\sqrt{\frac{\alpha|b||B_1|}{2\left(\frac{1+\lambda(p+1)+l}{1+l}\right)^n-\left(\frac{1+\lambda+l}{1+l}\right)^{2n}}}$$

and

$$|a_3|\leq\frac{\alpha|b||B_1|}{2}\left(\frac{1+l}{1+2\lambda+l}\right)^n+\frac{|b|^2}{4}\left(1+\alpha(2|B_1|-1)\right)^2\left(\frac{1+l}{1+\lambda+l}\right)^{2n}.$$

Suppose that $p=\alpha=1$ in Theorem 3.1, then the following corollary is obtained.

Corollary 3.3: Let $f_1(z)\in\Sigma^{n,1}(b,l,1,\lambda,\omega,\zeta)$, then for $\lambda\geq 0,\ l\geq 0, n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and

$z\in U$, then

$$|a_2|\leq\sqrt{\frac{|b||B_1|}{2\left(\frac{1+\lambda(p+1)+l}{1+l}\right)^n-\left(\frac{1+\lambda+l}{1+l}\right)^{2n}}}$$

and

$$|a_3|\leq\frac{|b||B_1|}{2}\left[\left(\frac{1+l}{1+2\lambda+l}\right)^n+2|b||B_1|\left(\frac{1+l}{1+\lambda+l}\right)^{2n}\right].$$

If $p=\alpha=1$ and $n=0$ in Theorem 3.1, then the following corollary is obtained.

Corollary 3.4: Let $f_1(z)\in\Sigma^{0,1}(b,l,1,\lambda,\omega,\zeta)$, then for $\lambda\geq 0,\ l\geq 0$ and $z\in U$, then

$$|a_2|\leq\sqrt{|b||B_1|}$$

and

$$|a_3|\leq\frac{|b||B_1|}{2}\left[1+2|b||B_1|\right].$$

Set $p=\alpha=\lambda=1$ and $l=0$.

Then, we obtain the following corollary.

Corollary 3.5: Let $f_1(z)\in\Sigma^{n,1}(b,0,1,1,0,\zeta)$, then for $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $z\in U$, then

$$|a_2|\leq\sqrt{\frac{|b||B_1|}{2.3^n-2^{2n}}}$$

and

$$|a_3|\leq\frac{|b||B_1|}{2}\left[\left(\frac{1}{3}\right)^n+2|b||B_1|\left(\frac{1}{2}\right)^{2n}\right].$$

Theorem 3.6: Let $f_p(z)\in\Sigma^{n,p}(b,l,\alpha,\lambda,\omega,\zeta)$, then for any complex number ψ

$$\left|a_{p+2}-\psi a_{p+1}^2\right|\leq\frac{\alpha|b||B_1|}{2\left(\frac{1+\lambda(p+1)+l}{1+l}\right)^n-}+\frac{(1-\psi)|b|^2\left(1+\alpha(2|B_1|-1)\right)^2}{4\left(\frac{1+\lambda p+l}{1+l}\right)^{2n}}.$$

Concluding Remarks:

Ultimately, it is pertinent to note that one of the prime significant of the bounds obtained for the initial coefficients $|a_{p+1}|$ and $|a_{p+2}|$ for function $f_p(z)\in\Sigma^{n,p}(b,l,\alpha,\lambda,\omega,\zeta)$ is the information about their geometric properties. For instance, the bounds can be usedin establishing the Fekete-Szego functional $\left|a_{p+2}-\psi a_{p+1}^2\right|$, Hankel determinant and so on. In the future, these bounds can also be used in putting information into a special code (i.e data encryption) among others.

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